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Bounded Toeplitz products on the Bergman space of the polydisk

Karel Stroethoff [a,*] and Dechao Zheng [b,1]

[a] *Mathematical Sciences, University of Montana, Missoula, MT 59812, USA*
[b] *Mathematics Department, Vanderbilt University, Nashville, TN 37240, USA*

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Abstract

We consider the question for which square integrable analytic functions f and g on the polydisk the densely defined products $T_f T_{\bar{g}}$ are bounded on the Bergman space. We prove results analogous to those we obtained in the setting of the unit disk [K. Stroethoff, D. Zheng, J. Funct. Anal. 169 (1999) 289–313].

1. Introduction

Throughout let n be a fixed integer $n \geqslant 2$. Denote the unit disk in $\mathbb{C}$ by $\mathbb{D}$, and let ν be Lebesgue volume measure on $\mathbb{D}^n$, normalized so that $\nu(\mathbb{D}^n) = 1$.

For $\lambda \in \mathbb{D}$, let φ_λ be the fractional linear transformation on $\mathbb{D}$ given by $\varphi_\lambda(z) = (\lambda - z)/(1 - \bar{\lambda} z)$. Each φ_λ is an automorphism on the disk, in fact, $\varphi_\lambda^{-1} = \varphi_\lambda$. For $w = (w_1, \ldots, w_n) \in \mathbb{D}^n$ the mapping φ_w on the polydisk $\mathbb{D}^n$ given by $\varphi_w(z) = (\varphi_{w_1}(z_1), \ldots, \varphi_{w_n}(z_n))$ is an automorphism on $\mathbb{D}^n$. The Bergman space $L_a^2(\mathbb{D}^n)$ is the space of analytic functions h on $\mathbb{D}^n$ which are square-integrable with respect to Lebesgue volume measure on $\mathbb{D}^n$. The reproducing kernel in $L_a^2(\mathbb{D}^n)$ is given by

* Corresponding author.
E-mail addresses: ma_kms@selway.umt.edu (K. Stroethoff), zheng@math.vanderbilt.edu (D. Zheng).
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$$K_w(z) = \prod_{j=1}^{n} \frac{1}{(1 - \bar{w}_j z_j)^2},$$

for $z, w \in \mathbb{D}^n$. If $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{D}^n)$, then $\langle h, K_w \rangle = h(w)$, for every $h \in L_a^2(\mathbb{D}^n)$ and $w \in \mathbb{D}^n$. The orthogonal projection P of $L^2(\mathbb{D}^n)$ onto $L_a^2(\mathbb{D}^n)$ is given by

$$(Pg)(w) = \langle g, K_w \rangle = \int_{\mathbb{D}^n} g(z) \prod_{j=1}^{n} \frac{1}{(1 - w_j \bar{z}_j)^2}\, d\nu(z),$$

for $g \in L^2(\mathbb{D}^n)$ and $w \in \mathbb{D}^n$. Given $f \in L^\infty(\mathbb{D}^n)$, the Toeplitz operator T_f is defined on $L_a^2(\mathbb{D}^n)$ by $T_f h = P(fh)$. We have

$$(T_f h)(w) = \int_{\mathbb{D}^n} f(z) h(z) \prod_{j=1}^{n} \frac{1}{(1 - w_j \bar{z}_j)^2}\, d\nu(z),$$

for $h \in L_a^2(\mathbb{D}^n)$ and $w \in \mathbb{D}^n$. Note that the above formula makes sense, and defines a function analytic on $\mathbb{D}^n$, also if $f \in L^2(\mathbb{D}^n)$. So, if $g \in L_a^2(\mathbb{D}^n)$ we define $T_{\bar{g}}$ by the formula

$$(T_{\bar{g}} h)(w) = \int_{\mathbb{D}^n} \overline{g(z)} h(z) \prod_{j=1}^{n} \frac{1}{(1 - w_j \bar{z}_j)^2}\, d\nu(z),$$

for $h \in L_a^2(\mathbb{D}^n)$ and $w \in \mathbb{D}^n$. If also $f \in L_a^2(\mathbb{D}^n)$, then $T_f T_{\bar{g}} h$ is the analytic function $f T_{\bar{g}} h$. We consider the following problem, which for $n = 1$ was raised by Sarason in [2].

