m-BEREZIN TRANSFORM ON THE POLYDISK

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ABSTRACT. m-Berezin transforms are introduced for bounded operators on the Bergman space of the polydisk. We show several properties of m-Berezin transform and use them to show that a radial operator in the Toeplitz algebra is compact iff its Berezin transform vanishes on the boundary of the polydisk. A useful and sharp approximate identity of its m-Berezin transforms is obtained for a bounded operator.

1. INTRODUCTION

Let D be the unit disk in the complex plane $\mathbb{C}$. For a fixed positive integer n, the unit polydisk D^n is the cartesian product of n copies of D and dz is the normalized Lebesque volume measure on the polydisk D^n. The Bergman space $L_a^2 = L_a^2(D^n, dz)$ is the set of all analytic functions on D^n which are square-integrable with respect to Lebesque volume measure.

Given $f \in L^\infty$, the Toeplitz operator T_f is defined on L_a^2 by $T_f h = P(fh)$ where P denotes the orthogonal projection P of L^2 onto L_a^2. Let $\mathfrak{L}(L_a^2)$ be the algebra of bounded operators on L_a^2. The Toeplitz algebra $\mathfrak{T}(L^\infty)$ is the closed subalgebra of $\mathfrak{L}(L_a^2)$ generated by $\{T_f : f \in L^\infty\}$. This paper is motivated by the problem when an operator in the Toeplitz algebra $\mathfrak{T}(L^\infty)$ is compact. The Berezin transforms will play an important role.

For $z = (z_1, \ldots, z_n) \in D^n$, let $\phi_z(w) = (\phi_{z_1}(w_1), \ldots, \phi_{z_n}(w_n))$ where $\phi_{z_i}(w_i) = (z_i - w_i)/(1 - w_i\overline{z_i})$. Then $\phi_z(w)$ is an automorphism on D^n that interchanges 0 and z. The pseudo-hyperbolic metric on D^n is defined as $\rho(z, w) = \max_{1 \le i \le n} |\phi_{z_i}(w_i)|$.

The reproducing kernel in L_a^2 is given by

$$K_z(w) = \prod_{i=1}^{n} \frac{1}{(1 - w_i\overline{z_i})^2},$$

for $z, w \in D^n$ and the normalized reproducing kernel k_z is $K_z(w)/\|K_z(\cdot)\|_2$. If $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2, then $\langle h, K_z \rangle = h(z)$, for every $h \in L_a^2$ and $z \in D^n$.

For $z \in D^n$, let U_z be the unitary operator given by

$$U_z f = (f \circ \phi_z) \prod_{i=1}^{n} \phi'_{z_i}.$$

For $S \in \mathfrak{L}(L_a^2)$, set

$$S_z = U_z S U_z.$$

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Since U_z is a selfadjoint unitary operator on L^2 and L^2_a, $U_zT_fU_z = T_{f\circ\phi_z}$ for every $f \in L^\infty$.

Let $\mathcal{T}$ denote the class of trace operators on L^2_a. For $T \in \mathcal{T}$, we will denote the trace of T by $tr[T]$ and $\|T\|_{C_1}$ denote the C_1 norm of T given by ([10])

$$\|T\|_{C_1} = tr[\sqrt{T^*T}].$$

Suppose f and g are in L^2_a. Consider the operator $f \otimes g$ on L^2_a defined by

$$(f \otimes g)h = \langle h, g\rangle f,$$

for $h \in L^2_a$. It is easily proved that $f \otimes g$ is in $\mathcal{T}$ and with norm equal to $\|f \otimes g\|_{C_1} = \|f\|_2 \, \|g\|_2$ and

$$tr[f \otimes g] = \langle f, g\rangle.$$

For the nonnegative integer m, the m-Berezin transform of an operator $S \in \mathfrak{L}(L^2_a)$ is defined by

$$B_mS(z) = (m+1)^n tr\left[S_z\left(\sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} C_{m,\alpha}u^\alpha \otimes u^\alpha\right)\right] \tag{1.1}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in N^n$ where N is the set of nonnegative integers, $|\alpha| = \sum_{i=0}^{n} \alpha_i$, $u^\alpha = u_1^{\alpha_1}\cdots u_n^{\alpha_n}$ and

$$C_{m,\alpha} = (-1)^{|\alpha|}\binom{m}{\alpha_1}\cdots\binom{m}{\alpha_n}.$$

Our definition of the m-Berezin transform is motivated by the fact that the reciprocal of the $\frac{m}{2}$-th power of the Bergman reproducing kernel is in the following form:

$$\frac{1}{K_z(z)^{\frac{m}{2}}} = \sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} C_{m,\alpha} z^\alpha \bar{z}^\alpha.$$

The m-Berezin transform depends only on the reproducing kernel and so it can be defined on many other reproducing kernel Hilbert spaces.

For a function $f \in L^\infty(D^n)$, the m-Berezin transform of f is defined by

$$B_m(f)(z) = B_m(T_f)(z).$$

Berezin first studied 0-Berezin transform for operators and m-Berezin transform for functions [5]. Usually the 0-Berezin transform is called the Berezin transform. Not only the Berezin transform plays an important role in studying Toeplitz and Hankle operators on the Bergman spaces ([3], [4], [9], and [14]), but the m-Berezin transforms are also useful in function theory on the unit ball ([1]).

We will show that the m-Berezin transforms B_m are invariant under the Möbious transform,

$$B_m(S_z) = (B_mS)\circ\phi_z, \tag{1.2}$$

and commuting with each other,

$$B_j(B_mS)(z) = B_m(B_jS)(z) \tag{1.3}$$

for any nonnegative integers j and m. Properties (1.2) and (1.3) were obtained for $S = T_f$ in [1] on the Bergman space of the unit ball and for operators S on the Bergman space of the unit disk [15]. Recently, they have been established for operators S on the Bergman space of the unit ball in [12]. We will show that for each m, $B_m S(z)$ is Lipschitz with respect to the pseudo-hyperbolic distance $\rho(z, w)$. This extends the Coburn result on the unit disk [8].

Using the m-Berezin transform, we will show that for a radial operator S in the Toeplitz algebra, S is compact iff $B_0 S(z) \to 0$ as $z \to \partial D^n$. This is obtained in [16] on the unit disk and in [12] on the unit ball.

We will obtain a useful and sharp approximate identity of the m-Berezin transforms (Theorem 3.7), which has been used to study compact products of Toeplitz operators [7].

Throughout the paper $C(m, n)$ will denote constant depending only on m and n, which may change at each occurrence.

2. m-BEREZIN TRANSFORM

In this section we will show some useful properties of the m-Berezin transform. First we give an integral representation of the m-Berezin transform $B_m(S)$. For $z \in D^n$ and a nonnegative integer m, let

$$K_z^m(u) = \prod_{i=1}^{n} \frac{1}{(1 - u_i \overline{z_i})^{m+2}}, \qquad u \in D^n.$$

For $u, \lambda \in D^n$, we know

$$\sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} C_{m,\alpha} u^\alpha \overline{\lambda^\alpha} = \prod_{i=1}^{n} (1 - u_i \overline{\lambda_i})^m. \tag{2.1}$$

From the definition of $\phi_{z_i}(w_i)$, we have the identity

$$1 - \phi_{z_i}(u_i) \overline{\phi_{z_i}(\lambda_i)} = \frac{(1 - |z_i|^2)(1 - u_i \overline{\lambda_i})}{(1 - u_i \overline{z_i})(1 - z_i \overline{\lambda_i})} \tag{2.2}$$

for $u_i, \lambda_i \in D$ and $i = 1, \ldots, n$. The following proposition gives an integral representation of the m-Berezin transform.

