The Dixmier Trace of Bergman Space Hankel Operators

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Singular Numbers and Schatten Ideals

- Suppose A is a compact operator on a Hilbert space $\mathcal{H}$.
- The *singular values* of A, $\{s_n(A)\}_{n=1}^{\infty}$, are the eigenvalues of the compact positive operator $|A| = (A^*A)^{1/2}$.
- They measure the size of the operator.
- If $\{s_n(A)\} \in l^p$ we say that the operator is in the Schatten ideal, $A \in \mathcal{S}_p$.

Classes of Operators

- Finite rank operator: $s_n(A) = 0$ for all $n > N$
- Trace class operator: $\{s_n(A)\} \in l^1$, $A = \sum c_i A_i$ with $\sum |c_i| < \infty$, each A_i of norm one and rank one
- Hilbert Schmidt operators: $\{s_n(A)\} \in l^2$, the mapping

$$f \times g \to \langle Af, g \rangle$$

  taking $\mathcal{H} \times \mathcal{H}^*$ to $\mathbb{C}$ extends to a bounded map $\mathcal{H} \otimes_{\text{Hilbert}} \mathcal{H}^* \to \mathbb{C}$.
- Compact operators: $s_n(A) \to 0$, A is a norm limit of finite rank operators.

Hankel Operators on the Hardy Space

- H^2, the Hardy space, the subspace of $L^2(\mathbb{T})$ formed by boundary values of functions holomorphic in the disk.
- P, the orthogonal projection, $P : L^2 \to H^2$.
- b, a function on the circle
- The Hankel operator with symbol b;

$$H_{\bar{b}} f = (I - P)(\bar{b} f) \simeq \bar{P}(\bar{b} f)$$

- Note that wlog we may assume $b \in \mathrm{Hol}(\mathbb{D})$
- There has been much study of the interrelation between the operator theory of the H_b's and the function theory of the b's. In particular the size of the $s_n(H_b)$ is determined by the smoothness of b
- Peller: *Hankel operators and their applications*.

Theorem (Kronecker, Nehari, Hartman, Peller, Semmes)

For $0 < p < \infty$, b *holomorphic*

$$H_{\bar{b}} \text{ is bounded iff } b \in BMO$$

$$H_{\bar{b}} \text{ is compact iff } b \in VMO$$

$$H_{\bar{b}} \in \mathcal{S}_p \text{ iff } b \in B^p(\mathbb{D}) \text{ (a Besov space)}$$

$$\text{iff } \exists m > 1/p \int_{\mathbb{D}} \left| (1 - |z|^2)^m b^{(m)}(z) \right|^p d\lambda(z) < \infty$$

$$\text{iff } \sum_{\{z_i\} \sim \text{ hyperbolic lattice}} Osc(b, z_i)^p < \infty$$

$$H_{\bar{b}} \text{ is finite rank iff } b \text{ is rational}$$

$d\lambda$ is invariant measure; $d\lambda = \left(1 - |z|^2\right)^{-2} dxdy$

- The relationship between b and $H_{\bar{b}}$, the previous theorem is only a small part, is one of the richest chapters in the interaction of operator theory and function theory.
- How does it generalize to other spaces? (of holomorphic functions ?) (with reproducing kernels ?)

Big and Small Hankel Operators on the Bergman Space

- The Bergman space, $A^2 = L^2(\mathbb{D}, dxdy) \cap \mathrm{Hol}$
- P, the orthogonal projection from L^2 to A^2
- How to define a Hankel operator? Given b defined on the disk do we want to define the Hankel operator by

$$\begin{aligned} H_b(f) &= (I - P)(\bar{b}f) = P^{\perp}(\bar{b}f), \text{ or} \\ h_b(f) &= \bar{P}(\bar{b}f) \ ? \end{aligned}$$

- These operators are quite different from each other, the first is big, the second is small. Hence the names $\cdots$
- Both are studied.

The Small Hankel Operator on the Bergman Space

- The methods and results for the small Hankel operator on the Bergman space similar to those used for Hardy space Hankel operators.