Problem of boundedness of Toeplitz products on $L_a^2(\mathbb{D}^n)$. For which f and g in $L_a^2(\mathbb{D}^n)$ is the operator $T_f T_{\bar{g}}$ bounded on $L_a^2(\mathbb{D}^n)$?

In this paper we extend our results for boundedness of these Toeplitz products on the Bergman space of the unit disk [4] to higher dimension. In the next section we will first give a necessary condition for boundedness of the Toeplitz product $T_f T_{\bar{g}}$ on $L_a^2(\mathbb{D}^n)$. A recent counter-example of Nazarov [1] for Toeplitz products on the Hardy space indicates that it may not be possible to prove that this necessary condition is also sufficient. In the final section of the paper we will show that this condition is, however, very close to being sufficient, as shown for Toeplitz products on the Hardy space of the unit circle in [5].

2. Necessary condition for boundedness

Suppose f and g are in $L^2(\mathbb{D}^n)$. Consider the operator $f \otimes g$ on $L_a^2(\mathbb{D}^n)$ defined by

$$(f \otimes g)h = \langle h, g \rangle f,$$

for $h \in L_a^2(\mathbb{D}^n)$. It is easily proved that $f \otimes g$ is bounded on $L_a^2(\mathbb{D}^n)$ with norm equal to $\|f \otimes g\| = \|f\| \|g\|$.

We will obtain an expression for the operator $f \otimes g$, where $f, g \in L_a^2(\mathbb{D}^n)$. This is most easily accomplished by using the Berezin transform: writing k_w for the normalized reproducing kernels, we define the Berezin transform of a bounded linear operator S on $L_a^2(\mathbb{D}^n)$ to be the function $\widetilde{S}$ defined on $\mathbb{D}^n$ by

$$\widetilde{S}(w) = \langle Sk_w, k_w\rangle,$$

for $w \in \mathbb{D}^n$. The boundedness of S implies that the function $\widetilde{S}$ is bounded on $\mathbb{D}^n$. The Berezin transform is injective, for $\widetilde{S}(w) = 0$, for all $w \in \mathbb{D}^n$, implies that $S = 0$, the zero operator on $L_a^2(\mathbb{D}^n)$ (see [3] for a proof). Using the reproducing property of K_w we have

$$\|K_w\|^2 = \langle K_w, K_w\rangle = K_w(w) = \prod_{j=1}^{n} \frac{1}{(1-|w_j|^2)^2},$$

thus

$$k_w(z) = \prod_{j=1}^{n} \frac{1-|w_j|^2}{(1-\bar{w}_j z_j)^2}, \tag{2.1}$$

for $z, w \in \mathbb{D}^n$. It follows from (2.1) that

$$\widetilde{S}(w) = \prod_{j=1}^{n} \bigl(1-|w_j|^2\bigr)^2 \langle SK_w, K_w\rangle,$$

for $w \in \mathbb{D}^n$. It is easily seen that $T_{\bar{g}}K_w = \overline{g(w)}K_w$. Thus $\langle T_f T_{\bar{g}} K_w, K_w\rangle = \langle T_{\bar{g}} K_w, T_{\bar{f}} K_w\rangle = \langle \overline{g(w)}K_w, \overline{f(w)}K_w\rangle = f(w)\overline{g(w)}\langle K_w, K_w\rangle$, and we see that

$$\widetilde{T_f T_{\bar{g}}}(w) = f(w)\overline{g(w)}. \tag{2.2}$$

Since $\langle (f\otimes g)K_w, K_w\rangle = \langle\langle K_w, g\rangle f, K_w\rangle = \langle K_w, g\rangle\langle f, K_w\rangle = f(w)\overline{g(w)}$, we also have

$$\widetilde{f\otimes g}(w) = \prod_{j=1}^{n} \bigl(1-|w_j|^2\bigr)^2 f(w)\overline{g(w)}. \tag{2.3}$$

