BOUNDEDNESS AND COMPACTNESS OF HANKEL OPERATORS ON THE SPHERE

1. Introduction

$S = \{z \in \mathbf{C}^n : |z| = 1\}$, the unit sphere in $\mathbf{C}^n$.

$\sigma =$ the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$.

Normalization: $\sigma(S) = 1$.

The *Cauchy projection* P is defined by the integral formula

$$(Pf)(w) = \int \frac{f(\zeta)}{(1 - \langle w, \zeta \rangle)^n} d\sigma(\zeta), \qquad |w| < 1.$$

P is the orthogonal projection from $L^2(S, d\sigma)$ onto the Hardy space $H^2(S)$.

Normalized reproducing kernel for $H^2(S)$:

$$k_z(w) = \frac{(1 - |z|^2)^{n/2}}{(1 - \langle w, z \rangle)^n}, \qquad |w| \leq 1, \quad |z| < 1.$$

The formula

$$d(\zeta,\xi) = |1-\langle\zeta,\xi\rangle|^{1/2}, \qquad \zeta,\xi\in S, \tag{1.1}$$

defines a metric on the sphere (anisotropic metric).

For $\zeta\in S$ and $r>0$, denote

$$B(\zeta,r) = \{x\in S : |1-\langle x,\zeta\rangle|^{1/2} < r\}$$

There is a constant $A_0\in(2^{-n},\infty)$ such that

$$2^{-n}r^{2n} \le \sigma(B(\zeta,r)) \le A_0 r^{2n}$$

for all $\zeta\in S$ and $0<r\le\sqrt{2}$.

A function $f\in L^1(S,d\sigma)$ is said to have *bounded mean oscillation* if

$$\|f\|_{\mathrm{BMO}} = \sup_{\substack{\zeta\in S\\ r>0}} \frac{1}{\sigma(B(\zeta,r))}\int_{B(\zeta,r)} |f-f_{B(\zeta,r)}|d\sigma < \infty,$$

where $f_B = \int_B f d\sigma/\sigma(B)$, the average of f over B. A function $f\in L^1(S,d\sigma)$ is said to have *vanishing mean oscillation* if

$$\lim_{\delta\downarrow 0}\sup_{\substack{\zeta\in S\\ 0<r\le\delta}} \frac{1}{\sigma(B(\zeta,r))}\int_{B(\zeta,r)} |f-f_{B(\zeta,r)}|d\sigma = 0.$$

BMO = all functions of bounded mean oscillation on S.

VMO = all functions of vanishing mean oscillation on S.

The *Hankel operator* $H_f : H^2(S) \to L^2(S, d\sigma)$ is defined by

$$H_f = (1 - P)M_f|H^2(S).$$

Relation between commutator and Hankel operators:

$$[P, M_f] = H^*_{\bar{f}} - H_f,$$

We can think of $[P, M_f]$ as a matrix with respect to the space decomposition

$$L^2(S, d\sigma) = H^2(S) \oplus \{H^2(S)\}^\perp.$$

That is, with respect to this space decomposition,

$$[P, M_f] = \begin{bmatrix} 0 & H^*_{\bar{f}} \\ -H_f & 0 \end{bmatrix}.$$

A fundamental result:

Theorem. (Coifman, Rochberg and Weiss, 1976)
(a) $[P, M_f]$ is bounded if and only if $f \in \mathrm{BMO}$.
(b) $[P, M_f]$ is compact if and only if $f \in \mathrm{VMO}$.
(c) Moreover, $\|[P, M_f]\| \le C\|f\|_{\mathrm{BMO}}$.

The "only if" part is easy; it follows from the inequality

$$\|(f - \langle fk_z, k_z\rangle)k_z\|^2 \le \|H_f k_z\|^2 + \|H_{\bar{f}} k_z\|^2.$$

The hard part of this theorem is the "if" part.

A basic fact: if $h \in H^2(S)$, then $H_h = 0$. Therefore

$$H_f = H_{f-Pf}.$$

Also,

$$f - Pf = H_f 1.$$

Recall that there is a famous $T1$-Theorem for singular inetgral operators on L^2. In analogy with that, the theorem of Coifman, Rochberg and Weiss implies what might be called

$H1$-Theorem.
(a) If $f - Pf \in$ BMO, then H_f is bounded.
(b) If $f - Pf \in$ VMO, then H_f is compact.

But in the $T1$-Theorem, the sufficient conditions for boundedness are well known to be necessary. So one naturally asks, what happens in the case of the $H1$-Theorem ?

This talk is about the various *converses* to the $H1$-Theorem stated above.

In general, there are two kinds of problems in the theory of Hankel operators, namely "two-sided" problems and "one-sided" problems. A "two-sided" problem concerns H_f and $H_{\bar{f}}$ simultaneously. "Two-sided" problems are equivalent to the study of the commutator $[P, M_f]$. Therefore there is a large body of literature on "two-sided" problems.

By contrast, a "one-sided" problem is the study of H_f alone. Almost invariably, a "one-sided" problem is more difficult than the corresponding "two-sided" problem. The reason for this is very simple: for a "one-sided" problem, the inequality

$$\|(f - \langle f k_z, k_z \rangle) k_z\|^2 \le \|H_f k_z\|^2 + \|H_{\bar{f}} k_z\|^2$$

is useless, because one assumes nothing about $H_{\bar{f}}$. To solve a "one-sided" problem, one must find a way to control mean oscillation by other methods.

