HYPERBOLIC DERIVATIVES AND GENERALIZED
SCHWARZ-PICK ESTIMATES

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Abstract. In this paper we use the beautiful formula of Faa di Bruno for the nth derivative of composition of two functions to obtain the generalized Schwarz-Pick estimates. By means of those estimates we show that the hyperbolic derivative of an analytic self-map of the unit disk is Lipschitz with respect to the pseudohyperbolic metric.

1. Introduction

For each \( z \in D \), let \( \varphi_z \) denote the Möbius transformation of \( D \)

\[
\varphi_z = \frac{z - w}{1 - zw},
\]

for \( w \in D \). The pseudo-hyperbolic distance on \( D \) is defined by

\[
\rho(z, w) = |\varphi_z(w)|, \quad z, w \in D.
\]

The pseudohyperbolic distance is Möbius invariant, that is

\[
\rho(gz, gw) = \rho(z, w),
\]

for all \( g \in \text{Aut}(D) \), the Möbius group of \( D \), and all \( z, w \in D \). It has the following useful property:

\[
1 - \rho(z, w)^2 = \frac{1 - |z|^2(1 - |w|^2)}{|1 - zw|^2}.
\]

The Bergman metric on \( D \) is the hyperbolic metric whose element of length is

\[
ds = \frac{|dz|}{1 - |z|^2}.
\]

In this metric every rectifiable arc \( \gamma \) has a length

\[
\int_{\gamma} \frac{|dz|}{1 - |z|^2}.
\]

It is easy to show that the induced distance on \( D \) is given by

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},
\]

for \( z, w \in D \).

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Let \( \varphi \) be an analytic self-map of the unit disk. By the classical Schwarz-Pick Lemma \([2],[5]\), \( \varphi \) decreases the hyperbolic distance between two points and the hyperbolic length of an arc. The explicit inequality is
\[
\left| \frac{\varphi(z_1) - \varphi(z_2)}{1 - \varphi(z_1)\varphi(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z}_1z_2} \right|
\]
for any \( z_1, z_2 \) in \( D \). In particular,
\[
(1.2) \quad \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2} \leq \frac{1}{1 - |z|^2}
\]
for \( z \) in \( D \). Let
\[
\tau_\varphi(z) = \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2},
\]
Then
\[
|\tau_\varphi(z)| \leq 1,
\]
for all \( z \in D \). Nontrivial equality \( |\tau_\varphi(z)| = 1 \) holds for some \( z \in D \) only when \( \varphi \) is a fractional linear transformation \( e^{i\theta} \varphi_a(z) \). For each \( z \in D \), the hyperbolic derivative of \( \varphi \) at \( z \) is defined by
\[
\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z,w)}.
\]
In Section 3 we will show that the hyperbolic derivative of \( \varphi \) equals \( |\tau_\varphi(z)| \) and \( \tau_\varphi(z) \) is Lipschitz with respect to the pseudohyperbolic metric. Hyperbolic derivatives have been used in studying composition operators on the Bloch space \([7],[9]\) and \([10]\).

Recently, MacCluer, Stroethoff, and Zhao \([8]\) used the formula of Faa di Bruno and the theory of the weighted composition operators \([11]\), to obtain the generalized Schwarz-Pick estimates:
\[
(1.3) \quad \sup_{z \in D} \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^n} < \infty
\]
for each analytic self-map \( \varphi \) of the unit disk. We obtain the following generalized Schwarz-Pick estimates: for each \( 0 < r < 1 \) and each positive integer \( n \), there is a positive constant \( M_{n,r} \) such that for each analytic self-map \( \varphi \) of the unit disk:
\[
(1.4) \quad \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_{n,r} \max_{\rho(w,z) < r} \frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2},
\]
for \( z \) in \( D \). Clearly, Combining (1.2) with (1.4) gives (1.3). Moreover, (1.4) directly leads to the result \([8]\) that if \( \varphi \) is in the little Bloch class, then
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0
\]
for each \( n \). The main tool is the beautiful formula of Faa di Bruno \([13]\) for the \( n \)th derivative of composition of two functions.