Proposition 2.1. *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 0$ and $z \in D^n$. Then*

$$B_m S(z) = \int_{D^n} \int_{D^n} \left[\prod_{i=1}^{n} (m+1)(1 - |z_i|^2)^{m+2} (1 - u_i \overline{\lambda_i})^m \right] K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} \, du \, d\lambda.$$

Proof. Let $\phi'_z(w) = \prod_{i=1}^n \phi'_{z_i}(w_i)$. For $\lambda \in D^n$, the definition of B_m implies

$$\begin{aligned} &B_m S(z) \\ &\quad = (m+1)^n \sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} C_{m,\alpha} \left\langle S_z \lambda^\alpha, \lambda^\alpha \right\rangle \\ &\quad = (m+1)^n \sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} C_{m,\alpha} \int_{D^n} S(\phi_z^\alpha \phi'_z)(\lambda) \overline{\phi_z^\alpha(\lambda) \phi'_z(\lambda)} d\lambda \\ &\quad = (m+1)^n \sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} C_{m,\alpha} \int_{D^n} \int_{D^n} \phi_z^\alpha(u) \phi'_z(u) \overline{\phi_z^\alpha(\lambda) \phi'_z(\lambda) S^* K_\lambda(u)} du d\lambda \end{aligned} \tag{2.3}$$

where the last equality holds by $S(\phi_z^\alpha \phi'_z)(\lambda) = \langle S(\phi_z^\alpha \phi'_z), K_\lambda \rangle = \langle \phi_z^\alpha \phi'_z, S^* K_\lambda \rangle$. Using (2.1) and (2.2), (2.3) equals

$$\begin{aligned} &(m+1)^n \int_{D^n} \int_{D^n} \left[\prod_{i=1}^{n} (1 - \phi_{z_i}(u_i) \overline{\phi_{z_i}(\lambda_i)})^m \right] \phi'_z(u) \overline{\phi'_z(\lambda) S^* K_\lambda(u)} du d\lambda \\ &\quad = \int_{D^n} \int_{D^n} \left[\prod_{i=1}^{n} (m+1)(1-|z_i|^2)^{m+2} (1 - u_i \overline{\lambda_i})^m \right] K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda \end{aligned}$$

as desired. ∎

Proposition 2.2 gives another form of B_m analogous to the definition of the m-Berezin transform on the unit disk [15].

Proposition 2.2. *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 0$ and $z \in D^n$. Then*

$$B_m S(z) = \left[\prod_{i=1}^{n} (m+1)(1-|z_i|^2)^{m+2} \right] \sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} C_{m,\alpha} \left\langle S(u^\alpha K_z^m), u^\alpha K_z^m \right\rangle . \tag{2.4}$$

Proof. Since

$$\begin{aligned} &\int_{D^n} \int_{D^n} \left[\prod_{i=1}^{n} (m+1)(1-|z_i|^2)^{m+2} (1 - u_i \overline{\lambda_i})^m \right] K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda \\ &\quad = \left[\prod_{i=1}^{n} (m+1)(1-|z_i|^2)^{m+2} \right] \sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} \int_B \int_B u^\alpha \overline{\lambda^\alpha} K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda \\ &\quad = \left[\prod_{i=1}^{n} (m+1)(1-|z_i|^2)^{m+2} \right] \sum_{i=1}^{n} \sum_{\alpha_i=0}^{m} C_{m,\alpha} \int_B S(u^\alpha K_z^m)(\lambda) \overline{\lambda^\alpha} \overline{K_z^m(\lambda)} d\lambda, \end{aligned}$$

by Proposition 2.1 we have (2.4) to complete the proof. ∎

On the unit disk the right hand side of (2.4) was used by Suarez in [15] to define the m-Berezin transforms.

Let $d\nu_m(u) = \prod_{i=1}^n (m+1)(1-|u_i|^2)^m du$. The following proposition gives a nice formula of $B_m(f)(z)$.

Proposition 2.3. *Let $z \in D^n$ and $f \in L^\infty$. Then*

$$B_m(f)(z) = \int_{D^n} f \circ \phi_z(u) d\nu_m(u).$$

Proof. Using the change of variables, (2.2) and (2.1), we have

$$\begin{aligned}
&\int_{D^n} f \circ \phi_z(u) d\nu_m(u) \\
&= \int_{D^n} f(u) \prod_{i=1}^{n} \frac{(m+1)(1-|z_i|^2)^{m+2}(1-|u_i|^2)^m}{|1-u_i\overline{z_i}|^{2(m+2)}} du \\
&= \left[\prod_{i=1}^{n}(m+1)(1-|z_i|^2)^{m+2}\right] \sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} \int_{D^n} f(u) \prod_{i=1}^{n} \frac{|u_i|^{2\alpha_i}}{|1-u_i\overline{z_i}|^{2(m+2)}} du \\
&= \left[\prod_{i=1}^{n}(m+1)(1-|z_i|^2)^{m+2}\right] \sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} C_{m,\alpha} \left\langle T_f(u^\alpha K_z^m), u^\alpha K_z^m \right\rangle \\
&= B_m(T_f)(z)
\end{aligned}$$

where the last equality holds by (2.4). ☐

Clearly, (1.1) implies $\|B_m S\|_\infty \leq C(m,n)\|S_z\| = C(m,n)\|S\|$ for $S \in \mathfrak{L}(L_a^2)$. Thus, $B_m : \mathfrak{L}(L_a^2) \to L^\infty$ is a bounded linear operator. The following theorem gives the norm of B_m.

Theorem 2.4. *Let $m \geq 0$. Then*

$$\|B_m\| = (m+1)^n \sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} |C_{m,\alpha}| \left(\prod_{i=1}^{n} \frac{1}{\alpha_i+1}\right).$$

Proof. From [6], we have the duality result $\mathfrak{L}(L_a^2) = \mathcal{T}^*$. So, the definition of B_m gives the norm of B_m. Since

$$\|u^\alpha\|^2 = \prod_{i=1}^{n} \frac{1}{\alpha_i+1},$$

we have

$$\begin{aligned}
\|B_m\| &= (m+1)^n \left\| \sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} C_{m,\alpha} \left(\prod_{i=1}^{n} \frac{1}{\alpha_i+1}\right) \frac{u^\alpha}{\|u^\alpha\|} \otimes \frac{u^\alpha}{\|u^\alpha\|} \right\|_{C^1} \\
&= (m+1)^n \sum_{i=1}^{n}\sum_{\alpha_i=0}^{m} |C_{m,\alpha}| \left(\prod_{i=1}^{n} \frac{1}{\alpha_i+1}\right)
\end{aligned}$$

as desired. ☐

Lemma 2.5. *For $z, w \in D^n$, put $t_i = (\phi_{z_i}(w_i)\overline{z_i} - 1)/(1 - z_i\overline{\phi_{z_i}(w_i)})$, $i = 1, \ldots, n$. Then $U_w U_z = V_t U_{\phi_z(w)}$ where $(V_t f)(u) = (\prod_{i=1}^{n} t_i) f(tu)$ for $f \in L_a^2$ and $tu = (t_1u_1, \ldots, t_nu_n)$.*

Proof. The map $\phi_{w_i} \circ \phi_{z_i} \circ \phi_{\phi_{z_i}(w_i)}$ is an automorphism of D that fixes 0, hence it is a rotation and maps t_i to 1. Since ϕ_{w_i} is an involution, $\phi_{z_i} \circ \phi_{\phi_{z_i}(w_i)}(t_i u_i) = \phi_{w_i}(u_i)$. Thus

$$
\begin{aligned}
&U_w U_z f(u)\\
&\quad = (f \circ \phi_z \circ \phi_w)(u) \prod_{i=1}^n \phi'_{z_i}(\phi_{w_i}(u_i))\phi'_{w_i}(u_i)\\
&\quad = (f \circ \phi_{\phi_z(w)})(tu) \prod_{i=1}^n \phi'_{z_i}(\phi_{z_i} \circ \phi_{\phi_{z_i}(w_i)}(t_i u_i))\phi'_{z_i}(\phi_{\phi_{z_i}(w_i)}(t_i u_i))\phi'_{\phi_{z_i}(w_i)}(t_i u_i) t_i\\
&\quad = (f \circ \phi_{\phi_z(w)})(tu) \prod_{i=1}^n \phi'_{\phi_{z_i}(w_i)}(t_i u_i) t_i\\
&\quad = V_t U_{\phi_z(w)} f(u)
\end{aligned}
$$

as desired. □

Theorem 2.6. *Let $S \in \mathfrak{L}(L^2_a)$, $m \geq 0$ and $z \in D^n$. Then $B_m S_z = (B_m S) \circ \phi_z$.*