"One-sided" problems are all about these *other methods*.

In the case $n = 1$, i.e., on the unit circle, because

$$\overline{f - Pf} \in H^2 \tag{1.3}$$

every "one-sided" problem is actually a "two-sided" problem. But when $n \ge 2$, (1.3) no longer holds, and a difference between "two-sided" problems and "one-sided" problems appears. The main difficulty in "one-sided" problems is the fact that the subspace

$$L^2(S, d\sigma) \ominus \{H^2(S) + \overline{H^2(S)}\} \tag{1.4}$$

is huge and intractable when $n \ge 2$.

A good example of a "one-sided" result is the following:

Theorem 1.1. (Dechao Zheng) *Let* $f \in \mathrm{BMO}$. *Then the Hankel operator* H_f *is compact if and only if*

$$\lim_{|z|\uparrow 1} \|H_f k_z\| = 0.$$

Although this is the best existing result on the compactness of H_f, questions still remain. Note that Theorem 1.1 is really a statement about the **FAMILY**

$$\{H_f : f \in \mathrm{BMO}\}$$

as a whole. We know that a necessary condition for any operator X to be compact is

$$\lim_{|z|\uparrow 1} \|X k_z\| = 0. \tag{1.5}$$

What Theorem 1.1 really says is that if

$$X \in \{H_f : f \in \mathrm{BMO}\},$$

then (1.5) is also a sufficient condition for X to be compact. This is certainly very nice, but it does not say much about f.

We would like to determine the compactness of H_f in terms of f, such as the membership of f in some easily-defined function class.

As it turns out, the Hankel operator H_f actually tells us a great deal about the commutator $[P, M_{f-Pf}]$. That is, in many situations, a "one-sided" problem actually has a "two-sided" solution! In other words, notwithstanding the size of

$$L^2(S, d\sigma) \ominus \{H^2(S) + \overline{H^2(S)}\},$$

the theory of Hankel operators in the case $n \geq 2$ resembles the case $n = 1$ in more ways than we previously realized.

What initially led to this investigation was the consideration of the subset

$$\mathcal{A} = \{f \in L^\infty(S, d\sigma) : H_f \text{ is compact}\}$$

of $L^\infty(S, d\sigma)$. As Davie and Jewell observed, $\mathcal{A}$ is in fact a Banach subalgebra of $L^\infty(S, d\sigma)$.

When $n = 1$, i.e., in the case of the unit circle, it is well known that

$$\mathcal{A} = H^\infty + C(\mathbf{T}),$$

which is unquestionably a direct condition for compactness. But when $n \geq 2$, $\mathcal{A}$ is known to be strictly larger than $H^\infty(S) + C(S)$ (Davie and Jewell).

So here at least, there is a genuine difference between the case $n = 1$ and the case $n \geq 2$. But wait, for *difference* is not the whole story. Even for $\mathcal{A}$, there is similarity between the case $n = 1$ and the case $n \geq 2$.

Let us also consider the subset

$$\mathcal{A}_1 = \{f \in L^\infty(S, d\sigma) : f - Pf \in \text{VMO}\}$$

of $L^\infty(S, d\sigma)$. By the $H1$-Theorem of Coifman, Rochberg and Weiss we have

$$\mathcal{A}_1 \subset \mathcal{A}.$$

One might say that $\mathcal{A}_1$ is the obvious part of $\mathcal{A}$. Our first result is the reverse inclusion, i.e., $\mathcal{A}$ consists of nothing but its obvious part.

Theorem 1.2. $\mathcal{A} \subset \mathcal{A}_1$.

As it turns out, this result can be refined and improved in many different ways.

For each $f \in L^1(S, d\sigma)$ and each $\zeta \in S$, denote

$$\mathrm{LMO}(f)(\zeta) = \lim_{\delta \downarrow 0} \sup_{B(\xi,r) \subset B(\zeta,\delta)} \frac{1}{\sigma(B(\xi, r))} \int_{B(\xi,r)} |f - f_{B(\xi,r)}| d\sigma,$$

which is called the *local mean oscillation* of f at ζ.

Theorem 1.3. *If f is a function in* BMO *and ζ is a point in S such that*

$$\lim_{\substack{z \to \zeta \\ |z|<1}} \|H_f k_z\| = 0, \tag{1.6}$$

then $\mathrm{LMO}(f - Pf)(\zeta) = 0$.

Corollary 1.4. *Suppose that $f \in$* BMO. *If*

$$\lim_{|z| \uparrow 1} \|H_f k_z\| = 0, \tag{1.7}$$

then $f - Pf \in$ VMO.

Corollary 1.4 explains why Theorem 1.1 holds: if f belongs to BMO and satisfies (1.7), then $f - Pf \in$ VMO, which implies the compactness of $[P, M_{f-Pf}]$, which in turn implies the compactness of $H_{f-Pf} = H_f$.

Corollary 1.5. *Suppose that $f \in$* BMO *and that*

$$f \perp H^2(S) + \overline{H^2(S)}.$$

Then H_f is compact if and only if $H_{\bar{f}}$ is compact.

