HYPERBOLIC DERIVATIVES
AND GENERALIZED SCHWARZ-PICK ESTIMATES

PRATIBHA GHATAGE AND DECHAO ZHENG

Abstract. In this paper we use the beautiful formula of Faa di Bruno for the \( n \)th derivative of composition of two functions to obtain the generalized Schwarz-Pick estimates. By means of those estimates we show that the hyperbolic derivative of an analytic self-map of the unit disk is Lipschitz with respect to the pseudohyperbolic metric.

1. Introduction

For each \( z \in D \), let \( \varphi_z \) denote the Möbius transformation of \( D \),

\[
\varphi_z = \frac{z - w}{1 - zw},
\]

for \( w \in D \). The pseudo-hyperbolic distance on \( D \) is defined by

\[
\rho(z, w) = |\varphi_z(w)|, \quad z, w \in D.
\]

The pseudohyperbolic distance is Möbius invariant, that is,

\[
\rho(gz, gw) = \rho(z, w),
\]

for all \( g \in \text{Aut}(D) \), the Möbius group of \( D \), and all \( z, w \in D \). It has the following useful property:

\[
1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - zw|^2}.
\]

The Bergman metric on \( D \) is the hyperbolic metric whose element of length is

\[
ds = \frac{|dz|}{1 - |z|^2}.
\]

In this metric every rectifiable arc \( \gamma \) has a length

\[
\int_{\gamma} \frac{|dz|}{1 - |z|^2}.
\]

It is easy to show that the induced distance on \( D \) is given by

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},
\]

for \( z, w \in D \).
Let \( \varphi \) be an analytic self-map of the unit disk. By the classical Schwarz-Pick Lemma \([2, 3]\), \( \varphi \) decreases the hyperbolic distance between two points and the hyperbolic length of an arc. The explicit inequality is

\[
\left| \frac{\varphi(z_1) - \varphi(z_2)}{1 - \varphi(z_1)\varphi(z_2)} \right| \leq \frac{|z_1 - z_2|}{|1 - z_1z_2|}
\]

for any \( z_1, z_2 \) in \( D \). In particular,

\[
(1.2) \quad \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2} \leq \frac{1}{1 - |z|^2}
\]

for \( z \) in \( D \). Let

\[
\tau_\varphi(z) = \frac{(1 - |z|^2)\varphi'(z)}{1 - |\varphi(z)|^2}.
\]

Then

\[
|\tau_\varphi(z)| \leq 1,
\]

for all \( z \in D \). Nontrivial equality \( |\tau_\varphi(z)| = 1 \) holds for some \( z \in D \) only when \( \varphi \) is a fractional linear transformation \( e^{i\theta} \varphi_\alpha(z) \). For each \( z \in D \), the hyperbolic derivative of \( \varphi \) at \( z \) is defined by

\[
\lim_{\beta(z, w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z, w)}.
\]

In Section \([8]\) we will show that the hyperbolic derivative of \( \varphi \) equals \( |\tau_\varphi(z)| \) and that \( \tau_\varphi(z) \) is Lipschitz with respect to the pseudohyperbolic metric. Hyperbolic derivatives have been used in studying composition operators on the Bloch space \([7, 9]\) and \([10]\).

Recently, MacCluer, Stroethoff, and Zhao \([8]\) used the formula of Faa di Bruno and the theory of the weighted composition operators \([11]\) to obtain the generalized Schwarz-Pick estimates:

\[
(1.3) \quad \sup_{z \in D} \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)} < \infty
\]

for each analytic self-map \( \varphi \) of the unit disk. We obtain the following generalized Schwarz-Pick estimates: for each \( 0 < r < 1 \) and each positive integer \( n \), there is a positive constant \( M_{n,r} \) such that for each analytic self-map \( \varphi \) of the unit disk:

\[
(1.4) \quad \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_{n,r} \max_{\rho(w,z) < r} \frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2},
\]

for \( z \) in \( D \). Clearly, Combining \((1.2)\) with \((1.4)\) gives \((1.3)\). Moreover, \((1.4)\) directly leads to the result \([8]\) that if \( \varphi \) is in the little Bloch class, then

\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0
\]

for each \( n \). The main tool is the beautiful formula of Faa di Bruno \([13]\) for the \( n \)th derivative of the composition of two functions.