For a multi-index $\beta = (\beta_1, \ldots, \beta_n)$, it follows from (2.2) that the Berezin transform of the Toeplitz product $T_{z^\beta} T_f T_{\bar{g}} T_{\bar{z}^\beta} = T_{z^\beta f} T_{\overline{gz^\beta}}$ is equal to the function $w \mapsto w^\beta f(w)\overline{g(w)w^\beta} = |w_1|^{2\beta_1} \ldots |w_n|^{2\beta_n} f(w)\overline{g(w)}$. Writing

$$\prod_{j=1}^{n} (1-x_j)^2 = \sum (-1)^k \binom{2}{k_1}\binom{2}{k_2}\cdots\binom{2}{k_n} x_1^{k_1}\ldots x_n^{k_n},$$

where $k = k_1 + k_2 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ from $\{0, 1, 2\}$, we see from (2.3) and the injectivity of the Berezin transform that

$$f\otimes g = \sum (-1)^k \binom{2}{k_1}\binom{2}{k_2}\cdots\binom{2}{k_n} T_{z_1^{k_1}\ldots z_n^{k_n}} T_f T_{\bar{g}} T_{\overline{z_1^{k_1}\ldots z_n^{k_n}}},$$

where $k = k_1 + k_2 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ ranging over the set $\{0, 1, 2\}$. Using that $\|T_{z_j}\| = 1$, for each j, it follows that $\|f\otimes g\| \leqslant 4^n \|T_f T_{\bar{g}}\|$, and thus

$$\|f\|\|g\| \leqslant 4^n \|T_f T_{\bar{g}}\|, \tag{2.4}$$

for f and g in $L_a^2(\mathbb{D}^n)$.

We will next apply the invariance under composition of the symbols with the fractional transformations. We need some preliminaries to make this precise. The mapping φ_w has real Jacobian $\prod_{j=1}^{n} \varphi'_{w_j}(z_j)\overline{\varphi'_{w_j}(z_j)} = \prod_{j=1}^{n} |\varphi'_{w_j}(z_j)|^2$, which by (2.1) is equal to $|k_w(z)|^2$, so we have the following change-of-variable formula

$$\int_{\mathbb{D}^n} h\big(\varphi_w(z)\big)\big|k_w(z)\big|^2 \, d\nu(z) = \int_{\mathbb{D}^n} h(u)\, d\nu(u), \tag{2.5}$$

for every $h \in L^1(\mathbb{D}^n)$. It follows from (2.5) that the mapping $U_w h = (h \circ \varphi_w)k_w$ is an isometry on $L_a^2(\mathbb{D}^n)$:

$$\|U_w h\|^2 = \int_{\mathbb{D}^n} \big|h\big(\varphi_w(z)\big)\big|^2 \big|k_w(z)\big|^2 \, d\nu(z) = \int_{\mathbb{D}^n} \big|h(u)\big|^2 \, d\nu(u) = \|h\|^2,$$

for all $h \in L_a^2(\mathbb{D}^n)$. It is easily verified that

$$k_w\big(\varphi_w(z)\big) = \frac{1}{k_w(z)}.$$

Since $\varphi_w \circ \varphi_w = id$, we see that

$$\big(U_w(U_w h)\big)(z) = (U_w h)\big(\varphi_w(z)\big)k_w(z) = h(z)k_w\big(\varphi_w(z)\big)k_w(z) = h(z),$$

for all $z \in \mathbb{D}^n$ and $h \in L_a^2(\mathbb{D}^n)$. Thus $U_w^{-1} = U_w$, and hence U_w is unitary. Furthermore,

$$T_{f\circ\varphi_w} U_w = U_w T_f. \tag{2.6}$$

Proof. For $h \in H^\infty$ and $g \in L_a^2(\mathbb{D}^n)$ we have

$$\begin{aligned}
\langle U_w T_f h, U_w g\rangle = \langle T_f h, g\rangle = \langle f h, g\rangle &= \int_{\mathbb{D}^n} f(u)h(u)\overline{g(u)}\, d\nu(z) \\
&= \int_{\mathbb{D}^n} f\big(\varphi_w(z)\big)h\big(\varphi_w(z)\big)\overline{g\big(\varphi_w(z)\big)}\big|k_w(z)\big|^2 \, d\nu(z) \\
&= \int_{\mathbb{D}^n} f\big(\varphi_w(z)\big)h\big(\varphi_w(z)\big)k_w(z)\overline{g\big(\varphi_w(z)\big)k_w(z)}\, d\nu(z) \\
&= \langle f U_w h, U_w g\rangle = \langle T_{f\circ\varphi_w} U_w h, U_w g\rangle,
\end{aligned}$$

establishing (2.6). □

It follows from (2.6), applied to f and $\bar{g}$, that

$$\begin{aligned}
T_{f\circ\varphi_w} T_{\bar{g}\circ\varphi_w} &= (T_{f\circ\varphi_w} U_w)U_w(T_{\bar{g}\circ\varphi_w} U_w)U_w \\
&= (U_w T_f)U_w(U_w T_{\bar{g}})U_w = U_w(T_f T_{\bar{g}})U_w.
\end{aligned}$$