Proof. By Proposition 2.2 and (1.1), we have

$$B_m(S_z)(0) = (m+1)^n \sum_{i=1}^n \sum_{\alpha_i=0}^m C_{m,\alpha} \langle S_z u^\alpha, u^\alpha \rangle = B_m S(z) = (B_m S) \circ \phi_z(0).$$

For any $w \in D^n$, Proposition 2.1 and Lemma 2.5 imply

$$
\begin{aligned}
(B_m S_z) \circ \phi_w(0) &= B_m((S_z)_w)(0)\\
&= \int_{D^n}\int_{D^n} \left[\prod_{i=1}^n (m+1)(1-u_i\overline{\lambda_i})^m\right] \overline{U_w U_z S^* U_z U_w K_\lambda(u)} du d\lambda\\
&= \int_{D^n}\int_{D^n} \left[\prod_{i=1}^n (m+1)(1-u_i\overline{\lambda_i})^m\right] \overline{V_t U_{\phi_z(w)} S^* U_{\phi_z(w)} V_t^* K_\lambda(u)} du d\lambda\\
&= B_m(S_{\phi_z(w)})(0).
\end{aligned}
$$

Thus $B_m S_z(w) = (B_m S) \circ \phi_z(w)$. □

Lemma 2.7. *Let $S \in \mathfrak{L}(L^2_a)$, $m \geq 0$ and $z \in D^n$. Then*

$$B_m S(z) = \left(\frac{m+1}{m}\right)^n B_{m-1}\left(\sum_{i=1}^n \sum_{\alpha_i=0}^1 C_{1,\alpha} T_{\overline{\phi_z^\alpha}} S T_{\phi_z^\alpha}\right)(z)$$

where ϕ_z^α is $\phi_{z_1}^{\alpha_1} \cdots \phi_{z_n}^{\alpha_n}$.

Proof. By Theorem 2.6, we only need to show that

$$B_m S(0) = \left(\frac{m+1}{m}\right)^n B_{m-1}\left(\sum_{i=1}^n \sum_{\alpha_i=0}^1 C_{1,\alpha} T_{\overline{u^\alpha}} S T_{u^\alpha}\right)(0).$$

From Proposition 2.1 and (2.1), we have

$$
\begin{aligned}
B_mS(0) &= \int_{D^n}\int_{D^n}\left[\prod_{i=1}^{n}(m+1)(1-u_i\overline{\lambda_i})^m\right]\overline{S^*K_\lambda(u)}dud\lambda \\
&= \sum_{i=1}^{n}\sum_{\alpha_i=0}^{1}C_{1,\alpha}\int_{D^n}\int_{D^n}u^\alpha\overline{\lambda^\alpha}\left[\prod_{i=1}^{n}(m+1)(1-u_i\overline{\lambda_i})^{m-1}\right]\overline{S^*K_\lambda(u)}dud\lambda \\
&= (m+1)^n\sum_{i=1}^{n}\sum_{\alpha_i=0}^{1}C_{1,\alpha}\sum_{i=1}^{n}\sum_{\beta_i=0}^{m-1}C_{m-1,\beta}\int_{D^n}\int_{D^n}u^{\alpha+\beta}\overline{\lambda^{\alpha+\beta}S^*K_\lambda(u)}dud\lambda \\
&= (m+1)^n\sum_{i=1}^{n}\sum_{\alpha_i=0}^{1}C_{1,\alpha}\sum_{i=1}^{n}\sum_{\beta_i=0}^{m-1}C_{m-1,\beta}\left\langle S(u^{\alpha+\beta}),u^{\alpha+\beta}\right\rangle \\
&= \left(\frac{m+1}{m}\right)^n\sum_{i=1}^{n}\sum_{\alpha_i=0}^{1}C_{1,\alpha}B_{m-1}\left(T_{\overline{u^\alpha}}ST_{u^\alpha}\right)(0).
\end{aligned}
$$

The proof is complete. □

Theorem 2.8. *Let* $S\in\mathfrak{L}(L_a^2)$ *and* $m\geq 0$*. Then there exists a constant* $C(m,n)>0$ *such that*

$$|B_mS(z)-B_mS(w)|\leq C(m,n)\|S\|\rho(z,w).$$

Proof. We will prove this theorem by induction on m. If $m=0$, (1.1) implies

$$
\begin{aligned}
|B_0S(z)-B_0S(w)| &= |tr[S_z(1\otimes 1)]-tr[S_w(1\otimes 1)]| \\
&= |tr[S_z(1\otimes 1)-SU_w(1\otimes 1)U_w]| \\
&= |tr[S_z(1\otimes 1)-SU_z(U_zU_w1\otimes U_zU_w1)U_z]|.
\end{aligned}
$$

From Lemma 2.5, the last term equals

$$
\begin{aligned}
|tr[S_z(1\otimes 1-U_{\phi_{w(z)}}1\otimes U_{\phi_{w(z)}}1)]| &\leq \|S_z\|\|1\otimes 1-U_{\phi_{w(z)}}1\otimes U_{\phi_{w(z)}}1\|_{C_1} \\
&\leq \sqrt{2}\|S\|\left(2-2\left|\left\langle 1,\prod_{i=1}^{n}\phi'_{\phi_{w_i}(z_i)}\right\rangle\right|^2\right)^{1/2} \\
&= 2\|S\|\left[1-\prod_{i=1}^{n}(1-|\phi_{w_i}(z_i)|^2)^2\right]^{1/2}
\end{aligned}
$$

where the second inequality holds by $\|T\|_{C^1} \le \sqrt{l}(tr[T^*T])^{1/2}$ where l is the rank of T. Let $\lambda_i = \phi_{w_i}(z_i)$. Since

$$\begin{aligned}\left[1-\prod_{i=1}^{n}(1-|\lambda_i|^2)^2\right] &= 1-(1-|\lambda_1|^2)^2+(1-|\lambda_1|^2)^2\left[1-\prod_{i=2}^{n}(1-|\lambda_i|^2)^2\right]\\ &\le C|\lambda_1|^2+C\left[1-\prod_{i=2}^{n}(1-|\lambda_i|^2)^2\right]\\ &\vdots\\ &\le C\max_{1\le i\le n}|\lambda_i|^2,\end{aligned}$$

we obtain

$$|B_0S(z)-B_0S(w)| \le C\|S\|\rho(z,w).$$

Suppose $|B_{m-1}S(z)-B_{m-1}S(w)| < C(m,n)\|S\|\rho(z,w)$. By Lemma 2.7, we have

$$\begin{aligned}&|B_mS(z)-B_mS(w)|\\ &\quad\le\left(\frac{m+1}{m}\right)^n\sum_{i=1}^{n}\sum_{\alpha_i=0}^{1}|C_{1,\alpha}|\left|B_{m-1}\left(T_{\overline{\phi_z^\alpha}}ST_{\phi_z^\alpha}\right)(z)-B_{m-1}\left(T_{\overline{\phi_w^\alpha}}ST_{\phi_w^\alpha}\right)(w)\right|.\end{aligned}$$

