ISOLATED POINTS AND ESSENTIAL COMPONENTS
OF COMPOSITION OPERATORS ON $H^\infty$

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ABSTRACT. We consider the topological space of all composition operators on the Banach algebra of bounded analytic functions on the unit disk. We obtain a function theoretic characterization of isolated points and show that each isolated composition operator is essentially isolated.

1. INTRODUCTION

Let $H^\infty$ be the set of all bounded analytic functions on the open unit disk $D$. Then $H^\infty$ is a Banach algebra under the supremum norm,

$$\|f\|_\infty = \sup\{|f(z)|; z \in D\}.$$

Every analytic self map $\varphi$ of $D$ induces through composition a linear composition operator $C_\varphi$ on $H^\infty$ defined by

$$C_\varphi(f) = f \circ \varphi$$

for $f \in H^\infty(D)$.

We consider here the set $\mathcal{C}(H^\infty)$ of composition operators on $H^\infty$ as a subset of the bounded linear operators on $H^\infty$, endowed with the operator norm. The basic problem we are interested in is the topological structure of $\mathcal{C}(H^\infty)$.

In [8], MacCluer, Ohno, and Zhao studied connected components and isolated points in $\mathcal{C}(H^\infty)$ and asked whether every isolated composition operator in $\mathcal{C}(H^\infty)$ is essentially isolated, that is, isolated in the space of composition operators with the topology induced by the essential semi-norm

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\|; K \text{ is compact on } H^\infty\}.$$

In this paper, we solve the above-mentioned problem affirmatively.

In [8 Corollary 9], it is proved that if

$$\int_0^{2\pi} \log(1 - |\varphi|) d\theta/2\pi > -\infty,$$

then $C_\varphi$ is not isolated in $\mathcal{C}(H^\infty)$. By [2], it is known that $\varphi$ satisfies condition (1.1) if and only if $\varphi$ is not an extreme point of the closed unit ball of $H^\infty$; see also [4, p. 138]. In Theorem 4.1 we prove that (1.1) holds if and only if $C_\varphi$ is not isolated in $\mathcal{C}(H^\infty)$. In Lemma 4.2 we prove that if $C_{\varphi}$ and $C_{\psi}$ are not in the...
same connected component of \(C(H^\infty)\), then \(1 \leq \|C_\varphi - C_\psi\|_e \leq 2\) for \(\psi \neq \varphi\). As a consequence we have that \(C_\varphi\) and \(C_\psi\) are in the same connected component if and only if \(C_\varphi\) and \(C_\psi\) are in the same essentially connected component. This answers MacCluer, Ohno, and Zhao’s problem posed in [8].

To prove our results, we need some preparation. A sequence \(\{z_k\}_k\) in \(D\) is called asymptotically interpolating if for every sequence of complex numbers \(\{a_k\}_k\) such that \(|a_k| \leq 1\) for every \(k\), there exists \(h \in H^\infty\) such that \(\|h\|_\infty \leq 1\) and \(|h(z_k) - a_k| \to 0\). In Section 3, we prove that for a given sequence \(\{w_n\}_n\) in \(D\) with \(|w_n| \to 1\) there exists an asymptotically interpolating subsequence. This is a key in this paper.

There are many studies of composition operators on the Hardy space \(H^2\); see [1] [7] [9] [11]. There are some differences in properties between \(H^\infty\) and \(H^2\). For example, there exists \(\varphi\) such that \(C_\varphi\) is not isolated in \(C(H^2)\) but \(\varphi\) does not satisfy (1.1); see [10]. This is contrary to our Theorem 4.1.

2. Preliminaries

First we introduce some notation. Let \(M(H^\infty)\) be the set of non-zero multiplicative linear functionals of \(H^\infty\). Then \(M(H^\infty)\) is a compact Hausdorff space with the weak*-topology. For a subset \(E\) of \(M(H^\infty)\), we denote by \(cl E\) the closure of \(E\) in \(M(H^\infty)\). We identify a function \(f\) in \(H^\infty\) with its Gelfand transform; \(\hat{f}(m) = m(f), m \in M(H^\infty)\).

For \(z\) and \(w\) in \(D\), we define the pseudohyperbolic distance \(\rho(z, w)\) by

\[
\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|.
\]

For a sequence \(\{z_n\}_n\) in \(D\) with \(\sum_{n=1}^{\infty} (1 - |z_n|) < \infty\), there corresponds a Blaschke product

\[
b(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{|z_n|} \frac{1}{1 - \bar{z}_n z}, \quad z \in D.
\]

A sequence \(\{z_n\}_n\) and an associated Blaschke product are called sparse or thin if

\[
\lim_{n \to \infty} \prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \overline{z}_k z_n} \right| = 1.
\]

If \(b\) is a sparse Blaschke product with zeros \(\{z_n\}_n\), then \(|b(w_j)| \to 1\) for every sequence \(\{w_j\}_j\) in \(D\) satisfying \(\rho(w_j, \{z_n\}_n) \to 1\) as \(j \to \infty\); see [5].

