Hilbert space compression of groups

M. Sapir

January 18, 2014

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The Hilbert space compression of a space is a q.i. invariant.

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- Expander families of graphs are not embeddable into Hilbert spaces.

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- Linear groups are embeddable into Hilbert spaces (they have finite decomposition complexity, hence property A).

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Problem. Is it true that the compression function of F is $\gg \sqrt{x}$?. Is it true that F satisfies property A? How about the group of piecewise fractional transformations of the circle fixing ∞ (Monod, Lodha-Moore)?

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- ρ is increasing but $\frac{\rho(x)}{x/\log x}$ is decreasing, $\lim \rho(x) = \infty$.
- ρ is subadditive ($\rho(x+y) \leq \rho(x) + \rho(y)$ for large enough x, y).

Theorem. (Arzhantseva-Druțu-S.) For every $\rho \in \mathcal{C}$ there exists a f.g. group $G$of asdim ≤ 2 and such that the Hilbert space compression function and the uniformly convex Banach space compression function of G is

- between $\sqrt{\rho}$ and ρ;
- between $\frac{\rho(x)}{\log x}$ and $\rho(x)$.

So one can find a f.g. group with an arbitrary small but non-zero compression function and with arbitrary Hilbert space (or uniformly convex Banach space) compression.

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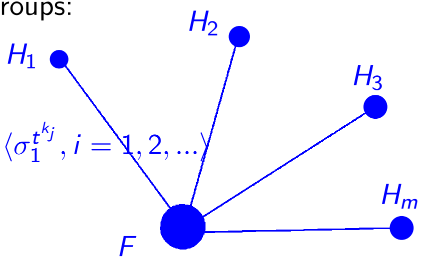

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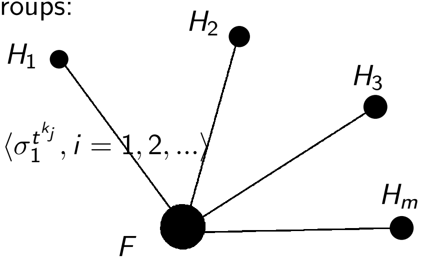

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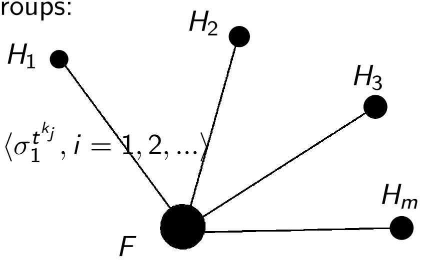

Each M_j is generated by $\langle \sigma_1(j), ..., \sigma_m(j)\rangle$. We identify $\sigma_i(j)$ with $\sigma^{t^{k_j}}$ of $H = \langle \sigma, t\rangle$.

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Aspherical manifolds

Theorem (S., Journal of AMS, 2014) There exists a smooth Riemannian aspherical closed manifold M^4 whose fundamental group contains an expander, hence is not coarsely embeddable into a Hilbert space, has Hilbert space compression 0 and infinite asymptotic dimension, does not satisfy the Baum-Connes conjecture with coefficients.

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Proof. Take the Gromov group G containing an expander. It is aspherical and recursively presented. Hence is inside a finitely presented group H by Higman. Unfortunately H is almost never aspherical if we use the previously known versions of the proof of Higman's theorem. So one needs a version preserving asphericity. Once an aspherical H is constructed one can use the Michael Davis construction to turn the group into a manifold.