This is reminds us a theorem about Hankel operators on the Segal-Bargmann space $H^2(\mathbf{C}^n, d\mu)$ due to Berger and Coburn.

Theorem 1.6. *There exists a constant* $0 < C < \infty$ *which depends only on the complex dimension* n *such that*

$$\|f - Pf\|_{\mathrm{BMO}} \leq C \sup_{|z|<1} \|H_f k_z\|$$

for every $f \in \mathrm{BMO}$.

This and the $H1$-Theorem of Coifman, Rochberg and Weiss together give us the inequality

$$\|[P, M_{f-Pf}]\| \leq C_1 \|H_f\|, \tag{1.8}$$

$f \in \mathrm{BMO}$.

Corollary 1.7. *There exists a constant* $0 < C < \infty$ *which depends only on the complex dimension* n *such that for* $f \in \mathrm{BMO}$ *satisfying the condition* $f \perp H^2(S) + \overline{H^2(S)}$, *we have*

$$C^{-1}\|H_f\| \leq \|H_{\bar{f}}\| \leq C\|H_f\|.$$

Suppose that A is a bounded operator on a Hilbert space $\mathcal{H}$. Recall that the *essential norm* of A is defined by the formula

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \text{ is compact on } \mathcal{H}\}.$$

An analogue of (1.8) holds for essential norms.

Theorem 1.8. *There exists a constant* $0 < C < \infty$ *which depends only on the complex dimension* n *such that*

$$\|[P, M_{f-Pf}]\|_{\mathcal{Q}} \leq C\|H_f\|_{\mathcal{Q}}$$

for every $f \in \mathrm{BMO}$.

Note that in all the results above the condition $f \in \mathrm{BMO}$ was a part of the assumption. But the bound provided by Theorem 1.6 enables us to deal with symbol functions which are not *a priori* assumed to be in BMO. For $\psi \in L^2(S, d\sigma)$, we can still define the Hankel operator H_ψ on the dense subset $H^\infty(S)$ of $H^2(S)$. That is, $H_\psi h = (1 - P)(\psi h)$ for $h \in H^\infty(S)$.

Theorem 1.9. *If $\psi \in L^2(S, d\sigma)$ and if*

$$\sup_{|z|<1} \|H_\psi k_z\| < \infty,$$

then $\psi - P\psi \in \mathrm{BMO}$.

Combining Theorem 1.9 and Corollary 1.4, and using the fact that $H_\psi = H_{\psi - P\psi}$, we have the following improvement of Theorem 1.1:

Corollary 1.10. *Suppose that $\psi \in L^2(S, d\sigma)$ and that*

$$\lim_{|z|\uparrow 1} \|H_\psi k_z\| = 0.$$

Then $\psi - P\psi \in \mathrm{VMO}$. Consequently H_ψ extends to a compact operator from $H^2(S)$ to $L^2(S, d\sigma) \ominus H^2(S)$.

Summarizing, we now have the

Complete Version of $H1$-Theorem.
Let $f \in L^2(S, d\sigma)$. Then
(a) H_f is bounded if and only if $f - Pf \in \mathrm{BMO}$;
(b) H_f is compact if and only if $f - Pf \in \mathrm{VMO}$.

Recall that the "if" part is due to Coifman, Rochberg and Weiss; our contribution is the "only if" part.

2. An Estimate of Mean Oscillation

Coifman, Rochberg and Weiss showed that the Cauchy projection P maps $L^\infty(S, d\sigma)$ into BMO. In fact, something slightly stronger is also true:

Proposition 2.1. *If* $f \in \mathrm{BMO}$, *then* $Pf \in \mathrm{BMO}$.

As it turns out, the key to the proofs of the results in Section 1 is the following quantitative refinement of Proposition 2.1.

Proposition 2.2. *There exists a constant* $0 < C_{2.2} < \infty$ *which depends only on the complex dimension* n *such that for all* $f \in L^2(S, d\sigma)$ *and* $B = B(\zeta, r)$, *where* $\zeta \in S$ *and* $r > 0$, *we have*

$$\left\{ \frac{1}{\sigma(B)} \int_B |Pf - (Pf)_B|^2 d\sigma \right\}^{1/2} \leq C_{2.2} \left\{ \frac{1}{\sigma(B_1)} \int_{B_1} |f - f_{B_1}|^2 d\sigma \right\}^{1/2} + C_{2.2} \sum_{k=2}^{\infty} \frac{2^{-k}}{\sigma(B_k)} \int_{B_k} |f - f_{B_k}| d\sigma,$$

where $B_k = B(\zeta, 2^k r)$ *for every* $k \geq 1$.