Based on the generalized Schwarz-Pick estimates we will show in Section \([8]\) that \( \tau_{\varphi,n}(z) = \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)} \) is Lipschitz with respect to the pseudohyperbolic metric. Thus \( \tau_{\varphi,n}(z) \) admits a continuous extension to the set of nontrivial Gleason parts of the maximal ideal space of \( H^\infty \).
2. Generalized Schwarz-Pick estimates

In this section, we will present a proof of the generalized Schwarz-Pick estimates. The generalized Schwarz-Pick estimates will be used in the proof of Theorem 6. The main tool is the beautiful formula of Faa di Bruno, which deals with the $n$th derivative of composition of an analytic function $f$ on the unit disk with a self-map $\varphi$ of the the unit disk [13].

**Theorem 1** (The Formula of Faa di Bruno). If $\varphi$ is an analytic function from the unit disk to the unit disk and if $f$ is an analytic function on the unit disk, then

$$(f \circ \varphi)^{(n)}(z) = \sum_{k_1!k_2!\cdots k_n!} k_1!k_2!\cdots k_n! f^{(k)}(\varphi(z)) \prod_{j=1}^{n} (\varphi^{(j)}(z))^{k_j}$$

where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.

The following result is well known [12]. We include a proof to motivate our Theorem 2.

**Proposition 1.** If $\varphi$ is a univalent analytic self-map of $D$, then

$$(1 - |z|^2)|\varphi''(z)| \leq 10|\varphi'(z)|$$

for all $z \in D$.

**Proof.** For a fixed $z$ in $D$, let $h$ be the Koebe transform of $\varphi$,

$$h(w) = \frac{\varphi\left(\frac{w}{1-\varphi(z)}\right) - \varphi(z)}{(1 - |z|^2)\varphi'(z)}$$

Then $h(0) = 0$ and $h'(0) = 1$. It follows from Bieberbach’s theorem ([12], page 8) that

$$|h''(0)| \leq 4.$$ 

On the other hand, an easy computation gives

$$h''(0) = \frac{1}{2} (1 - |z|^2) \frac{\varphi''(z)}{\varphi'(z)} - \overline{z}.$$ 

Hence

$$\frac{1}{2} (1 - |z|^2) \frac{\varphi''(z)}{\varphi'(z)} - \overline{z} \leq 4.$$ 

Since $|z| \leq 1$, we conclude that

$$|1 - |z|^2)\varphi''(z)| \leq 10|\varphi'(z)|.$$ 

This completes the proof. \qed

As a consequence of the proposition, we have

$$\frac{(1 - |z|^2)^2|\varphi''(z)|}{1 - |\varphi(z)|^2} \leq \frac{10(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}$$

for all $z \in D$ if $\varphi$ is a univalent self-map of the unit disk.
Example. Let $b$ be an interpolating Blaschke product with zeros $\{z_n\}$ in the unit disk and $\varphi = b^2$. Clearly, $\varphi'(z_n) = 0$ and $\varphi''(z_n) = 2|b'(z_n)|^2$. Let $\delta = \inf_{z_n}(1 - |z_n|^2)b'(z_n)]$. Thus
\[
\frac{(1 - |z_n|^2)|\varphi''(z_n)|}{1 - |\varphi(z_n)|^2} = 2(1 - |z_n|^2)|b'(z_n)|^2 \geq 2\delta |b'(z_n)| \geq \frac{2\delta^2}{1 - |z_n|^2}.
\]
So the inequality (2.1) does not hold for some analytic self-maps of the unit disk. But by means of the formula of Faa di Bruno we still have the generalized Schwarz-Pick estimates:

**Theorem 2.** For each positive integer $n$ and each number $0 < r < 1$, there is a positive constant $M_{n,r}$ such that for each analytic self-map $\varphi$ of the unit disk,
\[
\frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_{n,r} \max_{\rho(w,z)<r} \frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2}
\]
for $z$ in $D$.

As we mentioned in the introduction, by the Schwarz-Pick estimates (1.2), we have
\[
\frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2} \leq 1.
\]
Thus Theorem 2 implies the following generalized Schwarz-Pick Estimates:

**Theorem 3.** For each $n$, there is a positive constant $M_n$ such that for each analytic self-map $\varphi$ of the unit disk,
\[
\frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_n,
\]
for $z$ in $D$.