Since

$$\begin{aligned}&B_{m-1}\left(T_{\overline{\phi_z^\alpha}}ST_{\phi_z^\alpha}\right)(z)-B_{m-1}\left(T_{\overline{\phi_w^\alpha}}ST_{\phi_w^\alpha}\right)(w)\\ &\quad=B_{m-1}\left(T_{\overline{\phi_z^\alpha}}ST_{\phi_z^\alpha}-T_{\overline{\phi_w^\alpha}}ST_{\phi_z^\alpha}\right)(z)+B_{m-1}\left(T_{\overline{\phi_w^\alpha}}ST_{\phi_z^\alpha}-T_{\overline{\phi_w^\alpha}}ST_{\phi_w^\alpha}\right)(z)\\ &\qquad+B_{m-1}\left(T_{\overline{\phi_w^\alpha}}ST_{\phi_w^\alpha}\right)(z)-B_{m-1}\left(T_{\overline{\phi_w^\alpha}}ST_{\phi_w^\alpha}\right)(w),\end{aligned}$$

it is sufficient to show that for $|\alpha_i| \le 1$, $1 \le i \le n$,

$$\left|B_{m-1}\left(T_{\overline{\phi_z^\alpha}}ST_{\phi_z^\alpha}-T_{\overline{\phi_w^\alpha}}ST_{\phi_z^\alpha}\right)(z)\right| < C(m,n)\|S\|\rho(z,w).$$

(1.1) gives

$$\begin{aligned}&\left|B_{m-1}\left(T_{\overline{\phi_z^\alpha}}ST_{\phi_z^\alpha}-T_{\overline{\phi_w^\alpha}}ST_{\phi_z^\alpha}\right)(z)\right|\\ &\quad=m^n\left|tr\left[\left(T_{\overline{\phi_z^\alpha}-\overline{\phi_w^\alpha}}ST_{\phi_z^\alpha}\right)_z\left(\sum_{i=1}^{n}\sum_{\beta_i=0}^{m-1}C_{m-1,\beta}u^\beta\otimes u^\beta\right)\right]\right|\\ &\quad\le m^n\sum_{i=1}^{n}\sum_{\beta_i=0}^{m-1}C_{m-1,\beta}\left|\left\langle S_zT_{\phi_z^\alpha\circ\phi_z}u^\beta, T_{(\phi_z^\alpha-\phi_w^\alpha)\circ\phi_z}u^\beta\right\rangle\right|.\end{aligned}\tag{2.5}$$

Let $\lambda = \phi_w(z)$. Then

$$\begin{aligned}
&\left\|T_{(\phi_z^\alpha - \phi_w^\alpha)\circ\phi_z} u^\beta\right\|_2^2 \\
&\quad \le \int_{D^n} |(\phi_z \circ \phi_z)^\alpha(u) - (\phi_w \circ \phi_z)^\alpha(u)|^2 du \\
&\quad = \int_{D^n} |(\mathcal{U}u)^\alpha - \phi_\lambda(u)^\alpha|^2 du \\
&\quad \le 2\int_{D^n} |(\mathcal{U}u)^\alpha + (-1)^{|\alpha|+1}u^\alpha|^2 + |(-1)^{|\alpha|+1}u^\alpha + \phi_\lambda(u)^\alpha|^2 du \qquad (2.6)
\end{aligned}$$

where $\phi_w \circ \phi_z = \phi_\lambda \circ \mathcal{U}$ for some $\mathcal{U}u = (t_1u_1, \cdots, t_nu_n)$ and $|t_i| = 1$ for any $1 \le i \le n$. Lemma 2.5 gives that

$$t_i = \frac{\phi_{w_i}(z_i)\overline{w_i} - 1}{1 - w_i\overline{\phi_{w_i}(z_i)}} = \frac{\lambda_i\overline{w_i} - 1}{1 - w_i\overline{\lambda_i}}.$$

If $|\lambda| \le 1/2$ and $|w| > 1/2$, we have

$$|t_i + 1| \le 4|\lambda_i| \le 4|\lambda|$$

for any $1 \le i \le n$. So

$$\begin{aligned}
\int_{D^n} |(\mathcal{U}u)^\alpha + (-1)^{|\alpha|+1}u^\alpha|^2 du &\le \int_{D^n} \left| \left[\prod_{i=1}^n [(t_i+1)u_i - u_i]^{\alpha_i}\right] + (-1)^{|\alpha|+1}u^\alpha \right|^2 du \\
&\le C|\lambda|^2.
\end{aligned}$$

Also for $|\lambda| \le 1/2$,

$$|\phi_{\lambda_i}(u_i) + u_i| \le 4|\lambda_i|$$

and we have

$$\begin{aligned}
\int_{D^n} |(-1)^{|\alpha|+1}u^\alpha + \phi_\lambda(u)^\alpha|^2 du &= \int_{D^n} \left| (-1)^{|\alpha|+1}u^\alpha + \prod_{i=1}^n (-u_i + O(|\lambda|))^{\alpha_i} \right|^2 du \\
&\le C|\lambda|^2.
\end{aligned}$$

Thus (2.6) is less than or equal to $C|\lambda|^2$. Consequently, (2.5) is less than or equal to

$$C(m,n)\|S_z\||\lambda| = C(m,n)\|S\|\rho(z,w).$$

The proof is complete. □

Lemma 2.9. *Let* $S \in \mathfrak{L}(L_a^2)$ *and* $m, j \ge 0$. *If* $|S^*K_\lambda(z)| \le C$ *for any* $z \in D^n$ *and* $\lambda \in D^n$ *then* $(B_mB_j)(S) = (B_jB_m)(S)$.

Proof. By Theorem 2.6, it is enough to show that $(B_mB_j)S(0) = (B_jB_m)S(0)$. From Propositions 2.3 and 2.1, we have

$$
\begin{aligned}
&B_m(B_jS)(0)\\
&= (m+1)^n \int_{D^n} B_jS(z)\left[\prod_{i=1}^{n}(1-|z_i|^2)^m\right]dz\\
&= (m+1)^n(j+1)^n\\
&\qquad\times\int_{D^n}\int_{D^n}\int_{D^n}\left[\prod_{i=1}^{n}(1-|z_i|^2)^{m+j+2}(1-u_i\overline{\lambda_i})^j\right]K_z^j(u)\overline{K_z^j(\lambda)}\overline{S^*K_\lambda(u)}dud\lambda dz.
\end{aligned}
$$

Let

$$
F_{m,j}(u,\lambda) = \left[\prod_{i=1}^{n}(1-u_i\overline{\lambda_i})^j\right]\int_{D^n}\left[\prod_{i=1}^{n}(1-|z_i|^2)^{m+j+2}\right]K_z^j(u)\overline{K_z^j(\lambda)}dz.
$$

Then $F_{m,j}(u,\lambda) = \sum_{i=1}^{l} H_i(u)\overline{G_i}(\lambda)$ where H_i and G_i are holomorphic functions and for some $l \geq 0$. Thus, we only need to show $F_{m,j}(\lambda,\lambda) = F_{j,m}(\lambda,\lambda)$ for $\lambda \in D^n$. The change of variables implies

$$
\begin{aligned}
F_{m,j}(\lambda,\lambda) &= \left[\prod_{i=1}^{n}(1-|\lambda_i|^2)^j\right]\int_{D^n}\left[\prod_{i=1}^{n}(1-|z_i|^2)^{m+j+2}\right]|K_\lambda^j(z)|^2dz\\
&= \left[\prod_{i=1}^{n}(1-|\lambda_i|^2)^j\right]\int_{D^n}\left[\prod_{i=1}^{n}(1-|\phi_{\lambda_i}(z_i)|^2)^{m+j+2}\right]|K_\lambda^j(\phi_\lambda(z))|^2|k_\lambda(z)|^2dz\\
&= \left[\prod_{i=1}^{n}(1-|\lambda_i|^2)^m\right]\int_{D^n}\left[\prod_{i=1}^{n}(1-|z_i|^2)^{m+j+2}\right]|K_\lambda^m(z)|^2dz\\
&= F_{j,m}(\lambda,\lambda)
\end{aligned}
$$

as desired. □

Lemma 2.10. *For any $S \in \mathfrak{L}(L_a^2)$, there exists a sequence $\{S_\alpha\}$ satisfying $|S_\alpha^*K_\lambda(u)| \leq C(\alpha)$ for any $u \in D^n$ and $\lambda \in D^n$ such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwise.*