Proof. Given $f \in L^2(S, d\sigma)$ and $B = B(\zeta, r)$, we may assume $\|(Pf - (Pf)_B)\chi_B\| \neq 0$, for otherwise there is nothing to prove. Define

$$g = \frac{1}{\|(Pf - (Pf)_B)\chi_B\|}(Pf - (Pf)_B)\chi_B,$$

which is, of course, a unit vector in $L^2(S, d\sigma)$. Write 1 for the constant function of value 1 on S. Then obviously $\langle 1, g\rangle = 0$. Thus

$$\begin{aligned} \left\{\frac{1}{\sigma(B)}\int_B |Pf - (Pf)_B|^2 d\sigma\right\}^{1/2} &= \frac{\langle Pf - (Pf)_B, g\rangle}{\sigma^{1/2}(B)} \\ &= \frac{\langle Pf, g\rangle}{\sigma^{1/2}(B)}. \end{aligned} \tag{2.1}$$

To estimate $\langle Pf, g\rangle$, note that $P1 = 1$, which leads to $\langle 1, Pg\rangle = \langle 1, g\rangle = 0$. Hence

$$\begin{aligned} &\langle Pf, g\rangle = \langle f, Pg\rangle = \langle f - f_{B_1}, Pg\rangle \\ &\quad = \int_{B_1}(f - f_{B_1})\overline{Pg}d\sigma + \sum_{k=2}^{\infty}\int_{B_k\setminus B_{k-1}}(f - f_{B_1})\overline{Pg}d\sigma. \end{aligned} \tag{2.2}$$

Next we estimate the terms in (2.2), using the properties of g and P. For the first term in (2.2), we have

$$\int_{B_1}|f - f_{B_1}||Pg|d\sigma \leq \|(f - f_{B_1})\chi_{B_1}\|\|Pg\| \leq \|(f - f_{B_1})\chi_{B_1}\|.$$

Recall that $\sigma(B_1) \leq 2^{3n} A_0 \sigma(B)$. Let $C_1 = (2^{3n} A_0)^{1/2}$. Then

$$\begin{aligned}\int_{B_1} |f - f_{B_1}||Pg| d\sigma &\leq \|(f - f_{B_1})\chi_{B_1}\| \\ &= \sigma^{1/2}(B_1) \left\{ \frac{1}{\sigma(B_1)} \int_{B_1} |f - f_{B_1}|^2 d\sigma \right\}^{1/2} \\ &\leq C_1 \sigma^{1/2}(B) \left\{ \frac{1}{\sigma(B_1)} \int_{B_1} |f - f_{B_1}|^2 d\sigma \right\}^{1/2}. \end{aligned} \tag{2.3}$$

To estimate the other terms in (2.2), we need the fact that there is a constant C_2 which depends only on n such that

$$\left| \frac{1}{(1 - \langle x, y \rangle)^n} - \frac{1}{(1 - \langle x, \zeta \rangle)^n} \right| \leq C_2 \frac{|1 - \langle y, \zeta \rangle|^{1/2}}{|1 - \langle x, \zeta \rangle|^{n+(1/2)}} \tag{2.4}$$

if $y \in B$ and $x \in S \backslash B_1$.

Thus if $y \in B$ and $x \in B_k \backslash B_{k-1}$, $k \geq 2$, then

$$\begin{aligned}\left| \frac{1}{(1 - \langle x, y \rangle)^n} - \frac{1}{(1 - \langle x, \zeta \rangle)^n} \right| &\leq \frac{C_2 r}{(2^{k-1} r)^{2n+1}} \\ &= \frac{2^{2n+1} C_2}{2^k} \cdot \frac{1}{(2^k r)^{2n}} \leq \frac{C_3}{2^k \sigma(B_k)}. \end{aligned}$$

By the definition of g, we have $g = 0$ on $S \backslash B$ and

$$\int_B g d\sigma = 0.$$

Also, by the Cauchy-Schwarz inequality,

$$\int_B |g| d\sigma \leq \sigma^{1/2}(B) \|g\| = \sigma^{1/2}(B).$$

For $x \in S\backslash B_1$ we have

$$(Pg)(x) = \int_B \frac{g(y)}{(1-\langle x,y\rangle)^n} d\sigma(y)$$
$$= \int_B \left(\frac{1}{(1-\langle x,y\rangle)^n} - \frac{1}{(1-\langle x,\zeta\rangle)^n} \right) g(y) d\sigma(y).$$

Therefore
(2.6)
$$|(Pg)(x)| \le \frac{C_3}{2^k \sigma(B_k)} \int_B |g| d\sigma \le \frac{C_3 \sigma^{1/2}(B)}{2^k \sigma(B_k)} \quad \text{if } \ x \in B_k \backslash B_{k-1},$$

$k \ge 2$. Integrating the above over $B_k \backslash B_{k-1}$, we see that
(2.7)
$$\int_{B_k \backslash B_{k-1}} |Pg| d\sigma \le \frac{C_3 \sigma^{1/2}(B)}{2^k \sigma(B_k)} \sigma(B_k \backslash B_{k-1}) \le \frac{C_3}{2^k} \sigma^{1/2}(B)$$

if $k \ge 2$. Applying (2.6) and (2.7), for each $k \ge 2$ we have

$$\int_{B_k \backslash B_{k-1}} |f - f_{B_1}||Pg| d\sigma \le \int_{B_k \backslash B_{k-1}} |f - f_{B_k}||Pg| d\sigma$$
$$+ |f_{B_k} - f_{B_1}| \int_{B_k \backslash B_{k-1}} |Pg| d\sigma$$
$$\le \frac{C_3 \sigma^{1/2}(B)}{2^k \sigma(B_k)} \int_{B_k \backslash B_{k-1}} |f - f_{B_k}| d\sigma$$
$$+ \frac{C_3}{2^k} \sigma^{1/2}(B) |f_{B_k} - f_{B_1}|.$$