For \(z \in D\), and \(0 < r\), let

\[
\Delta(z, r) = \{w \in D; \rho(z, w) \leq r\}
\]

which is called the pseudo-hyperbolic disk. The pseudo-hyperbolic disk \(\Delta(z, r)\) is also a euclidean disk.

Let \(S(D)\) denote the set of analytic self-mapping of the unit disk \(D\). In [8] Theorems 1 and 2, MacCluer, Ohno, and Zhao proved the following.

Fact 2.1. Let \(\varphi, \psi \in S(D)\). Then the following hold:

(i) \(C_\varphi\) and \(C_\psi\) are in the same connected component in \(C(H^\infty)\) if and only if \(\|C_\varphi - C_\psi\| < 1\) if and only if

\[
\sup_{z \in D} \rho(\varphi(z), \psi(z)) < 1.
\]

(ii) Every connected component of \(C(H^\infty)\) is open and closed.
(iii) \( C_\varphi \) is isolated in \( C(H^\infty) \) if and only if the connected component containing \( C_\varphi \) consists of only \( C_\varphi \).

(iv) \( C_\varphi \) is isolated if and only if for all \( \psi \neq \varphi \) one has \( \| C_\varphi - C_\psi \| = 2 \).

Theorem 3 in [8] is restated as follows.

**Fact 2.2.** Let \( \varphi, \psi \in \mathcal{S}(D), \varphi \neq \psi \), and \( \| \varphi \|_\infty = 1 \). Then \( C_\varphi - C_\psi \) is a compact operator on \( H^\infty \) if and only if \( \limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z)) = 0 \).

**Proof.** By Theorem 3 in [8], \( C_\varphi - C_\psi \) is compact if and only if

\[
(2.1) \quad \partial \varphi(D) \cap \partial D = \partial \psi(D) \cap \partial D \neq \emptyset
\]

and

\[
(2.2) \quad \limsup_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z)) = 0.
\]

We need to show that (2.1) follows from (2.2). Suppose that \( \max\{|\varphi(z_n)|, |\psi(z_n)|\} \to 1 \). By (2.2), \( \rho(\varphi(z_n), \psi(z_n)) \to 0 \). Hence \( |\varphi(z_n) - \psi(z_n)| \to 0 \). Therefore (2.1) holds.

### 3. Asymptotically Interpolating Sequences

Let \( \mathcal{A} \) be the disk algebra, that is, \( \mathcal{A} \) is the space of continuous functions on the closed unit disk \( \overline{D} \) and analytic in \( D \).

**Theorem 3.1.** For every sequence \( \{w_n\}_n \) in \( D \) with \( |w_n| \to 1 \), there exists an asymptotically interpolating subsequence of \( \{w_n\}_n \).

**Proof.** We may assume that \( |w_n - 1| \to 0 \). Put \( f(z) = (z + 1)/2, z \in D \). Then \( f \in \mathcal{A} \),

\[
(3.1) \quad f(1) = 1 \quad \text{and} \quad |f| < 1 \quad \text{on} \quad \overline{D} \setminus \{1\}.
\]

Put \( g(z) = (z - 1)/2, z \in D \), and \( g_n = g^{1/n} \) for every positive integer \( n \). Then \( g_n \in \mathcal{A}, \|g_n\|_\infty = 1, g_n(1) = 0 \), and

\[
(3.2) \quad |g_n(z)| \to 1 \quad \text{for each} \quad z \in D.
\]

By induction, we shall find two sequences of increasing positive integers \( \{m_k\}_k \), \( \{n_k\}_k \), a sequence of complex numbers \( \{c_k\}_k \) with \( |c_k| < 1 \), and a subsequence \( \{z_k\}_k \) in \( \{w_n\}_n \) satisfying that

\[
(3.3) \quad \sup_{z \in D} \sum_{k=1}^N |(c_k f^{m_k} g_{n_k})(z)| < 1 \quad \text{for every} \quad N,
\]

\[
(3.4) \quad \sum_{k=1}^{N-1} |(c_k f^{m_k} g_{n_k})(z_N)| < (1/2)^N \quad \text{for every} \quad N \geq 2,
\]

\[
(3.5) \quad c_N (f^{m_N} g_{n_N})(z_N) > 1 - (1/2)^N \quad \text{for every} \quad N,
\]

and

\[
(3.6) \quad |f^{m_N}(z_j)| < (1/2)^N \quad \text{for} \quad 1 \leq j < N.
\]
First, take $m_1 = 1$. By (3.1), there exists $z_1 \in \{w_n\}_n$ such that $|f(z_1)| > 1/2$. By (3.2), there exists $n_1$ such that $|(f^{m_1}g_{n_1})(z_1)| > 1/2$. Take a complex number $c_1$ such as

$$c_1(f^{m_1}g_{n_1})(z_1) = |(f^{m_1}g_{n_1})(z_1)|.$$

Then (3.3) and (3.5) hold for $N = 1$.