Based on the generalized Schwarz-Pick estimates we will show in Section 3 that \( \tau_{\varphi,n}(z) = \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^n} \) is Lipschitz with respect to the pseudohyperbolic metric. Thus \( \tau_{\varphi,n}(z) \) admits a continuous extension to the set of nontrivial Gleason parts of the maximal ideal space of \( H^\infty \).
2. Generalized Schwarz-Pick estimates

In this section, we will present a proof of the generalized Schwarz-Pick estimates. The generalized Schwarz-Pick estimates will be used in the proof of Theorem 6. The main tool is the beautiful formula of Faa di Bruno, which deals with the $n$th derivative of composition of an analytic function $f$ on the unit disk with a self-map $\varphi$ of the unit disk \([13]\).

**Theorem 1.** (The Formula of Faa di Bruno) If $\varphi$ is an analytic function from the unit disk to the unit disk and $f$ is an analytic function on the unit disk, then

$$(f \circ \varphi)^{(n)}(z) = \sum_{k_1!k_2!\cdots k_n!} \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n}(\varphi^{(j)}(z))^{k_j}$$

where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \cdots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

The following result is well-known \([12]\). We include a proof to motivate our Theorem 2.

**Proposition 1.** If $\varphi$ is a univalent analytic self-map of $D$, then

$$(1 - |z|^2)|\varphi''(z)| \leq 10|\varphi'(z)|$$

for all $z \in D$.

Proof. For a fixed $z$ in $D$, let $h$ be the Koebe transform of $\varphi$

$$h(w) = \frac{\varphi\left(\frac{w+z}{1+z\bar{w}}\right) - \varphi(z)}{(1-|z|^2)\varphi'(z)}.$$ 

Then $h(0) = 0$ and $h'(0) = 1$. It follows from Bieberbach’s theorem (\([12]\), page 8) that

$$|h''(0)| \leq 4.$$ 

On the other hand, an easy computation gives

$$h''(0) = \frac{1}{2}(1-|z|^2)\frac{\varphi''(z)}{\varphi'(z)} - z.$$ 

Hence

$$\left|\frac{1}{2}(1-|z|^2)\frac{\varphi''(z)}{\varphi'(z)} - z\right| \leq 4.$$ 

Since $|z| \leq 1$, we conclude that

$$|(1-|z|^2)|\varphi''(z)| \leq 10|\varphi'(z)|.$$ 

This completes the proof.

As a consequence of the proposition, we have

$$(2.1) \quad \frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} \leq \frac{10(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}$$

for all $z \in D$ if $\varphi$ is a univalent self-map of the unit disk.

**Example:** Let $b$ be an interpolating Blaschke product with zeros $\{z_n\}$ in the unit disk and $\varphi = b^2$. Clearly, $\varphi'(z_n) = 0$ and $\varphi''(z_n) = 2|b'(z_n)|^2$. Let $\delta = \inf_{z_n}(1-|z_n|^2)|b'(z_n)|$. Thus

$$\frac{(1-|z_n|^2)|\varphi''(z_n)|}{1-|\varphi(z_n)|^2} = 2(1-|z_n|^2)|b'(z_n)|^2 \geq 2\delta|b'(z_n)| \geq \frac{2\delta^2}{1-|z_n|^2}.$$
So the inequality (2.1) does not hold for some analytic self-maps of the unit disk. But by means of the formula of Faa di Bruno we still have the generalized Schwarz-Pick Estimates:

**Theorem 2.** For each positive integer $n$ and each number $0 < r < 1$, there is a positive constant $M_{n,r}$ such that for each analytic self-map $\varphi$ of the unit disk,

$$\frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_{n,r} \max_{\rho(w,z)<r} \frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2},$$

for $z$ in $D$.

As we mentioned in introduction, by the Schwarz-Pick estimates (1.2), we have

$$\frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2} \leq 1.$$ 

Thus Theorem 2 implies the following generalized Schwarz-Pick Estimates [8].