But

$$|f_{B_k} - f_{B_1}| \le \sum_{j=2}^{k} |f_{B_j} - f_{B_{j-1}}| \le \sum_{j=2}^{k} \left(\frac{\sigma(B_j)}{\sigma(B_{j-1})} \right) \frac{1}{\sigma(B_j)} \int_{B_j} |f - f_{B_j}| d\sigma.$$

We see that if we set $C_4 = (1 + 2^{3n} A_0) C_3$, then

$$\int_{B_k \setminus B_{k-1}} |f - f_{B_1}||Pg| d\sigma \le \frac{C_4}{2^k} \sum_{j=2}^{k} \frac{\sigma^{1/2}(B)}{\sigma(B_j)} \int_{B_j} |f - f_{B_j}| d\sigma.$$

Therefore

$$\begin{aligned}
&\sum_{k=2}^{\infty} \int_{B_k \setminus B_{k-1}} |f - f_{B_1}||Pg| d\sigma \\
&\qquad \le C_4 \sigma^{1/2}(B) \sum_{k=2}^{\infty} \frac{1}{2^k} \sum_{j=2}^{k} \frac{1}{\sigma(B_j)} \int_{B_j} |f - f_{B_j}| d\sigma \\
&\qquad = C_4 \sigma^{1/2}(B) \sum_{j=2}^{\infty} \left(\sum_{k=j}^{\infty} \frac{1}{2^k} \right) \frac{1}{\sigma(B_j)} \int_{B_j} |f - f_{B_j}| d\sigma \\
&\qquad = 2 C_4 \sigma^{1/2}(B) \sum_{j=2}^{\infty} \frac{2^{-j}}{\sigma(B_j)} \int_{B_j} |f - f_{B_j}| d\sigma.
\end{aligned}$$

Combining this with (2.1-3), we find that $C_{2.2} = \max\{C_1, 2C_4\}$ will do for the proposition. $\square$

3. Möbius Transform

For each $z \in \mathbf{C}^n$ with $0 < |z| < 1$, let

$$\varphi_z(w) = \frac{1}{1 - \langle w, z \rangle} \left\{ z - \frac{\langle w, z \rangle}{|z|^2} z - (1 - |z|^2)^{1/2} \left(w - \frac{\langle w, z \rangle}{|z|^2} z \right) \right\},$$

$|w| \leq 1$. Then φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \mathrm{id}$.

The formula

$$(U_z g)(\xi) = g(\varphi_z(\xi)) k_z(\xi), \qquad \xi \in S \text{ and } g \in L^2(S, d\sigma),$$

defines a unitary operator with the property $[U_z, P] = 0$. Moreover, there exist constants $0 < \alpha < \beta < \infty$ such that

$$\alpha \| f \circ \varphi_a \|_{\mathrm{BMO}} \leq \beta \| f \|_{\mathrm{BMO}} \tag{3.3}$$

for all $f \in \mathrm{BMO}$ and $a \in \mathbf{C}^n$ with $0 < |a| < 1$.

Lemma 3.1. *Given any* $f \in \mathrm{BMO}$ *and* $z \in \mathbf{C}^n$ *with* $0 < |z| < 1$, *there exist functions* h_z *and* v_z *satisfying the following four conditions*:

(a) $h_z \in H^2(S)$.
(b) $h_z + v_z = f - Pf$.
(c) $\|v_z k_z\| = \|H_f k_z\|$.
(d) $\|v_z\|_{\mathrm{BMO}} \leq C_{3.1} \|f\|_{\mathrm{BMO}}$, *where the constant* $C_{3.1}$ *depends only on the complex dimension* n.

Proof. Given $f \in \mathrm{BMO}$ and $0 < |z| < 1$, set

$$h_z = (P(f \circ \varphi_z)) \circ \varphi_z - Pf$$

and

$$v_z = f - (P(f \circ \varphi_z)) \circ \varphi_z.$$

Then (a) and (b) are obvious. Using the identities $\varphi_z \circ \varphi_z = \mathrm{id}$ and $[U_z, P] = 0$, we have

$$\begin{aligned}
\|H_f k_z\| &= \|(1-P)M_f k_z\| \\
&= \|(1-P)M_{f\circ\varphi_z\circ\varphi_z} k_z\| \\
&= \|(1-P)U_z M_{f\circ\varphi_z} 1\| \\
&= \|U_z(1-P)M_{f\circ\varphi_z} 1\| \\
&= \|U_z\{f\circ\varphi_z - P(f\circ\varphi_z)\}\| \\
&= \|v_z k_z\|,
\end{aligned}$$

proving (c). To verify (d), note that Proposition 2.2 provides a constant C such that $\|P\eta\|_{\mathrm{BMO}} \le C\|\eta\|_{\mathrm{BMO}}$ for every $\eta \in \mathrm{BMO}$. Combining this with (3.3), we have

$$\begin{aligned}
\|v_z\|_{\mathrm{BMO}} &\le \|f\|_{\mathrm{BMO}} + \|(P(f\circ\varphi_z))\circ\varphi_z\|_{\mathrm{BMO}} \\
&\le \|f\|_{\mathrm{BMO}} + (\beta/\alpha)\|P(f\circ\varphi_z)\|_{\mathrm{BMO}} \\
&\le \|f\|_{\mathrm{BMO}} + (\beta/\alpha)C\|f\circ\varphi_z\|_{\mathrm{BMO}} \\
&\le \|f\|_{\mathrm{BMO}} + (\beta/\alpha)^2 C\|f\|_{\mathrm{BMO}}.
\end{aligned}$$