So if the Toeplitz product $T_f T_{\bar{g}}$ is bounded on $L_a^2(\mathbb{D}^n)$, then so is the product $T_{f\circ\varphi_w} T_{\bar{g}\circ\varphi_w}$, and $\|T_{f\circ\varphi_w} T_{\bar{g}\circ\varphi_w}\| = \|T_f T_{\bar{g}}\|$. By (2.4) we have

$$\|f \circ \varphi_w\|_2 \|g \circ \varphi_w\|_2 \leqslant 4^n \|T_{f\circ\varphi_w} T_{\bar{g}\circ\varphi_w}\| = 4^n \|T_f T_{\bar{g}}\|,$$

hence

$$\widetilde{|f|^2}(w)\widetilde{|g|^2}(w) \leqslant 4^n \|T_f T_{\bar g}\|^2,$$

for all $w \in \mathbb{D}^n$. So, for $f, g \in L_a^2(\mathbb{D}^n)$, a necessary condition for the Toeplitz product $T_f T_{\bar g}$ to be bounded on $L_a^2(\mathbb{D}^n)$ is

$$\sup_{w\in\mathbb{D}^n} \widetilde{|f|^2}(w)\widetilde{|g|^2}(w) < \infty. \tag{2.7}$$

In the next section we will show that this condition is very close to being sufficient for boundedness.

3. Sufficient condition

In this section we will prove that a condition slightly stronger than (2.7) is sufficient for boundedness of the Toeplitz product $T_f T_{\bar g}$. In fact we have the following result.

Theorem 3.1. *Let f and g be in $L_a^2(\mathbb{D}^n)$. If for $\varepsilon > 0$,*

$$\sup_{w\in\mathbb{D}^n} \widetilde{|f|^{2+\varepsilon}}(w)\widetilde{|g|^{2+\varepsilon}}(w) < \infty,$$

then the operator $T_f T_{\bar g}$ is bounded on $L_a^2(\mathbb{D}^n)$.

In the proof of Theorem 3.1 we will need estimates on $T_{\bar f}h$ and its derivatives, as well as an alternative way to write the inner product formula in $L_a^2(\mathbb{D}^n)$.

3.1. *Inner product formula in $L_a^2(\mathbb{D}^n)$*

In this subsection we will establish a formula for the inner product in $L_a^2(\mathbb{D}^n)$ needed to prove our sufficiency condition for boundedness of Toeplitz products. Our point of departure is the following inner product formula proved in [4]:

$$\int_{\mathbb{D}} u(z)\overline{v(z)}\,dA(z) = 3\int_{\mathbb{D}} u(z)\overline{v(z)}\big(1-|z|^2\big)^2\,dA(z) + \frac{1}{6}\int_{\mathbb{D}} u'(z)\overline{v'(z)}\big(1-|z|^2\big)^2\big(5-2|z|^2\big)\,dA(z),$$

for $u, v \in L_a^2(\mathbb{D})$. Let $f, g \in L_a^2(\mathbb{D}^2)$. For fixed $z_2 \in \mathbb{D}$ we have

$$\int_{\mathbb{D}} f(z_1,z_2)\overline{g(z_1,z_2)}\,dA(z_1) = 3\int_{\mathbb{D}} f(z_1,z_2)\overline{g(z_1,z_2)}\big(1-|z_1|^2\big)^2\,dA(z_1) + \frac{1}{6}\int_{\mathbb{D}} \frac{\partial f}{\partial z_1}(z_1,z_2)\overline{\frac{\partial g}{\partial z_1}(z_1,z_2)}\big(1-|z_1|^2\big)^2\big(5-2|z_1|^2\big)\,dA(z_1).$$