**Theorem 3.** For each $n$, there is a positive constant $M_n$ such that for each analytic self-map $\varphi$ of the unit disk,

$$\frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_n,$$

for $z$ in $D$.

If $\varphi$ is in the little Bloch class, i.e.,

$$\frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} \to 0$$

as $|z| \to 1$, then noting that for the given $0 < s < 1$, for every $w \in D$ with $\rho(w,z) < s$, $|w| \to 1$ as $|z| \to 1$, Theorem 2 gives the following result in [8].

**Theorem 4.** Let $\varphi$ be an analytic self-map of the unit disk. If

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0,$$

then for each $n$

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0.$$

**Proof of Theorem 2.** For a fixed $z$ in $D$, let $g = \varphi \circ \varphi_z$. Clearly, $g(0) = \varphi(z)$. By the formula of Faa di Bruno, we have

$$g^{(n)}(w) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}(\varphi_z(w)) \prod_{j=1}^{n} \left( \frac{\varphi_z^{(j)}(w)}{j!} \right)^{k_j},$$

where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \cdots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

Evaluating the value of $g^{(n)}$ at 0 gives

$$g^{(n)}(0) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}(\varphi_z(0)) \prod_{j=1}^{n} \left( \frac{\varphi_z^{(j)}(0)}{j!} \right)^{k_j}.$$
Noting that \( \varphi_z(0) = z \) and \( \varphi_z^{(j)}(w) = -(1 - |z|^2) \bar{z}^{j-1} j!(1 - \bar{z} w)^{-j-1} \), we have
\[
g^{(n)}(0) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} \varphi^{(k)}(z) \prod_{j=1}^{n} (-1 - |z|^2) \bar{z}^{j-1} k_j
\]
\[
= \sum (-1)^k \frac{n!}{k_1!k_2! \cdots k_n!} \varphi^{(k)}(z)(1 - |z|^2)^k \bar{z}^{n-k}.
\]
The last equality follows from that \( k_1 + \cdots + k_n = k \) and \( k_1 + 2k_2 + \cdots + nk_n = n \). Thus
\[
(-1)^n (1 - |z|^2)^n \varphi^{(n)}(z) = g^{(n)}(0) - \sum_{k<n} (-1)^k \frac{n!}{k_1!k_2! \cdots k_n!} \varphi^{(k)}(z)(1 - |z|^2)^k \bar{z}^{n-k}.
\]
So
\[
\frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} + \sum_{k<n} \frac{n!}{k_1!k_2! \cdots k_n!} |\varphi^{(k)}(z)|(1 - |z|^2)^k |\bar{z}^{n-k}|.
\]
Let \( M_k(z) = \frac{\varphi^{(k)}(z)(1 - |z|^2)^k}{1 - |\varphi(z)|^2} \). The above inequality gives
\[
M_n(z) \leq \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} + \sum_{k<n} \frac{n!}{k_1!k_2! \cdots k_n!} M_k(z).
\]
We need to estimate \( \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} \).