Proof. Since H^∞ is dense in L_a^2 and the set of finite rank operators is dense in the ideal $\mathcal{K}$ of compact operators on L^2, the set $\{\sum_{i=1}^{l} f_i \otimes g_i : f_i,\ g_i \in H^\infty\}$ is dense in the ideal $\mathcal{K}$ in the norm topology. Since $\mathcal{K}$ is dense in the space of bounded operators on L_a^2 in strong operator topology, (2.4) gives that for any $S \in \mathfrak{L}(L_a^2)$, there exists a finite rank operator sequences $S_\alpha = \sum_{i=1}^{l} f_i \otimes g_i$ such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwise for some $f_i,\ g_i$ in H^∞. Also, for $l \geq 0$, for such $S_\alpha = \sum_{i=1}^l f_i \otimes g_i$, we have

$$\begin{aligned} |S_\alpha^* K_\lambda(u)| &= \left| \sum_{i=1}^{l} (g_i \otimes f_i) K_\lambda(u) \right| \\ &= \left| \sum_{i=1}^{l} \langle K_\lambda(u), f_i(u) \rangle \, g_i(u) \right| \\ &\leq \sum_{i=1}^{l} |f_i(\lambda)||g_i(u)| \\ &\leq \sum_{i=1}^{l} \|f_i\|_\infty \|g_i\|_\infty \leq C(\alpha). \end{aligned}$$

The proof is complete. □

Proposition 2.11. *Let $S \in \mathfrak{L}(L_a^2)$ and $m, j \geq 0$. Then $(B_m B_j)(S) = (B_j B_m)(S)$.*

Proof. Let $S \in \mathfrak{L}(L_a^2)$. Then Lemma 2.10 implies that there exists a sequence $\{S_\alpha\}$ satisfying $|S_\alpha^* K_\lambda(u)| \leq C(\alpha)$, hence $B_m(B_j S_\alpha)(z) = B_j(B_m S_\alpha)(z)$ by Lemma 2.9. From Proposition 2.3 and (1.1), we know

$$B_m(B_j S_\alpha)(z) = \int_{D^n} (B_j S_\alpha) \circ \phi_z(u) d\nu_m(u)$$

and $\|(B_j S_\alpha) \circ \phi_z\|_\infty \leq C(j, n)\|S\|$. Also, $(B_j S_\alpha) \circ \phi_z(u)$ converges to $(B_j S) \circ \phi_z(u)$. Therefore $B_m(B_j S_\alpha)(z)$ converges to $B_m(B_j S)(z)$. By the uniqueness of the limit, we have $(B_m B_j)(S) = (B_j B_m)(S)$. □

Proposition 2.12. *Let $S \in \mathfrak{L}(L_a^2)$ and $m \geq 0$. If $B_0 S(z) \to 0$ as $z \to \partial D^n$ then $B_m S(z) \to 0$ as $z \to \partial D^n$.*

Proof. We use the standard duality result [6] that

$$\mathcal{T}^* = \mathfrak{L}(L_a^2)$$

where $\mathfrak{L}(L_a^2)$ is the space of all bounded operators on the Bergman space $L_a^2(D^n)$. The duality pairing is

$$\langle Y, X \rangle = tr(YX).$$

Suppose $B_0 S(z) \to 0$ as $z \to \partial D^n$. Then we will prove that $S_z \to 0$ as $z \to \partial D^n$ in the $\mathcal{T}^*$-topology. Suppose it is not true. Since for $z \in D^n$,

$$\|S_z\| \leq \|S\|,$$

we see that $\{S_z : z \in D^n\}$ is a compact subset of $\mathfrak{L}(L_a^2)$ in the $\mathcal{T}^*$-topology. Then for some net $\{w_\alpha\} \in D^n$ and an operator $V \neq 0$ in $\mathfrak{L}(L_a^2)$, there exists a net $\{S_{w_\alpha}\}$ such that $S_{w_\alpha} \to V$ as $w_\alpha \to \partial D^n$ in the $\mathcal{T}^*$-topology, hence $tr[S_{w_\alpha} T] \to tr[VT]$ for any

$T \in \mathcal{T}$. Let $T = k_\lambda \otimes k_\lambda$ for fixed $\lambda \in D^n$. Then Theorem 2.6 implies

$$\begin{aligned} tr[S_{w_\alpha} T] &= tr[S_{w_\alpha}(k_\lambda \otimes k_\lambda)] \\ &= \langle S_{w_\alpha} k_\lambda, k_\lambda \rangle \\ &= B_0 S_{w_\alpha}(\lambda) \\ &= (B_0 S) \circ \phi_{w_\alpha}(\lambda) \to 0 \end{aligned}$$

as $w_\alpha \to \partial D^n$. Since $tr[VT] = B_0 V(\lambda)$ and B_0 is one-to-one mapping, $V = 0$. This is the contradiction. Thus $S_z \to 0$ as $z \to \partial D^n$ in the $\mathcal{T}^*$-topology. (1.1) finishes the proof of this proposition. □

3. Compact Radial Operators

In this section first we will give a criterion for operators approximated by Toeplitz operators with symbol equal to their m-Berezin transforms. Theorem 3.7 extends and improves Theorem 2.4 in [16] and will be used to characterize compact radial operators in the Toeplitz algebra. We will show an example that the result in the theorem is sharp on the polydisk by the end of this section.

From Proposition 1.4.10 in [13], we have the following lemma.

Lemma 3.1. *Suppose $a < 1$ and $a + b < 2$. Then*

$$\sup_{z \in D^n} \int_{D^n} \frac{d\lambda}{\prod_{i=1}^n (1 - |\lambda_i|^2)^a |1 - \lambda_i \overline{z_i}|^b} < \infty.$$

This lemma gives the following lemma.

Let $1 < q < \infty$ and p be the conjugate exponent of q. If we take $p > 3$, then $q < 3/2$.

Lemma 3.2. *Let $S \in \mathfrak{L}(L^2_a)$ and $p > 3$. Then there exists $C(n,p) > 0$ such that $h(z) = \prod_{i=1}^n (1 - |z_i|^2)^{-2/3}$ satisfies*

$$\int_{D^n} |(SK_z)(w)| h(w) dw \le C(n,p) \|S_z 1\|_p h(z) \tag{3.1}$$

for all $z \in D^n$ and

$$\int_{D^n} |(SK_z)(w)| h(z) dz \le C(n,p) \|S^*_w 1\|_p h(w) \tag{3.2}$$

for all $w \in D^n$.