Thus $C_{3.1} = 1 + (\beta/\alpha)^2 C$ will do for (d). $\square$

Proof of Theorem 1.6. We first pick an integer $L > 2$ such that

$$C_{2.2}C_{3.1}\sum_{k=L+1}^{\infty}2^{-k} \le \frac{1}{4}, \tag{3.9}$$

where $C_{2.2}$ and $C_{3.1}$ are the constants in Proposition 2.2 and lemma 3.1 respectively. Let $f \in \mathrm{BMO}$ be given and write

$$u = f - Pf.$$

Proposition 2.2 tells us that $\|u\|_{\mathrm{BMO}} < \infty$ (crucial). By this finiteness, there exist $\xi \in S$ and $r > 0$ such that

$$\frac{1}{\sigma(B(\xi,r))}\int_{B(\xi,r)}|u - u_{B(\xi,r)}|d\sigma \ge \frac{1}{2}\|u\|_{\mathrm{BMO}}. \tag{3.10}$$

Write

$$B = B(\xi, r)$$

as in the proof of Theorem 1.3. Also, let $\rho = 2^L r$. Now the proof divides into two cases.

(1) Suppose that $\rho < 1/2$. In this case we define

$$z = (1-\rho^2)^{1/2}\xi.$$

Applying Lemma 3.1 to u and z, we obtain h_z and v_z such that

(i) $h_z \in H^2(S)$;
(ii) $h_z + v_z = u - Pu = u$;
(iii) $\|v_z k_z\| = \|H_u k_z\|$;
(iv) $\|v_z\|_{\mathrm{BMO}} \le C_{3.1}\|u\|_{\mathrm{BMO}}$.

By (i) and (ii), $h_z = -Pv_z$. Applying Proposition 2.2 to v_z and B, we have

$$\left\{ \frac{1}{\sigma(B)} \int_B |h_z - (h_z)_B|^2 d\sigma \right\}^{1/2}$$

$$= \left\{ \frac{1}{\sigma(B)} \int_B |Pv_z - (Pv_z)_B|^2 d\sigma \right\}^{1/2}$$

$$\leq C_{2.2} \sum_{k=1}^{L} \left\{ \frac{1}{\sigma(B_k)} \int_{B_k} |v_z - (v_z)_{B_k}|^2 d\sigma \right\}^{1/2}$$

$$+ C_{2.2} \sum_{k=L+1}^{\infty} 2^{-k} \|v_z\|_{\mathrm{BMO}},$$

where $B_k = B(\xi, 2^k r)$, $k \geq 1$. We have

$$\sum_{k=1}^{L} \left\{ \frac{1}{\sigma(B_k)} \int_{B_k} |v_z - (v_z)_{B_k}|^2 d\sigma \right\}^{1/2}$$

$$\leq L \frac{\sigma(B_L)}{\sigma(B_1)} \left\{ \frac{1}{\sigma(B_L)} \int_{B_L} |v_z - (v_z)_{B_L}|^2 d\sigma \right\}^{1/2}.$$

Combining the above with (iv) and with (3.9), we see that

$$\frac{1}{\sigma(B)} \int_B |h_z - (h_z)_B| d\sigma$$

$$\leq C_{2.2} C(n, L) \left\{ \frac{1}{\sigma(B_L)} \int_{B_L} |v_z - (v_z)_{B_L}|^2 d\sigma \right\}^{1/2} + \frac{1}{4} \|u\|_{\mathrm{BMO}},$$

where $C(n, L)$ depends only on n and L.

It is easy to show that

$$\left\{\frac{1}{\sigma(B_L)}\int_{B_L}|v_z-(v_z)_{B_L}|^2d\sigma\right\}^{1/2}\le\left\{\frac{1}{\sigma(B_L)}\int_{B_L}|v_z|^2d\sigma\right\}^{1/2}$$
$$\le 8^{n/2}\|v_zk_z\| = 8^{n/2}\|H_uk_z\|.$$

Since $u = h_z + v_z$, from the above we deduce

$$\frac{1}{\sigma(B)}\int_B |u-u_B|d\sigma \le (1+C_{2.2})C(n,L)8^{n/2}\|H_uk_z\|+\frac{1}{4}\|u\|_{\mathrm{BMO}}.$$

Recalling (3.10) and using the fact that $H_u = H_f$, we now have

$$\frac{1}{2}\|u\|_{\mathrm{BMO}} \le (1+C_{2.2})C(n,L)8^{n/2}\|H_fk_z\| + \frac{1}{4}\|u\|_{\mathrm{BMO}}.$$

Cancelling out $(1/4)\|u\|_{\mathrm{BMO}}$ form both sides, we obtain

$$\frac{1}{4}\|u\|_{\mathrm{BMO}} \le (1+C_{2.2})C(n,L)8^{n/2}\|H_fk_z\|$$

in the case $\rho < 1/2$. (Note that this last step required the fact $\|u\|_{\mathrm{BMO}} < \infty$.)