Let \( \lambda = g(0) \), \( h = \varphi_\lambda \circ g \). Then \( h \) is still an analytic self-map of the unit disk, \( h(0) = 0 \), and \( \|h\|_\infty \leq 1 \). As \( \varphi_\lambda \circ \varphi_\lambda(z) = z \), we obtain \( g = \varphi_\lambda \circ h \). The formula of Faa di Bruno again gives
\[
g^{(n)}(w) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} \varphi^{(k)}(h(w)) \prod_{j=1}^{n} \left( \frac{h^{(j)}(0)}{j!} \right)^{k_j}
\]
where \( k = k_1 + \cdots + k_n \) and the sum is over all \( k_1, \cdots, k_n \) for which \( k_1 + 2k_2 + \cdots + nk_n = n \).
Evaluating \( g^{(n)} \) at 0 gives
\[
g^{(n)}(0) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} \varphi^{(k)}(0) \prod_{j=1}^{n} \left( \frac{h^{(j)}(0)}{j!} \right)^{k_j},
\]
since \( h(0) = 0 \). Noting \( \varphi^{(k)}(w) = -(1 - |\lambda|^2) \bar{\lambda}^{k-1} k!(1 - \bar{\lambda} w)^{-k-1} \), the above equality leads to
\[
g^{(n)}(0) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} [- (1 - |\lambda|^2) \bar{\lambda}^{k-1} k!] \prod_{j=1}^{n} \left( \frac{h^{(j)}(0)}{j!} \right)^{k_j}.
\]
Hence
\[
\frac{|g^{(n)}(0)|}{1 - |g(0)|^2} \leq \sum \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^{n} \left( \frac{|h^{(j)}(0)|}{j!} \right)^{k_j}.
\]
Let \( a_n = \sum_{k<n} \frac{n!}{k_1!k_2! \cdots k_n!} \). So far we have shown
\[
M_n(z) \leq a_n \max_{k<n} M_k(z) + \sum \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^{n} \left( \frac{|h^{(j)}(0)|}{j!} \right)^{k_j}.
\]
Note $h = \varphi_\lambda \circ g$, $g = \varphi \circ \varphi_z$, and $\lambda = g(0) = \varphi(z)$. Then

$$h'(w) = \frac{(1 - |\lambda|^2)g'(w)}{(1 - \lambda g(w))^2},$$

and

$$h'(w) = \frac{\sum_{j=1}^{\infty} h^{(j)}(0) w^{j-1}}{(j-1)!}.$$ 

Let $0 < r < 1$. Thus

$$h^{(j)}(0) = r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} h'(re^{i\theta})e^{-i(j-1)\theta} d\theta.$$ 

So

$$|h^{(j)}(0)| \leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|h'(re^{i\theta})|}{|1 - \lambda g(re^{i\theta})|^2} d\theta$$

$$\leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 - |\lambda|^2)|g'(re^{i\theta})|}{|1 - \lambda g(re^{i\theta})|^2} d\theta$$

$$\leq r^{-(j-1)}(1 - r^2)^{-1}(j-1)! \max_{\rho(z,\omega) \leq r} \frac{(1 - |u|^2)|g'(u)|}{1 - |\varphi(u)|^2}.$$ 

for some constant $C_r > 0$. The third inequality follows from

$$\frac{(1 - |\lambda|^2)(1 - |g(re^{i\theta})|^2)}{|1 - \lambda g(re^{i\theta})|^2} = 1 - |\varphi_\lambda g(re^{i\theta})|^2.$$ 

The last inequality follows from making the change of variable $u = \varphi_z(w)$ and the fact that

$$(1 - |w|^2)|g'(w)| = (1 - |u|^2)|\varphi'(\varphi_z(w))\varphi_z'(w)|$$

$$= \frac{(1 - |w|^2)(1 - |\lambda|^2)}{|1 - \lambda w|^2} |\varphi'(\varphi_z(w))| = (1 - |\varphi_z(w)|^2)|\varphi'(\varphi_z(w))|.$$ 

Hence

$$\frac{|h^{(j)}(0)|}{j!} \leq (r^{j-1})(1 - r^2)^{-1} \max_{\rho(z,\omega) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2}.$$ 

The Schwarz-Pick estimate gives

$$\frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2} \leq 1$$

for each $u \in D$. Thus}

\[
\sum_{k_1|k_2| \cdots k_n} \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-n} \prod_{j=1}^{n} \left( \frac{|h^{(j)}(0)|}{j!} \right)^{k_j} \leq \sum_{k_1|k_2| \cdots k_n} \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-n} r^{k-n} (1 - r^2)^{-k} \max_{\rho(z,\omega) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2}.
\]
Let $b_{n,r} = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} k! r^{k-n} (1 - r^2)^{-k}$. The above inequality gives

$$M_n(z) \leq a_n \max_{k<n} M_k(z) + b_{n,r} \max_{\rho(z,w) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2}.$$  

By the induction, we conclude that

$$M_n(z) \leq M_{n,r} \max_{\rho(z,w) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2}$$

to complete the proof.