Theorem (Various people, early '80's)

For $0 < p < \infty$,

$$\begin{aligned} h_{\bar{b}} \in \mathcal{S}_p \qquad & \textit{iff } b \in B^p(\mathbb{D}) \\ & \textit{iff } \int_{\mathbb{D}} \left| (1 - |z|^2)^m b^{(m)}(z) \right|^p d\lambda(z) < \infty \\ & \textit{iff } \sum Osc(b, z_i)^p < \infty \end{aligned}$$

The Big Hankel Operator on the Bergman Space

Theorem (Arazy, Fisher, Peetre, 1988)

For $1 < p < \infty$ $H_{\bar{b}} \in \mathcal{S}_p$ *iff* $b \in B^p(\mathbb{D})$.
$H_{\bar{b}} \in \mathcal{S}_1$ *iff b is constant.*

- The change in behavior at $p = 1$ was a surprise.
- This is a more difficult result, in the previous case, for $p \geq 1$ one could study $p = 1$ and $p = 2$, then use interpolation and duality.
- There is a weak-type endpoint result.

Theorem (Nowak, 1991)

If b is smooth then $s_n\left(H_{\bar{b}}\right) = O(\frac{1}{n})$.

- Once you have this result can do interpolation.

Traces

- A trace on a space of operators is a densely defined positive linear functional τ so that for all A, B we have

$$\tau(AB) = \tau(BA).$$

- Example, for M_n, $n \times n$ matrices; $\tau(A) = \sum_1^n a_{jj} = \sum_1^n \langle Ae_i, e_i \rangle$ where $\{e_i\}$ is <u>*any*</u> ONB
- Exercise: up to positive scalar multiple that is the only trace on M_n.
- Example, for K the compact operators on a Hilbert space, we can try to do the same thing

$$\tau(A) = \sum_1^n \langle Ae_i, e_i \rangle \,; \quad \{e_i\} \text{ is } \underline{\textit{some}} \text{ ONB}$$

but this need not converge.

Theorem (Lidski)

For A in the Trace class, $\mathcal{S}_1$, the series converges absolutely and the sum in indepent of the choice of ONB.

The Dixmier Trace

- The Dixmier trace is a linear functional defined on a space of compact operators
- It is particularly well suited for studying operators which are not trace class, i.e. $\{s_n(A)\} \notin l^1$ but we do have $s_n(A) = O(\frac{1}{n})$
- In fact, singular value asymptotics of the sort $s_n(A) \sim c(A)n^{-d}$ are quite common among basic pseudodifferential operators.
- Another class of examples, $H_{\bar{b}}$, for smooth b. (But this class only somewhat different, see below.)
- The definition is. roughly,

$$Tr_\omega(|A|) = \lim_\omega \frac{1}{\log N} \sum_{k=1}^{N} s_k(A) := \omega\left(\frac{1}{\log N} \sum_{k=1}^{N} s_k(A)\right)$$

Questions

- "roughly"?
- What is ω ?
- Why is this interesting or important?

Answers

- "roughly" $= \lim_\omega$
- $\omega \in (l^\infty)^*$, $\omega \in (l_0^\infty)^\perp$ $\omega(\{h_n\}) \geq 0$ if all $h_n \geq 0$, and a certain technical property.
- Note that there are lots of such ω. This is a technical issue that we won't mention again.
- Why interesting? We were trying to understand better the behavior of the singular values of Hankel operators and in particular the failure of those operators to be in the trace class.
- However there is a much much bigger story here, but I'm not a good person to tell it.
- Connes: *Noncommutative Geometry*

Toeplitz Operators

If a is a function on the disk then the Toeplitz operator with symbol a. $T_a : A^2 \longrightarrow A^2$ is defined by $T_a(f) = P(af)$. Toeplitz operators on the Hardy space are defined analogously.