Proof. Fix $z \in D^n$. Since

$$U_z 1 = \left[\prod_{i=1}^n (|z_i|^2 - 1) \right] K_z,$$

we have

$$SK_z = \left[\prod_{i=1}^n (|z_i|^2 - 1)^{-1} \right] SU_z 1 = \left[\prod_{i=1}^n (|z_i|^2 - 1)^{-1} \right] (S_z 1 \circ \phi_z) \prod_{i=1}^n \phi'_{z_i}.$$

Thus, letting $\lambda = \phi_z(w)$, the change of variables and (2.2) imply

$$\begin{aligned}\int_{D^n} \frac{|(SK_z)(w)|dw}{\prod_{i=1}^n (1-|w_i|^2)^{2/3}} &= \frac{1}{\prod_{i=1}^n (1-|z_i|^2)} \int_{D^n} \frac{|(S_z 1 \circ \phi_z)(w)||k_z(w)|}{\prod_{i=1}^n (1-|w_i|^2)^{2/3}} dw \\ &= \frac{1}{\prod_{i=1}^n (1-|z_i|^2)^{2/3}} \int_{D^n} \frac{|S_z 1(\lambda)|}{\prod_{i=1}^n (1-|\lambda_i|^2)^{2/3} |1-\lambda_i \overline{z_i}|^{2/3}} d\lambda \\ &\leq \frac{\|S_z 1\|_p}{\prod_{i=1}^n (1-|z_i|^2)^{2/3}} \left(\int_{D^n} \frac{d\lambda}{\prod_{i=1}^n (1-|\lambda_i|^2)^{2q/3} |1-\lambda_i \overline{z_i}|^{2q/3}} \right)^{\frac{1}{q}}.\end{aligned}$$

The last inequality comes from Holder's inequality. Since $2q/3 < 1$, Lemma 3.1 implies (3.1).

To prove (3.2), replace S by S^* in (3.1), interchange w and z in (3.1) and then use the equation

$$(S^* K_w)(z) = \langle S^* K_w, K_z \rangle = \langle K_w, SK_z \rangle = \overline{SK_z}(w) \tag{3.3}$$

to obtain the desired result. □

Lemma 3.3. *Let $S \in \mathfrak{L}(L_a^2)$ and $p > 3$. Then*

$$\|S\| \leq C(n,p) \left(\sup_{z \in D^n} \|S_z 1\|_p \right)^{1/2} \left(\sup_{z \in D^n} \|S_z^* 1\|_p \right)^{1/2}$$

where $C(n,p)$ is the constant of Lemma 3.2.

Proof. (3.3) implies

$$\begin{aligned}(Sf)(w) &= \langle Sf, K_w \rangle \\ &= \int_{D^n} f(z) \overline{(S^* K_w)}(z) dz \\ &= \int_{D^n} f(z) (SK_z)(w) dz\end{aligned}$$

for $f \in L_a^2$ and $w \in D^n$. Thus, Lemma 3.2 and the classical Schur's theorem finish the proof. □

Lemma 3.4. *Let S_m be a bounded sequence in $\mathfrak{L}(L_a^2)$ such that $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$. Then*

$$\sup_{z \in D^n} |\langle (S_m)_z 1, f \rangle| \to 0 \tag{3.4}$$

as $m \to \infty$ for any $f \in L_a^2$ and

$$\sup_{z \in D^n} |(S_m)_z 1| \to 0 \tag{3.5}$$

uniformly on compact subsets of D^n as $m \to \infty$.

Proof. To prove (3.4), we only need to have

$$\sup_{z \in D^n} \left| \left\langle (S_m)_z 1, w^k \right\rangle \right| \to 0 \tag{3.6}$$

as $m \to \infty$ for any multi-index k.

Since

$$K_z(w) = \sum_{|\alpha|=0}^{\infty} \left[\prod_{i=1}^{n} (\alpha_i + 1) \right] \overline{z}^{\alpha} w^{\alpha}, \tag{3.7}$$

we have

$$\begin{aligned} B_0 S_m(\phi_z(\lambda)) &= B_0 (S_m)_z(\lambda) \\ &= \left[\prod_{i=1}^{n} (1 - |\lambda_i|^2)^2 \right] \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \left[\prod_{i=1}^{n} (\alpha_i + 1)(\beta_i + 1) \right] \left\langle (S_m)_z w^{\alpha}, w^{\beta} \right\rangle \overline{\lambda}^{\alpha} \lambda^{\beta} \end{aligned}$$

where α, β are multi-indices.

Then for any fixed k and $0 < r < 1$,

$$\begin{aligned} &\int_{rD^n} \frac{B_0 S_m(\phi_z(\lambda)) \overline{\lambda}^k}{\prod_{i=1}^{n} (1 - |\lambda_i|^2)^2} d\lambda \\ &= \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \left[\prod_{i=1}^{n} (\alpha_i + 1)(\beta_i + 1) \right] \left\langle (S_m)_z w^{\alpha}, w^{\beta} \right\rangle \int_{rD^n} \overline{\lambda}^{\alpha+k} \lambda^{\beta} d\lambda \\ &= r^{2n+2|k|} \left(\left\langle (S_m)_z 1, w^k \right\rangle + \sum_{|\alpha|=1}^{\infty} \left[\prod_{i=1}^{n} (\alpha_i + 1) \right] \left\langle (S_m)_z w^{\alpha}, w^{\alpha+k} \right\rangle r^{2|\alpha|} \right). \end{aligned}$$

Since S_m is bounded sequence, we have

$$\begin{aligned} &\left| \left\langle (S_m)_z 1, w^k \right\rangle \right| \\ &\leq r^{-2n-2|k|} \left| \int_{rD^n} \frac{B_0 S_m(\phi_z(\lambda)) \overline{\lambda}^k}{\prod_{i=1}^{n} (1 - |\lambda_i|^2)^2} d\lambda \right| + \sum_{|\alpha|=1}^{\infty} \left[\prod_{i=1}^{n} (\alpha_i + 1) \right] \|S_m\| \|w^{\alpha}\| \|w^{\alpha+k}\| r^{2|\alpha|} \\ &\leq r^{-2n-2|k|} \|B_0 S_m\|_{\infty} \int_{rD^n} \frac{|\lambda^k|}{\prod_{i=1}^{n} (1 - |\lambda_i|^2)^2} d\lambda + C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|}, \end{aligned}$$

hence, by assumption

$$\limsup_{m \to \infty} \sup_{z \in D^n} \left| \left\langle (S_m)_z 1, w^k \right\rangle \right| \leq C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|}.$$

Letting $r \to 0$, we have (3.6).

Now we prove (3.5). From (3.7), we get

$$\begin{aligned}|(S_m)_z 1(\lambda)| &= |\,\langle (S_m)_z 1, K_\lambda\rangle\,| \\ &\le \sum_{|\alpha|=0}^{\infty}\left[\prod_{i=1}^{n}(\alpha_i+1)\right]|\langle (S_m)_z 1, w^\alpha\rangle|\,|\lambda^\alpha| \\ &\le \sum_{|\alpha|=0}^{l-1}\left[\prod_{i=1}^{n}(\alpha_i+1)\right]|\langle (S_m)_z 1, w^\alpha\rangle| + \sum_{|\alpha|=l}^{\infty}\left[\prod_{i=1}^{n}(\alpha_i+1)\right]\|S_m\|\|w^\alpha\||\lambda^\alpha|\end{aligned}$$

for $z \in D^n$, $\lambda \in rD^n$ and $l \ge 1$. Since the second summation is less than or equals to

$$\sum_{j=l}^{\infty}(j+1)^{n/2}\sum_{|\alpha|=j}\left[\prod_{i=1}^{n}\left(\frac{\alpha_i+1}{j+1}\right)^{1/2}\right]|\lambda^\alpha| \le \sum_{j=l}^{\infty}\frac{(j+1)^{n/2}(n+j)!}{n!j!}r^j,$$

for any $\epsilon > 0$, we can find sufficiently large l such that the second summation is less than ϵ. Thus, (3.6) imply $\sup_{z\in D^n}|(S_m)_z 1| \to 0$ uniformly on compact subsets of D^n as $m \to \infty$. □

Lemma 3.5. *Let $\{S_m\}$ be a sequence in $\mathfrak{L}(L^2_a)$ such that for some $p > 3$, $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$,*

$$\sup_{z\in D^n}\|(S_m)_z 1\|_p \le C \qquad \text{and} \qquad \sup_{z\in D^n}\|(S^*_m)_z 1\|_p \le C$$

where $C > 0$ is independent of m, then $S_m \to 0$ as $m \to \infty$ in $\mathfrak{L}(L^2_a)$-norm.