3. Hyperbolic derivatives

In this section we will first show that the hyperbolic derivative of an analytic self-map $\varphi$ of the unit disk equals $|\tau_\varphi(z)|$. Then we will show that $\tau_\varphi(z)$ is Lipschitz with respect to the pseudo-hyperbolic metric.

**Theorem 5.** Let $\varphi : D \to D$ be an analytic self-map. For each point $z \in D$, then the hyperbolic derivative of $\varphi$

$$\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z,w)} = |\tau_\varphi(z)|.$$  

Proof. Assume that $\varphi$ is not constant. For each fixed $z \in D$, $\rho(\varphi(z), \varphi(w))$ converges to zero as $\beta(w,z)$ converges to zero because $\varphi$ is continuous in $D$ and $|\varphi(z)| < 1$. Thus

$$\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z,w)} = \lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w)) \rho(\varphi(z), \varphi(w))}{\rho(z,w) \beta(z,w)}.$$  

Both the first and third factors of the product in the right converge to one. Now second factor is

$$\frac{\rho(\varphi(z), \varphi(w))}{\rho(z,w)} = \frac{|\varphi(z) - \varphi(w)|}{|z-w|} \frac{|1 - \overline{z} w|}{|1 - \varphi(z) \overline{\varphi(w)}|}.$$  

Thus

$$\lim_{\beta(z,w) \to 0} \frac{\rho(\varphi(z), \varphi(w))}{\rho(z,w)} = \frac{|\varphi'(z)| (1 - |z|^2)}{1 - |\varphi(z)|^2},$$  

so

$$\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z,w)} = \frac{|\varphi'(z)| (1 - |z|^2)}{1 - |\varphi(z)|^2}.$$  

This completes the proof.

For each $n$, define

$$\tau_{\varphi,n}(z) = \frac{(1 - |z|^2)^n \varphi^{(n)}(z)}{1 - |\varphi(z)|^2}.$$  

**Theorem 6.** Let $\varphi$ be an analytic self-map of the unit disk $D$. Then for each $n$, $\tau_{\varphi,n}(z)$ is Lipschitz with respect to the pseudohyperbolic metric. More precisely,

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq C_n \rho(z,w),$$  

for any $z, w \in D$. Here $C_n$ is a positive constant only depending on $n$.

Proof. Suppose that $f$ is a differentiable function on the unit disk. Let $\partial_z f = \frac{\partial f}{\partial z}$ and $\partial_z f = \frac{\partial f}{\partial \overline{z}}$. Note that $\tau_{\varphi,n}(z)$ is differentiable on the unit disk. Easy calculations
give
\[
\partial_t \tau_{\varphi,n}(z) = -zn(1 - |z|^2)^{n-1}\varphi^{(n)}(z)(1 - |\varphi(z)|^2) + (1 - |z|^2)^n\varphi^{(n)}(z)\varphi'(z)\varphi(z),
\]
and
\[
\partial_z \tau_{\varphi,n}(z) = \frac{1}{(1 - |\varphi(z)|^2)^2} \left\{ \left| (1 - |z|^2)^n\varphi^{(n+1)}(z) - zn(1 - |z|^2)^n\varphi^{(n)}(z) \right| \left| (1 - |\varphi(z)|^2)^n\right| \right. + \left. (1 - |z|^2)^n\varphi^{(n)}(z)\varphi'(z)\varphi(z) \right\}.
\]
Thus
\[
|\partial_t \tau_{\varphi,n}(z)| \leq \frac{1}{1 - |z|^2} \left| n(1 - |z|^2)^n|\varphi^{(n)}(z)| \right| + \left| \varphi(z) \right| \left| (1 - |z|^2)^n|\varphi^{(n)}(z)| \right| \left| (1 - |\varphi(z)|^2)^n \right| \leq \frac{(n + 1)M_n}{1 - |z|^2},
\]
which the last inequality follows from Theorem 3, and
\[
|\partial_z \tau_{\varphi,n}(z)| \leq \frac{1}{1 - |z|^2} \left| n(1 - |z|^2)^n|\varphi^{(n+1)}(z)| \right| + \left| \varphi(z) \right| \left| (1 - |z|^2)^n|\varphi^{(n)}(z)| \right| \left| (1 - |\varphi(z)|^2)^n \right| \leq \frac{M_{n+1} + (n + 1)M_n}{1 - |z|^2},
\]
where the last inequality follows from Theorem 3. Given \( z \) and \( w \) in \( D \), let \( \gamma(t) : [0, 1] \to D \) be a smooth curve to connect \( z \) and \( w \).