If $\varphi$ is in the little Bloch class, i.e.,
\[
\frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} \to 0
\]
as $|z| \to 1$, then noting that for the given $0 < s < 1$, for every $w \in D$ with $\rho(w,z) < s$, $|w| \to 1$ as $|z| \to 1$, Theorem 2 gives the following result in $\mathbb{N}$.

**Theorem 4.** Let $\varphi$ be an analytic self-map of the unit disk. If
\[
\lim_{|z|\to 1^-} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0,
\]
then for each $n$,
\[
\lim_{|z|\to 1^-} \frac{(1 - |z|^2)^n|\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0.
\]

**Proof of Theorem 2.** For a fixed $z$ in $D$, let $g = \varphi \circ \varphi_z$. Clearly, $g(0) = \varphi(z)$. By the formula of Faa di Bruno, we have
\[
g^{(n)}(w) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} \varphi^{(k)}(\varphi_z(w)) \prod_{j=1}^{n} \left( \frac{\varphi^{(j)}(w)}{j!} \right)^{k_j}
\]
where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$. 
Evaluating the value of $g^{(n)}(0)$ at 0 gives

$$g^{(n)}(0) = \sum_{k_1!k_2!\cdots k_n!} \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}(\varphi_z(0)) \prod_{j=1}^{n} \left( \frac{\varphi_z^{(j)}(0)}{j!} \right)^{k_j}.$$  

Noting that $\varphi_z(0) = z$ and $\varphi_z^{(j)}(w) = -((1 - |z|^2)\bar{z}^{j-1}j!(1 - \bar{z}w)^{-j-1}$, we have

$$g^{(n)}(0) = \sum_{k_1!k_2!\cdots k_n!} \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}(z) \prod_{j=1}^{n} ((1 - |z|^2)\bar{z}^{j-1}j!)^{k_j} = \sum_{k_1!k_2!\cdots k_n!} (-1)^{k} \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}(z)(1 - |z|^2)^{k}z^{n-k}.$$  

The last equality follows from $k_1 + \cdots + k_n = k$ and $k_1 + 2k_2 + \cdots + nk_n = n$. Thus

$$(-1)^n (1 - |z|^2)^n \varphi^{(n)}(z) = g^{(n)}(0) - \sum_{k < n} (-1)^{k} \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}(z)(1 - |z|^2)^{k}z^{n-k}.$$  

So

$$\frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} + \sum_{k < n} \frac{n!}{k_1!k_2!\cdots k_n!} |\varphi^{(k)}(z)| (1 - |z|^2)^k |z|^{n-k}.$$

Let $M_k(z) = \frac{|\varphi^{(k)}(z)| (1 - |z|^2)^k}{1 - |\varphi(z)|^2}$. The above inequality gives

$$M_n(z) \leq \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} + \sum_{k < n} \frac{n!}{k_1!k_2!\cdots k_n!} M_k(z).$$  

We need to estimate $|g^{(n)}(0)|$.  

Let $\lambda = g(0),$ $h = \varphi_\lambda \circ g$. Then $h$ is still an analytic self-map of the unit disk, $h(0) = 0$, and $\|h\|_{\infty} \leq 1$. Since $\varphi_\lambda \circ \varphi_\lambda(z) = z$, we obtain $g = \varphi_\lambda \circ h$. The formula of Faa di Bruno again gives

$$g^{(n)}(w) = \sum_{k_1!k_2!\cdots k_n!} \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}_\lambda(h(w)) \prod_{j=1}^{n} \left( \frac{h^{(j)}(w)}{j!} \right)^{k_j}$$

where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$.  

Evaluating $g^{(n)}$ at 0 gives

$$g^{(n)}(0) = \sum_{k_1!k_2!\cdots k_n!} \frac{n!}{k_1!k_2!\cdots k_n!} \varphi^{(k)}_\lambda(0) \prod_{j=1}^{n} \left( \frac{h^{(j)}(0)}{j!} \right)^{k_j}$$

since $h(0) = 0$. Noting $\varphi^{(k)}_\lambda(w) = -(1 - |\lambda|^2)\bar{\lambda}^{k-1}k!(1 - \bar{\lambda}w)^{-k-1}$, the above equality leads to

$$g^{(n)}(0) = \sum_{k_1!k_2!\cdots k_n!} \frac{n!}{k_1!k_2!\cdots k_n!} \left[ -(1 - |\lambda|^2)\bar{\lambda}^{k-1}k! \right] \prod_{j=1}^{n} \left( \frac{h^{(j)}(0)}{j!} \right)^{k_j}.$$.  