(2) Suppose that $\rho \ge 1/2$. This the trivial case. $\square$

Remark. In the above proof, the fact $\|u\|_{\mathrm{BMO}} < \infty$ was used non-trivially in two places. This is the reason why Theorem 1.9 requires a separate proof.

4. Smoothing

Let $\mathcal{U} = \mathcal{U}(n)$ denote the collection of unitary transformations on $\mathbf{C}^n$. For each $U \in \mathcal{U}$, define the operator $W_U : L^2(S, d\sigma) \to L^2(S, d\sigma)$ by the formula

$$(W_U g)(\zeta) = g(U\zeta),$$

$g \in L^2(S, d\sigma)$. By the invariance of σ, W_U is a unitary operator on $L^2(S, d\sigma)$. Obviously, $[P, W_U] = 0$ for every $U \in \mathcal{U}$.

With the usual multiplication and topology, $\mathcal{U}$ is a compact group. Write dU for the Haar measure on $\mathcal{U}$.

It is easy to see that for each $g \in L^2(S, d\sigma)$, the map $U \mapsto W_U g$ is continuous with respect to the norm topology of $L^2(S, d\sigma)$. Let Φ be a continuous function on $\mathcal{U}$. Then for each $g \in L^2(S, d\sigma)$ we can define the integral

$$Y_\Phi g = \int_{\mathcal{U}} \Phi(U) W_U g dU$$

in the sense that

$$\langle Y_\Phi g, f \rangle = \int_{\mathcal{U}} \Phi(U) \langle W_U g, f \rangle dU$$

for every $f \in L^2(S, d\sigma)$.

Lemma 4.1. *If $\Phi \in C(\mathcal{U})$, then $\|Y_\Phi g\|_\infty < \infty$ for every $g \in L^2(S, d\sigma)$.*

Proof. Recall that the equality

$$\int_{\mathcal{U}} f(U\zeta)dU = \int f d\sigma$$

holds for all $f \in C(S)$ and $\zeta \in S$. Thus for $q, p \in C(S)$ we have

$$\begin{aligned}
|\langle Y_\Phi q, p\rangle| &= \left| \int_{\mathcal{U}} \Phi(U)\langle W_U q, p\rangle dU \right| \\
&= \left| \int_{\mathcal{U}} \Phi(U) \left\{ \int q(U\zeta)\overline{p(\zeta)}d\sigma(\zeta) \right\} dU \right| \\
&= \left| \int \left\{ \int_{\mathcal{U}} \Phi(U) q(U\zeta) dU \right\} \overline{p(\zeta)} d\sigma(\zeta) \right| \\
&\leq \|\Phi\|_\infty \int \left\{ \int_{\mathcal{U}} |q(U\zeta)| dU \right\} |p(\zeta)| d\sigma(\zeta) \\
&= \|\Phi\|_\infty \int |q| d\sigma \int |p| d\sigma.
\end{aligned}$$

Since Y_Φ is obviously a bounded operator on $L^2(S, d\sigma)$ and since $C(S)$ is dense in $L^2(S, d\sigma)$, the above implies

$$|\langle Y_\Phi g, f\rangle| \leq \|\Phi\|_\infty \int |g| d\sigma \int |f| d\sigma$$

for all $g, f \in L^2(S, d\sigma)$. This obviously means $\|Y_\Phi g\|_\infty < \infty$. □

Proof of Theorem 1.9. Let $\eta : [0, \infty) \to [0, 1]$ be a continuous function satisfying the conditions that $\eta = 1$ on $[0, 1]$ and that $\eta = 0$ on $[2, \infty)$. For each $j \in \mathbf{N}$, define

$$\Phi_j(U) = \frac{\eta(j\|1 - U\|)}{\int_{\mathcal{U}} \eta(j\|1 - V\|)dV}, \qquad U \in \mathcal{U}.$$

Then the following properties are obvious:

(1) $\Phi_j \in C(\mathcal{U})$.

(2) $\Phi_j \geq 0$ on $\mathcal{U}$.

(3) $\Phi_j(U) = 0$ if $\|1 - U\| \geq 2/j$.

(4) $\int_{\mathcal{U}} \Phi_j(U)dU = 1$.

Let ψ be given as in the statement of the theorem and denote

$$R = \sup_{|z|<1} \|H_\psi k_z\|.$$

Furthermore, for each $j \in \mathbf{N}$ denote

$$\psi_j = Y_{\Phi_j}\psi.$$

By Lemma 4.1, $\|\psi_j\|_\infty < \infty$. Thus we can apply Theorem 1.6 to obtain

$$\|\psi_j - P\psi_j\|_{\mathrm{BMO}} \leq C \sup_{|z|<1} \|H_{\psi_j} k_z\|, \tag{4.1}$$

where C depends only on the complex dimension n. We claim that

$$\sup_{|z|<1} \|H_{\psi_j} k_z\| \leq R \tag{4.2}$$

for every $j \in \mathbf{N}$.