Proof. Lemma 3.3 implies

$$\|S_m\| \le C(n,p)\left(\sup_{z\in D^n}\|(S_m)_z 1\|_p\right)^{1/2}\left(\sup_{z\in D^n}\|(S^*_m)_z 1\|_p\right)^{1/2} \le C(n,p),$$

hence, Lemma 3.4 gives

$$\sup_{z\in D^n}|(S_m)_z 1| \to 0 \tag{3.8}$$

uniformly on compact subsets of D^n as $m \to \infty$.

Here, for $3 < s < p$, Holder's inequality gives

$$\begin{aligned}\sup_{z\in D^n}\|(S_m)_z 1\|_s^s &\le \sup_{z\in D^n}\int_{D^n\setminus r\overline{D^n}}|(S_m)_z 1(w)|^s dw + \sup_{z\in D^n}\int_{r\overline{D^n}}|(S_m)_z 1(w)|^s dw \\ &\le C\sup_{z\in D^n}\|(S_m)_z 1\|_p^s(1-r)^{(1-s/p)} + \sup_{z\in D^n}\int_{r\overline{D^n}}|(S_m)_z 1(w)|^s dw\end{aligned}$$

and (3.8) implies the second term tends to 0 as $m \to \infty$. Also, the first term is less than or equals to $C^s(1-r)^{(1-s/p)}$ which converges to 0 as r goes to 1. Consequently, Lemma

3.3 gives

$$\begin{aligned}\|S_m\| &\le C(n,s)\left(\sup_{z\in D^n}\|(S_m)_z 1\|_s\right)^{1/2}\left(\sup_{z\in D^n}\|(S_m^*)_z 1\|_s\right)^{1/2}\\ &\le C(n,s)\left(\sup_{z\in D^n}\|(S_m)_z 1\|_s\right)^{1/2}\to 0\end{aligned}$$

where the last inequality holds by $\|\|_s \le \|\|_p$. $\square$

Corollary 3.6. *Let $S \in \mathfrak{L}(L_a^2)$ such that for some $p > 3$,*

$$\sup_{z\in D^n}\|S_z 1 - (T_{B_m S})_z 1\|_p \le C \qquad \text{and} \qquad \sup_{z\in D^n}\|S_z^* 1 - (T_{B_m(S^*)})_z 1\|_p \le C, \tag{3.9}$$

where $C > 0$ is independent of m. Then $T_{B_m S} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. Let $S_m = S - T_{B_m S}$. Then Proposition 2.11 and Theorem 2.8 imply

$$B_0(S_m) = B_0 S - B_0(T_{B_m S}) = B_0 S - B_0(B_m S) = B_0 S - B_m(B_0 S)$$

which tends uniformly to 0 as $m \to \infty$, hence $\|B_0(S_m)\|_\infty \to 0$. Consequently, Lemma 3.5 finishes the proof. $\square$

Theorem 3.7. *Let $S \in \mathfrak{L}(L_a^2)$. If there is $p > 3$ such that*

$$\sup_{z\in D^n}\|T_{(B_m S)\circ\phi_z} 1\|_p < C \quad \textit{and} \quad \sup_{z\in D^n}\|T^*_{(B_m S)\circ\phi_z} 1\|_p < C \tag{3.10}$$

where $C > 0$ is independent of m, then $T_{B_m S} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. By Corollary 3.6, we only need to show that (3.10) implies (3.9). Since

$$T_{(B_m S)\circ\phi_z} = (T_{B_m S})_z$$

and

$$T^*_{(B_m S)\circ\phi_z} = T_{\overline{B_m S_z}} = T_{B_m(S_z^*)} = T_{(B_m(S^*))\circ\phi_z},$$

it is sufficient to show that

$$\sup_{z\in D^n}\|S_z 1\|_p < \infty.$$

By Lemma 3.3, we get

$$\|T_{B_m S}\| \le C(n,p)\left(\sup_{z\in D^n}\|T_{B_m S\circ\phi_z} 1\|_p\right)^{1/2}\left(\sup_{z\in D^n}\|T^*_{B_m S\circ\phi_z} 1\|_p\right)^{1/2} < C$$

where C is independent of m, hence writing $S_m = S - T_{B_m S}$, we have $\|S_m\| \le C$ where C is independent of m. Also, the proof of Corollary 3.6 implies $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$.

Let f be a polynomial with $\|f\|_q = 1$. Then Lemma 3.4 implies

$$\sup_{z\in D^n}|\langle (S_m)_z 1, f\rangle| \to 0$$

as $m \to \infty$. Thus, for any $\epsilon > 0$ and $z_0 \in D^n$, we have

$$|\langle S_{z_0} 1, f\rangle| \le \sup_{z\in D^n}|\langle (S_m)_z 1, f\rangle| + |\langle (T_{B_m S})_{z_0} 1, f\rangle| \le \epsilon + C$$

for sufficiently large m, where C is independent of m. Since ϵ is arbitrary, we get

$$\sup_{z\in D^n} \|S_z 1\|_p < \infty$$

as desired. □

A radial operator S on L_a^2 is a radial operator if it is diagonal with respect to the orthonormal base $\{\prod_{i=1}^n \sqrt{\alpha_i+1} z^\alpha : \alpha \in N^n\}$. Define $Uf(w) = \left(\prod_{j=1}^n e^{i\theta_j}\right) f(\mathcal{U}w)$ for $f \in L_a^2$ where $\mathcal{U}w = (e^{i\theta_1}w_1, \ldots, e^{i\theta_n}w_n)$. Then U is a unitary operator on L_a^2. Clearly, for $S \in \mathfrak{L}(L_a^2)$, S is a radial operator iff $SU = US$ for any U.

If $S \in \mathfrak{L}(L_a^2)$, the radialization of S is defined by

$$S^\sharp = \int_{T^n} U^* S U d\theta$$

where the integral is taken in the weak sense. Then $S^\sharp = S$ if S is radial and $\mathcal{U}$-invariance of $d\theta$ shows that $S^\sharp$ is indeed a radial operator.

If $f \in L^\infty$ and $g, h \in L_a^2$ then

$$\begin{aligned}\langle U^* T_f U g, h\rangle &= \int_{D^n} f(w) Ug(w) \overline{Uh(w)} dw \\ &= \int_{D^n} f(\mathcal{U}^* w) g(w) \overline{h(w)} dw.\end{aligned}$$

Thus $U^* T_f U = T_{f\circ\mathcal{U}^*}$ and

$$U^* T_{f_1} \cdots T_{f_l} U = T_{f_1\circ\mathcal{U}^*} \cdots T_{f_l\circ\mathcal{U}^*}$$

for $f_1, \ldots, f_l \in L^\infty$, $l \geq 0$.

Lemma 3.8. *Let $S \in \mathfrak{L}(L_a^2)$ be a radial operator. Then*

$$T_{B_m(S)} = \int_{D^n} S_w d\nu_m(w).$$

Proof. Let $z \in D^n$. By (1.1) and Lemma 2.5, we obtain

$$\begin{aligned}B_0\left(\int_{D^n} S_w d\nu_m(w)\right)(z) &= \left\langle \left(\int_{D^n} S_w d\nu_m(w)\right)_z 1, 1\right\rangle \\ &= \int_{D^n} \langle U_z U_w S U_w U_z 1, 1\rangle \, d\nu_m(w) \\ &= \int_{D^n} \left\langle U_{\phi_z(w)} V_t^* S V_t U_{\phi_z(w)} 1, 1\right\rangle d\nu_m(w)\end{aligned}$$

where V_t is in Lemma 2.5. Since S is a radial operator, Theorem 2.6, Proposition 2.3 and Proposition 2.11 imply that the last integral equals

$$\begin{aligned}\int_{D^n} \left\langle U_{\phi_z(w)} S U_{\phi_z(w)} 1, 1\right\rangle d\nu_m(w) &= \int_{D^n} B_0 S \circ \phi_z(w) d\nu_m(w) \\ &= B_m B_0 S(z) \\ &= B_0 B_m S(z) \\ &= B_0(T_{B_m(S)})(z).\end{aligned}$$

Since B_0 is one-to-one mapping, the proof is complete. □

Theorem 3.9. *Let $S \in \mathfrak{T}(L^\infty)$ be a radial opeartor. Then S is compact if and only if $B_0 S \equiv 0$ on ∂D^n.*

Proof. Suppose $B_0 S \equiv 0$ on ∂D^n. Then $B_m S \equiv 0$ on ∂D^n by Proposition 2.12, hence $T_{B_m S}$ is compact for all $m \geq 0$.