Theorem 1

Theorem (Ingliš, Rochberg, '07)

If a is a smooth function on the closed disk then

$$Tr_\omega\left(|H_a|\right) = \frac{1}{2\pi}\int_{\mathbb{T}} \left|\bar{\partial} a\right| d\theta.$$

If $a_1, \ldots, a_n$ *are smooth then*

$$Tr_\omega\left(T_{a_1} \ldots T_{a_n} |H_a|\right) = \frac{1}{2\pi}\int_{\mathbb{T}} a_1 \ldots a_n \left|\bar{\partial} a\right| d\theta.$$

- Working with Tr_ω frequently leads to elegant formulas with equal signs.
- This is an example of Connes' quantized calculus. The operator theoretic constructs on the left hand side can be used to replace and then generalize the calculus expressions on the right.

Context

If H_f is the big Hankel operator acting on the Bergman space of the unit ball in $\mathbf{C}^d$ and f is holomorphic

Theorem (Engliš, Guo, and Zhang, 2007)

$$\operatorname{Tr}_\omega(|H_{\bar{f}}|^{2d}) = \int_S \left(|\nabla f|^2 - |Rf|^2 \right)^d d\sigma.$$

S is the boundary of the ball, $d\sigma$ is its normalized surface measure and R is radial differentiation.

- Surprisingly the $d = 1$ result is more delicate, will mention why later.

Suppose ϕ is a holomorphic univalent map of the disk to a domain Ω.

Theorem

$$\mathrm{Tr}(|H_{\bar{\phi}}|^2)^{1/2} = \mathrm{Area}(\Omega)$$

Theorem

$$\mathrm{Tr}_\omega(|H_{\bar{\phi}}|) = \mathrm{Length}(\partial\Omega).$$

Geometry

If the $\partial\Omega$ has finite length and f is holomorphic on $\overline{\Omega}$ then

Theorem

$$\mathrm{Tr}_\omega(T_{f\circ\phi}|H_{\bar{\phi}}|) = \int_{\partial\Omega} f(\zeta)\,|d\zeta|\,.$$

- Suppose instead that $\partial\Omega$ has Hausdorff dimension $p > 1$ (and some other technical requirements). Let $d\Lambda_p$ be p-dimensional Hausdorff measure on $\partial\Omega$ and let T_*^{H} and H_*^{H} be the Toeplitz and Hankel operators on the *Hardy* space.

Theorem (Connes, Sullivan, 1994)

There is a $c \neq 0$ so that if f is holomorphic on $\overline{\Omega}$ then

$$\mathrm{Tr}_\omega(T^H_{f\circ\phi}|H^H_{\bar{\phi}}|^p) = c\int_{\partial\Omega} f(\zeta)d\Lambda_p(\zeta).$$

Theorem 2, Regularity

Theorem 1 is proved for smooth functions but in some cases we could identify the optimal regularity.

Theorem

Suppose f is holomorphic on the disk, TFAE:

1. $\int_{\mathbb{T}} |f'| \, d\theta < \infty$.
2. $Osc(f) \in l^1_{weak}$.
3. $Tr_\omega \left(|H_{\bar{f}}|\right) < \infty$.
   When these conditions hold we have

$$Tr_\omega \left(|H_{\bar{f}}|\right) = \frac{1}{2\pi} \int_{\mathbb{T}} |f'| \, d\theta.$$

(Recall $\mathrm{Osc}(f) \in l^1$ is the Besov space B^1; $\int |f''| dA < \infty$.)

A Digression on the First Two Conditions.

- The equivalence of the first two conditions is the equivalence of a Sobolev style space (integrability of a derivative) and a Besov style space (global control of local oscillation estimates of harmonic extension).
- This is unusual for $p \neq 2$
- However Connes, Sullivan, Teleman and Semmes noted that in d dimensions the Sobolev space of functions with gradient in L^d coincides with the Besov space $B^{d,\infty}$.
- The equivalence of the two conditions in the theorem is the $d = 1$ version of their result, which is false
- for general functions but true for holomorphic ones. Their proof gives that f' is a finite measure. In the presence of holomorphy one then has the theorem of F. and M. Riesz theorem.

Theorem 3, Multiply Connected Domains

Suppose Ω is a multiply connected domain with smooth boundary $\partial\Omega$. Let $d\gamma$ be arclength measure on each boundary component rescaled to put mass 1 on each boundary component.