\[
|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| = \left| \int_0^1 \frac{d\tau_{\varphi,n}(\gamma(t))}{dt} dt \right|
\]
\[
\leq \int_0^1 \left| \frac{d}{dt} \tau_{\varphi,n}(\gamma(t)) \right| dt
\]
\[
\leq \int_0^1 \left| \partial_t \tau_{\varphi,n}(\gamma(t)) \right| \left| \frac{d\gamma(t)}{dt} \right| + \left| \partial_z \tau_{\varphi,n}(\gamma(t)) \right| \left| \frac{d\gamma(t)}{dt} \right| dt,
\]
where the last inequality follows from the first chain rule:
\[
\frac{d}{dt} \tau_{\varphi,n}(\gamma(t)) = \partial_t \tau_{\varphi,n}(\gamma(t)) \frac{d\gamma(t)}{dt} + \partial_z \tau_{\varphi,n}(\gamma(t)) \frac{d\gamma(t)}{dt}.
\]
Combining the above estimates gives
\[
|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq \int_0^1 \frac{M_{n+1} + 2(n + 1)M_n}{1 - |\gamma(t)|^2} d|\gamma(t)|.
\]
If we choose \( \gamma \) to be a geodesic to connect \( z \) and \( w \), the above inequality gives
\[
|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq (M_{n+1} + 2(n + 1)M_n)\beta(z, w)
\]
\[
\leq \frac{(M_{n+1} + 2(n + 1)M_n)\rho(z, w)}{1 - \rho(z, w)^2}.
\]
The last inequality comes from that for all \( 0 < x < 1, \)
\[
\frac{1}{2} \ln \frac{1 + x}{1 - x} \leq \frac{x}{1 - x^2}.
\]
If \( |\rho(z, w)| < 1/8 \), the above inequality gives
\[
|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq 2(M_{n+1} + 2(n + 1)M_n)\rho(z, w).
\]
If $|\rho(z, w)| \geq 1/8$, we have $8|\rho(z, w)| \geq 1$, and

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq \max\{|\tau_{\varphi,n}(z)|, |\tau_{\varphi,n}(w)|\} \leq M_n \leq 8M_n \rho(z, w).$$

Choosing $C_n = \max\{2(M_{n+1} + 2(n+1)M_n), 8M_n\}$, we have

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq C_n \rho(z, w),$$

to complete the proof.

Theorem 6 has an application to closed-range composition operators on the Bloch space.

Hoffman [6] showed that $(1 - |z|^2)^n \varphi^{(n)}(z)$ continuously extends to the maximal ideal space of $H^\infty$. Let $\mathcal{G}$ be the subset of the maximal ideal space of $H^\infty$ consisting of non-trivial Gleason parts. As a corollary of a result in [1] and Theorem 1.2 [4], we have the following result.

**Corollary 1.** Suppose that $\varphi$ is an analytic self-mapping of the unit disk. Then $\tau_{\varphi,n}(z)$ admits a continuous extension to $\mathcal{G}$.

**Addendum:** After we finished this paper, we got K. Stroethoff’s paper [14], which showed that

$$\rho(|\tau_{\varphi}(z)|, |\tau_{\varphi}(w)|) \leq 2\rho(z, w),$$

for $z, w \in D$. This generalizes Beardon’s result [3]: if $\varphi(0) = 0$,

$$\rho(\tau_{\varphi}(0), \tau_{\varphi}(w)) < 2\rho(0, w)$$

for $w \in D$. We thank K. Stroethoff.

**REFERENCES**


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