Let

$$Q = \int_{T^n} T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*} d\theta$$

with $f_1, \ldots, f_l \in L^\infty$ for some $l \geq 0$. Then $Q \in \mathfrak{L}(L_a^2)$. By Lemma 3.8, for any $z \in D^n$, we have

$$\begin{aligned}T_{(B_m(Q))\circ\phi_z} &= \int_{D^n} ((Q)_z)_w d\nu_m(w) \\ &= \int_{D^n}\int_{T^n} T_{f_1\circ\mathcal{U}^*\circ\phi_z\circ\phi_w} \cdots T_{f_l\circ\mathcal{U}^*\circ\phi_z\circ\phi_w} d\theta d\nu_m(w).\end{aligned}$$

Consequently,

$$\begin{aligned}\|T_{(B_m(Q))\circ\phi_z}\| &\leq C(l)\|f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \cdots \|f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \\ &= C(l)\|f_1\|_\infty \cdots \|f_l\|_\infty.\end{aligned}$$

Similarly, we have

$$\|T^*_{(B_m(Q))\circ\phi_z}\| \leq C(l)\|f_1\|_\infty \cdots \|f_l\|_\infty.$$

Thus, Theorem 3.7 gives that

$$T_{B_m(Q)} \to Q \tag{3.11}$$

in $\mathfrak{L}(L_a^2)$-norm.

Since $S \in \mathfrak{T}(L^\infty)$, there exists a sequence $\{S_k\}$ such that $S_k \to S$ in $\mathfrak{L}(L_a^2)$-norm where each S_k is a finite sum of finite products of Toeplitz operators. Since the radialization is continuous and S is radial, $S_k^\sharp \to S^\sharp = S$. From Lemma 3.8, we have

$$\|T_{B_m S}\| = \left\|\int_{D^n} S_w d\nu_m(w)\right\| \leq \int_{D^n} \|S_w\| d\nu_m(w) = \|S\|.$$

Thus

$$\begin{aligned}\|S - T_{B_m S}\| &\le \|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| + \|T_{B_m(S_k^\sharp)} - T_{B_m S}\| \\ &\le 2\|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\|\end{aligned}$$

and (3.11) imply $T_{B_m(S)} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm, hence S is compact.

The other direction is trivial. □

Example. This example shows that the number 3 in Theorem 3.7 is sharp. We show that there is a bounded operator S on L_a^2 such that

$$\sup_{z \in D^n} \max\{\|T_{(B_m S)\circ\phi_z} 1\|_3, \|T^*_{(B_m S)\circ\phi_z} 1\|_3\} < \infty,$$

and for each $m \ge 0$, $B_m(S)(z) \to 0$ as $z \to \partial D^n$, but S is not compact on L_a^2.

Let S be defined on L_a^2 by

$$S\left(\sum_{|\alpha|=0}^{\infty} a_\alpha w^\alpha\right) = \sum_{l=0}^{\infty} a_{(2^l,0,\cdots,0)} w_1^{2^l}.$$

It is clear that S is a self-adjoint projection with infinite-dimensional range. Thus S is not compact on L_a^2. Since

$$SK_z(w) = S\left(\sum_{|\alpha|=0}^{\infty}\left(\prod_{i=1}^{n}(\alpha_i+1)\right)\bar{z}^\alpha w^\alpha\right) = \sum_{l=0}^{\infty}(2^l+1)\bar{z}_1^{2^l} w_1^{2^l},$$

we have

$$\begin{aligned}B_0(S)(z) &= \langle Sk_z, k_z\rangle \\ &= \left(\prod_{i=1}^{n}(1-|z_i|^2)^2\right)\sum_{l=0}^{\infty}(2^l+1)(|z_1|^2)^{2^l}.\end{aligned}$$

It is easy to see that $B_0(S)(z) \to 0$ as $z \to \partial D^n$. By Proposition 2.12, we see that $B_m(S)(z) \to 0$ as $z \to \partial D^n$. This gives that $T_{B_m(S)}$ is compact. Hence $T_{B_m(S)}$ does not converge to S in the norm topology.

Now we show

$$\sup_{z\in D^n} \|S_z 1\|_3 < \infty.$$

For $z \in D^n$, we know

$$\begin{aligned}(U_z 1)(w) &= \prod_{i=1}^{n}(|z_i|^2 - 1)\frac{1}{(1-\bar{z}_i w_i)^2} \\ &= \left(\prod_{i=1}^{n}(|z_i|^2-1)\right)\sum_{|\alpha|=0}^{\infty}\left(\prod_{i=1}^{n}(\alpha_i+1)\right)\bar{z}^\alpha w^\alpha.\end{aligned}$$

Thus we get

$$(SU_z1)(w) = \left(\prod_{i=1}^{n}(|z_i|^2 - 1)\right)\sum_{l=0}^{\infty}(2^l + 1)\bar{z}_1^{2^l} w_1^{2^l},$$

hence

$$(S_z1)(w) = (U_zSU_z1)(w) = \left(\prod_{i=1}^{n}\frac{(1 - |z_i|^2)^2}{(1 - \overline{z_i}w_i)^2}\right)\sum_{l=0}^{\infty}(2^l + 1)\bar{z}_1^{2^l}(\phi_{z_1}(w_1))^{2^l}.$$

By change of variables $w = \phi_z(\lambda)$, we obtain

$$\begin{aligned}\|S_z1\|_3^3 &= \left(\prod_{i=1}^{n}(1 - |z_i|^2)^2\right)\int_{D^n}\left(\prod_{i=1}^{n}|1 - \bar{z}_i\lambda_i|^2\right)\left|\sum_{l=0}^{\infty}(2^l + 1)(\bar{z}_1\lambda_1)^{2^l}\right|^3 d\lambda \\ &\le 4^n\left(\prod_{i=1}^{n}(1 - |z_i|^2)^2\right)\int_{D}\left|\sum_{l=0}^{\infty}(2^l + 1)(\bar{z}_1\lambda_1)^{2^l}\right|^3 d\lambda_1 < C\end{aligned}$$

where the last inequality holds by means of the Zygmund theorem on gap series [17], it was proved in [11]. Since $S_z^* = S_z$, we have

$$C = \sup_{z\in D^n}\max\{\|S_z1\|_3, \|S_z^*1\|_3\} < \infty.$$

Clearly, S is a radial operator. By Lemma 3.8, we have

$$\begin{aligned}T_{(B_mS)\circ\phi_z}1 &= \int_{D^n}(S_w)_z 1 d\nu_m(w) \\ &= \int_{D^n}S_{\phi_z(w)}1 d\nu_m(w) \\ &= \int_{D^n}S_\lambda 1 d\nu_m\circ\phi_z(\lambda).\end{aligned}$$

Noting that for each $z \in D^n$, $d\nu_m \circ \phi_z$ is a probability measure on D^n, we have

$$\|T_{(B_mS)\circ\phi_z}1\|_3 \le \int_{D^n}\|S_\lambda 1\|_3 d\nu_m\circ\phi_z(\lambda) \le C.$$

Similarly, we also have

$$\|T^*_{(B_mS)\circ\phi_z}1\|_3 \le C.$$

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