Integrating both sides of the above equation with respect to z_2 results in

$$\begin{aligned}
&\int_{\mathbb{D}}\int_{\mathbb{D}} f(z_1,z_2)\overline{g(z_1,z_2)}\,dA(z_1)\,dA(z_2)\\
&\quad = 9\int_{\mathbb{D}}\int_{\mathbb{D}} f(z_1,z_2)\overline{g(z_1,z_2)}\big(1-|z_2|^2\big)^2\big(1-|z_1|^2\big)^2\,dA(z_1)\,dA(z_2)\\
&\qquad + \frac{1}{2}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\partial f}{\partial z_1}\overline{\frac{\partial g}{\partial z_1}}\big(1-|z_1|^2\big)^2\big(1-|z_2|^2\big)^2\big(5-2|z_1|^2\big)\,dA(z_1)\,dA(z_2)\\
&\qquad + \frac{1}{2}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\partial f}{\partial z_2}\overline{\frac{\partial g}{\partial z_2}}\big(1-|z_1|^2\big)^2\big(1-|z_2|^2\big)^2\big(5-2|z_2|^2\big)\,dA(z_1)\,dA(z_2)\\
&\qquad + \frac{1}{36}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\partial^2 f}{\partial z_1\partial z_2}\overline{\frac{\partial^2 g}{\partial z_1\partial z_2}}\big(1-|z_1|^2\big)^2\big(1-|z_2|^2\big)^2\\
&\qquad\qquad \times \big(5-2|z_1|^2\big)\big(5-2|z_2|^2\big)\,d\nu(z_1,z_2).
\end{aligned}$$

To formulate a formula for the inner product in $L_a^2(\mathbb{D}^n)$ we first introduce some notation. For a nonempty subset $\alpha = \{\alpha_1,\dots,\alpha_m\}$ of $\{1,\dots,n\}$ with $\alpha_1 < \dots < \alpha_m$ let μ_α be the measure on $\mathbb{D}^n$ defined by

$$d\mu_\alpha(z) = \frac{3^{n-m}}{6^m}\big(1-|z_1|^2\big)^2\dots\big(1-|z_n|^2\big)^2\prod_{j\in\alpha}\big(5-2|z_j|^2\big)\,dA(z_1)\dots dA(z_n),$$

for $z = (z_1,\dots,z_n)$, where m is the cardinality of α, and let

$$D^\alpha h = D_{\alpha_1}\dots D_{\alpha_m}h,$$

where $D_j h(z) = \partial h/\partial z_j$. Define $D^\emptyset h = h$. Note that

$$d\mu_\emptyset(z) = 3^n\big(1-|z_1|^2\big)^2\dots\big(1-|z_n|^2\big)^2\,dA(z_1)\dots dA(z_n)$$

and

$$d\mu_\alpha(z) \leqslant 3^n\big(1-|z_1|^2\big)^2\dots\big(1-|z_n|^2\big)^2\,dA(z_1)\dots dA(z_n),$$

for all subsets α of $\{1,\dots,n\}$. Then the following formula for the inner product in $L_a^2(\mathbb{D}^n)$ is proved by repeating the above procedure:

$$\int_{\mathbb{D}^n} f(z)\,\overline{g(z)}\,d\nu(z) = \sum_\alpha\int_{\mathbb{D}^n} D^\alpha f(z)\,\overline{D^\alpha g(z)}\,d\mu_\alpha(z), \tag{3.2}$$

where α runs over all subsets of $\{1,\dots,n\}$.

3.2. Estimates

In the proof of our sufficiency condition for boundedness of Toeplitz products we will also need the estimates contained in the following lemmas.

Lemma 3.3. *For $f \in L_a^2(\mathbb{D}^n)$ and $h \in H^\infty(\mathbb{D}^n)$ we have*

$$\left|(T_{\bar f}h)(w)\right| \leqslant \prod_{j=1}^n \frac{1}{1-|w_j|^2}\,\widetilde{|f|^2}(w)^{1/2}\,\|h\|,$$

for all $w \in \mathbb{D}^n$.