Theorem

Suppose a is smooth on $\bar{\Omega}$ then

$$Tr_{\omega}\left(|H_a|\right) = \int_{\partial\Omega} \left|\bar{\partial}a\right| d\gamma.$$

Theorem 4, a Matrix

- Suppose $b = \sum b_n z^n$ is holomorphic on the disk. Let $M_b = (m_{ij})$ be the matrix given by

$$
\begin{aligned}
m_{i,j} &= \frac{(j+1)^{1/2}(i-j)\, b_{i-j}}{(i+1)^{3/2}} \text{ if } i \geq j \\
&= 0 \qquad\qquad\qquad \text{otherwise}
\end{aligned}
$$

- M_b is the matrix of a modified Toeplitz operator on the Hardy space. There is certainly a family of such, replacing 1/2 and 3/2 by other numbers. I don't know anything about them.

Theorem

$$Tr_\omega(|M_b|) = \frac{1}{2\pi}\int_{\mathbb{T}} |b'|\, d\theta.$$

Proof of Theorem

- $H^*H : A^2 \to A^2$
- $J = I^{1/2} H^* H I^{-1/2} : H^2 \to H^2$ is unitarily equivalent.
- J is a positive ΨDO with principal symbol $\sigma^2 \geq 0$.
- H^*H has some cancellation, recall $(I - P)$ is part of H.
- In higher dimensions there is cancellation in the computation of (the analog of) σ^2.
- In one dimensional there is also a second cancellation.
- Can compute its principal symbol σ of $J^{1/2}$

Theorem (Wodzicki, Connes, 1980's,)

$$Tr_\omega(J^{1/2}) = Residue_{s=1} Tr\left((J^{1/2})^s\right) = \frac{1}{2\pi}\int \sigma$$

Proof of Theorem 2, Regularity

- Numbers related to the singular values can be used to control local oscillation.

$$\left(1-|z_i|^2\right)\left|f'(z_i)\right| \leq C\left|\langle S^* H_{\bar{f}} Te_i, e_i\rangle\right|.$$

- Sums of those oscillation estimates are Riemann sums for certain integrals.

$$\begin{aligned} \left\|f'\right\|_{H^1} &= \overline{\lim}_{r\to 0}\,(1-r)^{-1}\iint_{1-r<|z|<1}\left|f'\right| \\ &\leq C\ \underline{\lim}_{N\to\infty} N^{-1}\sum^{2^N} s_k\left(H_{\bar{f}}\right) \\ &\leq C\ Tr_\omega\left(\left|H_{\bar{f}}\right|\right) \end{aligned}$$

- The holomorphy is used in the proof:
  - subharmonicity of $|f'|$ used in comparing operator to symbol and in justifying the Riemann sum argument.
  - the monotonicity of the integral means of $|f'|$ is used in the previous estimates.

Proof of Theorem 3, Multiply Connected Domains

- The result for other simply connected domains follows by conformal mapping.
- For general domains

$$\begin{aligned} H_b^{\Omega} &\sim \sum H_{b_i}^{\Omega} \\ &\sim \oplus H_{b_i}^{\Omega_i} \end{aligned}$$

Here $\Omega = \cap \Omega_i$ where $\Omega_i = \ldots\ldots..$

- Hence $Tr_\omega(\left|H_b^{\Omega}\right|) = \sum Tr_\omega(|H_{b_i}^{\Omega_i}|)$.
- Evaluate $Tr_\omega(|H_{b_i}^{\Omega_i}|)$ by reducing to the simply connected case.

Proof of Theorem 4, a Matrix

$$\begin{aligned} P^{\perp} &: \quad bf \longrightarrow \text{ a very large space} \\ \widetilde{P} &: \quad bf \longrightarrow \ker\left(\left(\bar{z}\bar{\partial}\right)^2\right) \ominus A^2 \end{aligned}$$

is a projection onto a much smaller space and $\tilde{H}_{\bar{b}} f = \widetilde{P}\left(bf\right)$ is a simpler

operator and has the same trace. Now compute the matrix of the new operator.

- (My first thought had been to obtain the result for $\tilde{H}_{\bar{b}}$ directly and then use that to get Theorem 1.)