To prove (4.2), note that for all $U \in \mathcal{U}$ and $z \in \mathbf{C}^n$ with $|z| < 1$, we have $W_U H_\psi W_{U^*} k_z = H_{W_U \psi} k_z$ and $W_{U^*} k_z = k_{Uz}$. Thus for all $j \in \mathbf{N}$, $|z| < 1$ and $f \in L^2(S, d\sigma) \ominus H^2(S)$ we have

$$\begin{aligned}
\langle H_{\psi_j} k_z, f\rangle = \langle \psi_j k_z, f\rangle = \langle \psi_j, \bar{k}_z f\rangle &= \langle Y_{\Phi_j} \psi, \bar{k}_z f\rangle \\
&= \int_{\mathcal{U}} \Phi_j(U) \langle W_U \psi, \bar{k}_z f\rangle dU \\
&= \int_{\mathcal{U}} \Phi_j(U) \langle k_z W_U \psi, f\rangle dU \\
&= \int_{\mathcal{U}} \Phi_j(U) \langle H_{W_U \psi} k_z, f\rangle dU \\
&= \int_{\mathcal{U}} \Phi_j(U) \langle W_U H_\psi k_{Uz}, f\rangle dU.
\end{aligned}$$

By properties (2) and (4) we now have

$$|\langle H_{\psi_j} k_z, f\rangle| \le \int_{\mathcal{U}} \Phi_j(U) \|H_\psi k_{Uz}\| \|f\| dU \le R\|f\|$$

for all $j \in \mathbf{N}$, $|z| < 1$ and $f \in L^2(S, d\sigma) \ominus H^2(S)$. This proves (4.2).

Now consider an arbitrary $B = B(\zeta, r)$, where $\zeta \in S$ and $r > 0$. By (4.1) and (4.2),

$$\frac{1}{\sigma(B)} \int_B |\psi_j - P\psi_j - (\psi_j - P\psi_j)_B| d\sigma \le CR \tag{4.3}$$

for every $j \in \mathbf{N}$. Clearly, the proof will be complete if we can show $\lim_{j\to\infty} \|\psi_j - \psi\| = 0$, for this convergence and (4.3) together will give us

$$\frac{1}{\sigma(B)} \int_B |\psi - P\psi - (\psi - P\psi)_B| d\sigma \le CR.$$

Thus the proof is now reduced to that of the convergence

$$\text{s-}\lim_{j\to\infty} Y_{\Phi_j} = 1 \tag{4.4}$$

on the Hilbert space $L^2(S, d\sigma)$. But this works just like in the case of convolution and is absolutely routine.

It is easy to see that if $q \in C(S)$, then

$$(Y_{\Phi_j} q)(\zeta) = \int_{\mathcal{U}} \Phi_j(U) q(U\zeta) dU, \qquad \zeta \in S.$$

Applying properties (1)-(4), we have

$$\lim_{j\to\infty} \|Y_{\Phi_j} q - q\|_\infty = 0, \qquad q \in C(S). \tag{4.5}$$

Also, by (2) and (4), the norm of the operator Y_{Φ_j} on the Hilbert space $L^2(S, d\sigma)$ satisfies the estimate $\|Y_{\Phi_j}\| \leq 1$. Obviously, (4.4) follows from (4.5) and this norm bound. □

5. Open Questions

Suppose that $n \geq 2$. Recall that what initially lead to this investigation was the consideration of the Banach subalgebra

$$\mathcal{A} = \{f \in L^\infty(S, d\sigma) : H_f \text{ is compact}\}$$

of $L^\infty(S, d\sigma)$. Davie and Jewell showed that

$$\mathcal{A} \neq H^\infty(S) + C(S).$$

We have figured out that

$$\mathcal{A} = \{f \in L^\infty(S, d\sigma) : f - Pf \in \mathrm{VMO}\}, \tag{5.1}$$

which is progress. One can see that (5.1) and the fact that $\mathcal{A}$ is a Banach algebra together have interesting *multiplicative* consequences.

On the other hand, there is plenty of unknown about $\mathcal{A}$. To discuss the unknown, observe that

$$\mathcal{A} \supset H^\infty(S) + \{\mathrm{VMO} \cap L^\infty(S, d\sigma)\}.$$

The following questions were raised by Davie and Jewell in 1977. More than thirty years later, these questions remain open.

Question 1. Is it true that

$$\mathcal{A} = H^\infty(S) + \{\text{VMO} \cap L^\infty(S, d\sigma)\}?$$

Note that an affirmative answer to Question 1 would imply that $H^\infty(S) + \{\text{VMO} \cap L^\infty(S, d\sigma)\}$ is a Banach subalgebra of $L^\infty(S, d\sigma)$. Therefore the following are weaker versions of Question 1.

Question 2. Is the subset

$$H^\infty(S) + \{\text{VMO} \cap L^\infty(S, d\sigma)\}$$

closed in $L^\infty(S, d\sigma)$ with respect to the norm $\|.\|_\infty$?

Question 3. Is the $\|.\|_\infty$-closure of

$$H^\infty(S) + \{\text{VMO} \cap L^\infty(S, d\sigma)\}$$

an algebra?

Question 4. Does the Banach algebra generated by

$$H^\infty(S) + \{\text{VMO} \cap L^\infty(S, d\sigma)\}$$

coincide with $\mathcal{A}$?