Proof. By the inequality of Cauchy–Schwarz,

$$\begin{aligned}\left|(T_{\bar f}h)(w)\right|^2 &\leqslant \left(\int_{\mathbb{D}^n} \left|f(z)\right|\left|h(z)\right| \prod_{j=1}^n \frac{1}{|1-w_j\bar z_j|^2}\,d\nu(z)\right)^2 \\ &\leqslant \int_{\mathbb{D}^n} \left|f(z)\right|^2 \frac{1}{|1-w_j\bar z_j|^4}\,d\nu(z) \int_{\mathbb{D}^n} \left|h(z)\right|^2 d\nu(z) \\ &= \prod_{j=1}^n \frac{1}{(1-|w_j|^2)^2}\widetilde{|f|^2}(w)\|h\|^2,\end{aligned}$$

and the stated inequality follows. □

Lemma 3.4. *Let $f \in L_a^2(\mathbb{D}^n)$, $h \in H^\infty(\mathbb{D}^n)$ and $\varepsilon > 0$. If $\alpha = \{\alpha_1, \ldots, \alpha_m\}$ is a subset of $\{1, \ldots, n\}$ with $\alpha_1 < \cdots < \alpha_m$, then*

$$\begin{aligned}\left|D^\alpha(T_{\bar f}h)(w)\right| \leqslant 2^{2n} &\prod_{j=1}^n \frac{1}{1-|w_j|^2}\widetilde{|f|^{2+\varepsilon}}(w)^{1/(2+\varepsilon)} \\ &\times \left(\int_{\mathbb{D}^n} \left|h(z)\right|^\delta \prod_{j=1}^n \frac{1}{|1-w_j\bar z_j|^2}\,d\nu(z)\right)^{1/\delta},\end{aligned}$$

for all $w \in \mathbb{D}^n$, where $\delta = (2+\varepsilon)/(1+\varepsilon)$.

Proof. We will first prove the estimate for $\alpha = \{1, \ldots, n\}$. For $f \in L_a^2(\mathbb{D}^n)$ and $h \in H^\infty(\mathbb{D}^n)$ we have

$$(T_{\bar f}h)(w) = \int_{\mathbb{D}^n} \overline{f(z)}h(z) \prod_{j=1}^n \frac{1}{(1-w_j\bar z_j)^2}\,d\nu(z),$$

thus

$$\frac{\partial^n}{\partial w_1 \ldots \partial w_n}(T_{\bar f}h)(w) = 2^n \int_{\mathbb{D}^n} \overline{f(z)}h(z) \prod_{j=1}^n \frac{\bar z_j}{(1-w_j\bar z_j)^3}\,d\nu(z).$$

Let $\varepsilon > 0$. Applying Hölder's inequality we get

$$\begin{aligned}
&\left|\frac{\partial^n}{\partial w_1 \dots \partial w_n}(T_{\bar f} h)(w)\right| \\
&\quad \leqslant 2^n \int_{\mathbb{D}^n} |f(z)| \prod_{j=1}^n \frac{1}{|1 - w_j \bar z_j|^{4/(2+\varepsilon)}} |h(z)| \prod_{j=1}^n \frac{1}{|1 - w_j \bar z_j|^{(2+3\varepsilon)/(2+\varepsilon)}}\, d\nu(z) \\
&\quad \leqslant 2^n \left(\int_{\mathbb{D}^n} |f(z)|^{2+\varepsilon} \prod_{j=1}^n \frac{1}{|1 - w_j \bar z_j|^4}\, d\nu(z) \right)^{1/(2+\varepsilon)} \\
&\qquad \times \left(\int_{\mathbb{D}^n} |h(z)|^\delta \prod_{j=1}^n \frac{1}{|1 - w_j \bar z_j|^{(2+3\varepsilon)/(1+\varepsilon)}}\, d\nu(z) \right)^{1/\delta} \\
&\quad = 2^n \left(\prod_{j=1}^n \frac{1}{(1 - |w_j|^2)^2} \widetilde{|f|^{2+\varepsilon}}(w) \right)^{1/(2+\varepsilon)} \\
&\qquad \times \left(\int_{\mathbb{D}^n} |h(z)|^\delta \prod_{j=1}^n \frac{1}{|1 - w_j \bar z_j|^{(2+3\varepsilon)/(1+\varepsilon)}}\, d\nu(z) \right)^{1/\delta}.
\end{aligned}$$

