Hilbert space compression of groups

G. Arzhantseva, C. Druţu, V. Guba and M. Sapir

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The Hilbert space compression of a space is a q.i. invariant.

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- Expander families of graphs are not embeddable into Hilbert spaces.

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- (Gromov) Expanders embed into f.g. groups. So there are groups that are not coarsely embeddable into Hilbert spaces. (Uniformly convex Banach spaces?) Their Hilbert space compression $= 0$.

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Problem. Is it true that the compression function of F is $\gg \sqrt{x}$?

Wreath products

Theorem (Arzhantseva-Guba-Sapir)

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So one can find a f.g. group with an arbitrary small but non-zero compression function.

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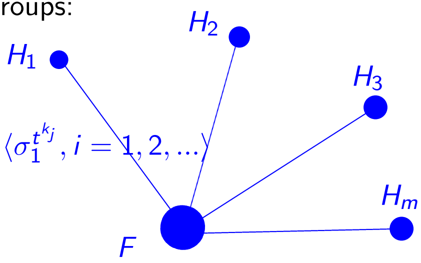

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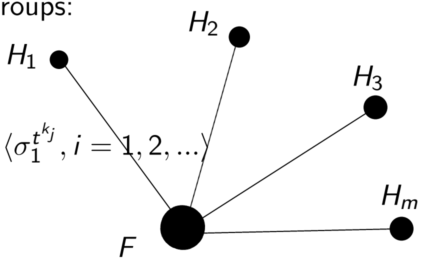

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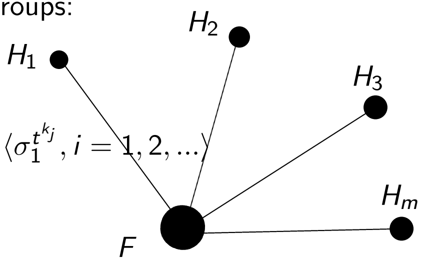

Each M_j is generated by $\langle \sigma_1(j), ..., \sigma_m(j) \rangle$. We identify $\sigma_i(j)$ with $\sigma^{t^{k_j}}$ of $H = \langle \sigma, t \rangle$.