Hence
\[ \frac{|g^{(n)}(0)|}{1 - |g(0)|^2} \leq \sum \frac{n!}{k_1!k_2!\cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^{n} \left( \frac{|h^{(j)}(0)|}{j!} \right)^{k_j}. \]

Let \( a_n = \sum_{k<n} \frac{n!}{k_1!k_2!\cdots k_n!} \). So far we have shown
\[ M_n(z) \leq a_n \max_{k<n} M_k(z) + \sum \frac{n!}{k_1!k_2!\cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^{n} \left( \frac{|h^{(j)}(0)|}{j!} \right)^{k_j}. \]

Note that \( h = \varphi \circ g, g = \varphi \circ \varphi_z, \) and \( \lambda = g(0) = \varphi(z) \). Then
\[ h'(w) = \frac{(1 - |\lambda|^2)g'(w)}{(1 - \lambda g(w))^2} \]
and
\[ h'(w) = \sum_{j=1}^{\infty} \frac{h^{(j)}(0)}{(j-1)!} w^{j-1}. \]

Let \( 0 < r < 1 \). Thus
\[ h^{(j)}(0) = r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} h'(re^{i\theta}) e^{-i(j-1)\theta} d\theta. \]

So
\[ |h^{(j)}(0)| \leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} |h'(re^{i\theta})| d\theta \]
\[ \leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 - |\lambda|^2)|g'(re^{i\theta})|}{|1 - \lambda g(re^{i\theta})|^2} d\theta \]
\[ \leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 - |\varphi \lambda g(re^{i\theta})|^2)|g'(re^{i\theta})|}{1 - |g(re^{i\theta})|^2} d\theta \]
\[ \leq r^{-(j-1)}(1 - r^2)^{-1} \frac{1}{(j-1)!} \max_{|u| \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2} \]
for some constant \( C_r > 0 \). The third inequality follows from
\[ \frac{(1 - |\lambda|^2)(1 - |g(re^{i\theta})|^2)}{1 - \lambda g(re^{i\theta})^2} = 1 - |\varphi \lambda (g(re^{i\theta}))|^2. \]

The last inequality follows from making the change of variable \( u = \varphi_z(w) \) and the fact that
\[ (1 - |w|^2)|g'(w)| = (1 - |w|^2)|\varphi'(\varphi_z(w))\varphi_z'(w)| \]
\[ = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{z}w|^2} |\varphi'(\varphi_z(w))| = (1 - |\varphi_z(w)|^2)|\varphi'(\varphi_z(w))|. \]

Hence
\[ \frac{|h^{(j)}(0)|}{j!} \leq [jr^{(j-1)}(1 - r^2)^{-1}] \max_{\rho(z,u) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2}. \]

The Schwarz-Pick estimate gives
\[ \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2} \leq 1. \]
for each \( u \in D \). Thus
\[
\sum_{k_1 \ldots k_n} \frac{n!}{k_1! k_2! \ldots k_n!} |\lambda^{k-1} k! \prod_{j=1}^{n} \left( \frac{|h^{(j)}(0)|}{j!} \right)^{k_j} n \rho(z,u) \leq \sum_{k_1 \ldots k_n} \frac{n!}{k_1! k_2! \ldots k_n!} |\lambda^{k-1} k! | \rho(z,u)^{k-n} (1 - r^2)^{-k} \max_{\rho(z,u) \leq r} \frac{(1 - |u|^2) |\varphi'(u)|}{1 - |\varphi(u)|^2}.
\]

Let \( b_{n,r} = \sum_{k_1 \ldots k_n} \frac{n!}{k_1! k_2! \ldots k_n!} k! | \rho(z,u)^{k-n} (1 - r^2)^{-k} \). The above inequality gives
\[
M_n(z) \leq a_n \max_{k \leq n} M_k(z) + b_{n,r} \max_{\rho(z,u) \leq r} \frac{(1 - |u|^2) |\varphi'(u)|}{1 - |\varphi(u)|^2}.
\]