For each j,

$$\frac{1}{|1 - w_j \bar z_j|^{(2+3\varepsilon)/(1+\varepsilon)}} \leqslant \frac{2}{|1 - w_j \bar z_j|^2 (1 - |w_j|^2)^{\varepsilon/(1+\varepsilon)}},$$

so

$$\begin{aligned}
&\left|\frac{\partial^n}{\partial w_1 \dots \partial w_n}(T_{\bar f} h)(w)\right| \\
&\quad \leqslant 2^{2n} \prod_{j=1}^n \frac{1}{1 - |w_j|^2} \widetilde{|f|^{2+\varepsilon}}(w)^{1/(2+\varepsilon)} \left(\int_{\mathbb{D}^n} |h(z)|^\delta \prod_{j=1}^n \frac{1}{|1 - w_j \bar z_j|^2}\, d\nu(z) \right)^{1/\delta},
\end{aligned}$$

proving the estimate for $\alpha = \{1, \dots, n\}$.

Now suppose that $\alpha = \{\alpha_1, \dots, \alpha_m\}$, where $\alpha_1 < \dots < \alpha_m$. For $f \in L^2_a(\mathbb{D}^n)$ and $h \in H^\infty(\mathbb{D}^n)$ we have

$$D^\alpha (T_{\bar f} h)(w) = 2 \int_{\mathbb{D}^n} \prod_{\ell \in \alpha} \frac{\bar z_\ell}{1 - w_\ell \bar z_\ell}\, \overline{f(z)}\, h(z) \prod_{j=1}^n \frac{1}{(1 - w_j \bar z_j)^2}\, d\nu(z).$$

Noting that

$$\begin{aligned}
\prod_{\ell \in \alpha} \frac{1}{|1 - w_\ell \bar z_\ell|} &= \frac{1}{|1 - w_1 \bar z_1| \dots |1 - w_n \bar z_n|} \prod_{j \in \{1,\dots,n\} \setminus \alpha} |1 - w_j \bar z_j| \\
&\leqslant \frac{2^{n-m}}{|1 - w_1 \bar z_1| \dots |1 - w_n \bar z_n|},
\end{aligned}$$

we get

$$\left|D^{\alpha}(T_{\bar f}h)(w)\right| \leqslant 2^{n-m+1}\int_{\mathbb{D}^n}\left|f(z)\right|\left|h(z)\right|\prod_{j=1}^{n}\frac{1}{|1-w_j\bar z_j|^3}\,d\nu(z),$$

and the stated inequality follows from the proof of the first part of the lemma. □

3.3. Sufficient condition for boundedness

We are now in a position to prove our sufficiency condition for boundedness of Toeplitz products.

Proof of Theorem 3.1. Let h and k be bounded analytic functions on $\mathbb{D}^n$. It follows from Lemma 3.3 that

$$\left|(T_{\bar f}h)(w)(T_{\bar g}k)(w)\right| \leqslant \prod_{j=1}^{n}\frac{1}{(1-|w_j|^2)^2}\widetilde{|f|^2}(w)^{1/2}\,\widetilde{|g|^2}(w)^{1/2}\|h\|\|k\|,$$

thus

$$\int_{\mathbb{D}^n}\left|(T_{\bar f}h)(z)(T_{\bar g})(z)\right|d\mu_{\emptyset}(z) \leqslant 3^n\|h\|\|k\|\sup_{w\in\mathbb{D}^n}\widetilde{|f|^2}(w)^{1/2}\widetilde{|g|^2}(w)^{1/2}.$$

Using Lemma 3.4 we have

$$\left|D^{\alpha}T_{\bar f}h(w)\,D^{\alpha}T_{\bar g}k(w)\right| \leqslant \prod_{j=1}^{n}\frac{1}{(1-|w_j|^2)^2}\widetilde{|f|^{2+\varepsilon}}(w)^{1/(2+\varepsilon)}\widetilde{|g|^{2+\varepsilon}}(w)^{1/(2+\varepsilon)}$$
$$\times\, Q\big[|h|^{\delta}\big](w)^{1/\delta}Q\big[|k|^{\delta}\big](w)^{1/\delta},$$

where Q is the integral operator defined by

$$Q[u](w) = \int_{\mathbb{D}^n}u(z)\prod_{j=1}^{n}\frac{1}{|1-w_j\bar z_j|^2}\,d\nu(z),$$

for $u\in L^1(\mathbb{D}^n,d\nu)$ and $w\in\mathbb{D}^n$. If

$$\widetilde{|f|^{2+\varepsilon}}(w)^{1/(2+\varepsilon)}\,\widetilde{|g|^{2+\varepsilon}}(w)^{1/(2+\varepsilon)} \leqslant M,$$

for all $w\in\mathbb{D}^n$, then the above inequality implies

$$\int_{\mathbb{D}^n}\left|D^{\alpha}T_{\bar f}h(w)\,D^{\alpha}T_{\bar g}k(w)\right|d\mu_{\alpha}(z) \leqslant 3^n M\int_{\mathbb{D}^n}Q\big[|h|^{\delta}\big](z)^{1/\delta}Q\big[|k|^{\delta}\big](z)^{1/\delta}\,d\nu(z).$$

That the operator Q is L^p bounded, for each $1<p<\infty$, is easily shown as in the one-dimensional case (see, for example, Chapter 4 in [6]). Since $\delta<2$, $p=2/\delta>1$, so there exists a constant C such that

$$\int_{\mathbb{D}^n}Q[u](z)^p\,d\nu(z) \leqslant C^p\int_{\mathbb{D}^n}\left|u(z)\right|^p d\nu(z).$$

In particular,

$$\int_{\mathbb{D}^n} Q\big[|h|^\delta\big](z)^p \, d\nu(z) \leqslant C^p \|h\|^2,$$

and a similar inequality for function k. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\int_{\mathbb{D}^n} Q\big[|h|^\delta\big](z)^{1/\delta} Q\big[|k|^\delta\big](z)^{1/\delta} \, d\nu(z) \\
&\quad \leqslant \left(\int_{\mathbb{D}^n} Q\big[|h|^\delta\big](z)^{2/\delta} \, d\nu(z) \right)^{1/2} \left(\int_{\mathbb{D}^n} Q\big[|k|^\delta\big](z)^{2/\delta} \, d\nu(z) \right)^{1/2} \\
&\quad \leqslant \big(C^p \|h\|^2\big)^{1/2} \big(C^p \|k\|^2\big)^{1/2} = C^{2/\delta} \|h\| \|k\|.
\end{aligned}$$

Thus

$$\int_{\mathbb{D}^n} \big| D^\alpha T_{\bar{f}} h(w) \, D^\alpha T_{\bar{g}} k(w) \big| \, d\mu_\alpha(z) \leqslant 3^n \, M C^{2/\delta} \|h\| \|k\|,$$

for every subset α of $\{1, \ldots, n\}$. Using (3.2) we conclude that there is a finite constant C' such that

$$\big| \langle T_f T_{\bar{g}} k, h \rangle \big| \leqslant C' \|h\| \|k\|,$$

for all bounded analytic functions h and k on $\mathbb{D}^n$. Hence the operator $T_f T_{\bar{g}}$ is bounded on $L_a^2(\mathbb{D}^n)$. □

3.4. Compact Toeplitz products

The following theorem states that the Toeplitz product $T_f T_{\bar{g}}$ is only compact in the trivial case that it is the zero operator.

Theorem 3.5. *Let f and g be in $L_a^2(\mathbb{D}^n)$. Then $T_f T_{\bar{g}}$ is compact if and only if $f \equiv 0$ or $g \equiv 0$.*

Proof. If $T_f T_{\bar{g}}$ is compact on $L_a^2(\mathbb{D}^n)$, then its Berezin transform vanishes near the boundary of $\mathbb{D}^n$:

$$\widetilde{T_f T_{\bar{g}}}(w) \to 0$$

as w in $\mathbb{D}^n$ approaches $\partial(\mathbb{D}^n)$. We have seen that $\widetilde{T_f T_{\bar{g}}}(w) = f(w)\overline{g(w)}$, so

$$\big| f(w) g(w) \big| = \big| \widetilde{T_f T_{\bar{g}}}(w) \big| \to 0$$

as w in $\mathbb{D}^n$ approaches $\partial(\mathbb{D}^n)$, and it follows from the maximum modulus principle that $fg \equiv 0$. □

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