By the induction, we conclude that
\[
M_n(z) \leq M_{n,r} \max_{\rho(z,u) \leq r} \frac{(1 - |u|^2) |\varphi'(u)|}{1 - |\varphi(u)|^2}
\]
to complete the proof. \( \square \)

3. Hyperbolic derivatives

In this section we will first show that the hyperbolic derivative of an analytic self-map \( \varphi \) of the unit disk equals \( |\tau_\varphi(z)| \). Then we will show that \( \tau_\varphi(z) \) is Lipschitz with respect to the pseudo-hyperbolic metric.

**Theorem 5.** Let \( \varphi : D \to D \) be an analytic self-map. Then, for each point \( z \in D \), the hyperbolic derivative of \( \varphi \) is equal to
\[
\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z,w)} = |\tau_\varphi(z)|.
\]

**Proof.** Assume that \( \varphi \) is not constant. For each fixed \( z \in D \), \( \rho(\varphi(z), \varphi(w)) \) converges to zero as \( \beta(w,z) \) converges to zero because \( \varphi \) is continuous in \( D \) and \( |\varphi(z)| < 1 \). Thus
\[
\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z,w)} = \lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\rho(\varphi(z), \varphi(w))} \frac{\rho(\varphi(z), \varphi(w))}{\rho(z, w)} \frac{\rho(z, w)}{\beta(z, w)}.
\]
Both the first and third factors of the product on the right converge to one. Now the second factor is
\[
\frac{\rho(\varphi(z), \varphi(w))}{\rho(z, w)} = \frac{|\varphi(z) - \varphi(w)|}{|z - w|} \frac{|1 - \overline{\varphi(z)} \overline{\varphi(w)}|}{|1 - \overline{\varphi(z)} \overline{\varphi(w)}|}.
\]
Thus
\[
\lim_{\beta(z,w) \to 0} \frac{\rho(\varphi(z), \varphi(w))}{\rho(z, w)} = \frac{|\varphi'(z)(1 - |z|^2)|}{1 - |\varphi(z)|^2}.
\]
So
\[
\lim_{\beta(z,w) \to 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z, w)} = \frac{|\varphi'(z)(1 - |z|^2)|}{1 - |\varphi(z)|^2}.
\]
This completes the proof. \( \square \)

For each \( n \), define
\[
\tau_{\varphi,n}(z) = \frac{(1 - |z|^2)^n \varphi^{(n)}(z)}{1 - |\varphi(z)|^2}.
\]
Theorem 6. Let φ be an analytic self-map of the unit disk D. Then for each n, τφ,n(z) is Lipschitz with respect to the pseudohyperbolic metric. More precisely,

$$|τ_{φ,n}(z) - τ_{φ,n}(w)| ≤ C_nρ(z, w)$$

for any z, w ∈ D. Here C_n is a positive constant only depending on n.

Proof. Suppose that f is a differentiable function on the unit disk. Let ∂z f = ∂f/∂z and ∂φ f = ∂f/∂φ. Note that τ_{φ,n}(z) is differentiable on the unit disk. Easy calculations give

$$∂z τ_{φ,n}(z) = \overline{z^n(1 - |z|^2)^{n-1}φ^{(n)}(z)(1 - |φ(z)|^2) + (1 - |z|^2)^nφ^{(n)}(z)φ'(z)φ(z)} \overline{(1 - |φ(z)|^2)^2}$$

and

$$∂φ τ_{φ,n}(z) = \frac{1}{(1 - |φ(z)|^2)^2} \left\{ \left[ (1 - |z|^2)^nφ^{(n+1)}(z) - z^n(1 - |z|^2)^{n-1}φ^{(n)}(z) \right] \right. \times \left[ (1 - |φ(z)|^2) + (1 - |z|^2)^nφ^{(n)}(z)φ'(z)φ(z) \right\}.$$ 

Thus

$$|∂z τ_{φ,n}(z)| \leq \frac{1}{1 - |z|^2} \left\{ \frac{n(1 - |z|^2)^n|φ^{(n)}(z)|}{1 - |φ(z)|^2} \right\} + |φ(z)| \left\{ \frac{(1 - |z|^2)^n|φ'(z)|}{1 - |φ(z)|^2} \right\} \leq \frac{(n + 1)M_n}{1 - |z|^2},$$

where the last inequality follows from Theorem 3 and

$$|∂φ τ_{φ,n}(z)| \leq \frac{1}{1 - |z|^2} \left\{ \frac{(1 - |z|^2)^n|φ^{(n+1)}(z)|}{1 - |φ(z)|^2} \right\} + n\frac{(1 - |z|^2)^n|φ^n(z)|}{1 - |φ(z)|^2} + \frac{M_{n+1} + (n + 1)M_n}{1 - |z|^2},$$

where the last inequality follows from Theorem 3. Given z and w in D, let γ(t) : [0, 1] → D be a smooth curve to connect z and w, i.e.,

$$|τ_{φ,n}(z) - τ_{φ,n}(w)| = \left| \int_0^1 \frac{dτ_{φ,n}(γ(t))}{dt} dt \right| \leq \int_0^1 \left| \frac{d}{dt} τ_{φ,n}(γ(t)) \right| dt \leq \int_0^1 \left[ |∂z τ_{φ,n}(γ(t))| |\frac{dγ(t)}{dt}| + |∂φ τ_{φ,n}(γ(t))| |\frac{dγ(t)}{dt}| \right] dt,$$

where the last inequality follows from the first chain rule:

$$\frac{d}{dt} τ_{φ,n}(γ(t)) = ∂z τ_{φ}(γ(t)) \frac{dγ(t)}{dt} + ∂φ τ_{φ,n}(γ(t)) \frac{dγ(t)}{dt}.$$

Combining the above estimates gives

$$|τ_{φ,n}(z) - τ_{φ,n}(w)| \leq \int_γ \frac{M_{n+1} + 2(n + 1)M_n}{1 - |γ(t)|^2} d|γ(t)|.$$
If we choose $\gamma$ to be a geodesic to connect $z$ and $w$, then the above inequality gives
\[ |\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq (M_{n+1} + 2(n+1)M_n)\beta(z,w) \]
\[ \leq \frac{(M_{n+1} + 2(n+1)M_n)\rho(z,w)}{1 - \rho(z,w)^2}. \]

The last inequality comes from the fact that for all $0 < x < 1$,
\[ \frac{1}{2} \ln \frac{1 + x}{1 - x} \leq \frac{x}{1 - x^2}. \]

If $|\rho(z,w)| < 1/8$, the above inequality gives
\[ |\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq 2(M_{n+1} + 2(n+1)M_n)\rho(z,w). \]

If $|\rho(z,w)| \geq 1/8$, we have $8|\rho(z,w)| \geq 1$, and
\[ |\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq \max\{|\tau_{\varphi,n}(z)|, |\tau_{\varphi,n}(w)|\} \leq M_n \leq 8M_n\rho(z,w). \]

Choosing $C_n = \max\{2(M_{n+1} + 2(n+1)M_n), 8M_n\}$, we have
\[ |\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq C_n\rho(z,w) \]
to complete the proof.

Theorem 6 has an application to closed-range composition operators on the Bloch space.

Homan [6] showed that $(1 - |z|^2)^n\varphi^{(n)}(z)$ continuously extends to the maximal ideal space of $H^\infty$. Let $G$ be the subset of the maximal ideal space of $H^\infty$ consisting of nontrivial Gleason parts. As a corollary of a result in [1] and Theorem 1.2 [4], we have the following result.

**Corollary 1.** Suppose that $\varphi$ is an analytic self-mapping of the unit disk. Then $\tau_{\varphi,n}(z)$ admits a continuous extension to $G$.

**Addendum**

After we finished this paper, we obtained K. Stroethoff’s paper [14], which showed that
\[ \rho(|\tau_{\varphi}(z)|, |\tau_{\varphi}(w)|) \leq 2\rho(z,w), \]
for $z, w \in D$. This generalizes Beardon’s result [3]: If $\varphi(0) = 0$, then
\[ \rho(\tau_{\varphi}(0), \tau_{\varphi}(w)) < 2\rho(0, w) \]
for $w \in D$. We thank K. Stroethoff.

**References**


DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND, OHIO 44115
E-mail address: p.ghatage@csuohio.edu

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37240
E-mail address: zheng@math.vanderbilt.edu