Next, suppose that $\{m_k\}_{k=1}^N, \{n_k\}_{k=1}^N, \{c_k\}_{k=1}^N, \{z_k\}_{k=1}^N$ are chosen satisfying our conditions. Put

$$F_N = \sum_{k=1}^N |c_k f^{m_k} g_{n_k}| \quad \text{on } \overline{D}.$$

Since $g_n(1) = 0$, $F_N(1) = 0$. Take an open subset $U_N$ of $\overline{D}$ such that $1 \in U_N$,

$$\{z_1, z_2, \ldots, z_N\} \cap U_N = \emptyset,$$

and

$$(3.7) \quad F_N < (1/2)^{N+2} \quad \text{on } U_N.$$

By (3.1) and (3.3), there exists $m_{N+1}$ such that $m_N < m_{N+1}$,

$$(3.9) \quad |f^{m_{N+1}}| < (1/2)^{N+1} \quad \text{on } \overline{D} \setminus U_N,$$

and

$$(3.10) \quad F_N + |f^{m_{N+1}}| < 1 \quad \text{on } \overline{D} \setminus U_N.$$

By (3.1) again, there is a point $z_{N+1}$ in $\{w_n\}_n \cap U_N$ such that

$$|f^{m_{N+1}}(z_{N+1})| > \frac{1 - (1/2)^{N+1}}{1 - (1/2)^N}.$$

By (3.2), there exists $n_{N+1}$ such that $n_N < n_{N+1}$ and

$$(3.11) \quad |(f^{m_{N+1}}g_{n_{N+1}})(z_{N+1})| > \frac{1 - (1/2)^{N+1}}{1 - (1/2)^N}.$$

By (3.10),

$$(3.12) \quad F_N + |f^{m_{N+1}}g_{n_{N+1}}| < 1 \quad \text{on } \overline{D} \setminus U_N.$$

Since $\|f^{m_{N+1}}g_{n_{N+1}}\|_\infty < 1$, by (3.8) and (3.12)

$$(3.13) \quad \sup_{z \in \overline{D}} |F_N(z) + (1 - (1/2)^{N+2})|(f^{m_{N+1}}g_{n_{N+1}})(z)| < 1.$$

Take a complex number $b_{N+1}$ such that

$$b_{N+1}(1 - (1/2)^{N+2})(f^{m_{N+1}}g_{n_{N+1}})(z_{N+1}) = (1 - (1/2)^{N+2})|(f^{m_{N+1}}g_{n_{N+1}})(z_{N+1})|.$$

Put $c_{N+1} = b_{N+1}(1 - (1/2)^{N+2})$. Then $|c_{N+1}| = 1 - (1/2)^{N+2}$, and by (3.13) we get (3.3) for $N + 1$. Also, by (3.11)

$$c_{N+1}(f^{m_{N+1}}g_{n_{N+1}})(z_{N+1}) = (1 - (1/2)^{N+2})|(f^{m_{N+1}}g_{n_{N+1}})(z_{N+1})|,$$

$$> 1 - (1/2)^{N+1}.$$

Thus we get (3.5) for $N + 1$. Since $z_{N+1} \in U_N$, by (3.8) we have (3.4) for $N + 1$. By (3.7) and (3.9), (3.6) holds. This completes the induction.

By (3.6),

$$(3.14) \quad \sum_{k=N+1}^{\infty} |(c_k f^{m_k} g_{n_k})(z_N)| < \sum_{k=N+1}^{\infty} (1/2)^k = 1/2^N.$$
Let \( \{a_k\}_k \) be a sequence of complex numbers such that \(|a_k| \leq 1 \) for every \( k \). Put

\[
h(z) = \sum_{k=1}^{\infty} a_k (c_k f^{m_k} g_{n_k})(z), \quad z \in D.
\]

By (3.3), \( h \in B(H^\infty) \), and

\[
|h(z_N) - a_N| \leq (|1 - (c_n f^{m_n} g_n)(z_N)|) + \sum_{k=1}^{N-1} |(c_k f^{m_k} g_{n_k})(z_N)| + \sum_{k=N+1}^{\infty} |(c_k f^{m_k} g_{n_k})(z_N)|
\]

\[
< 3(1/2)^N \quad \text{by (3.4), (3.5), and (3.14)}
\]

\[
\rightarrow 0 \quad \text{as } N \rightarrow \infty.
\]

This completes the proof.

4. Main results

By Fact 2.1(iii), a composition operator \( C_\varphi \) is an isolated point if and only if the connected component containing \( C_\varphi \) in \( C(H^\infty) \) consists of only \( C_\varphi \). Our first main result is the following theorem which gives a function theoretic characterization of isolated points in \( C(H^\infty) \).

**Theorem 4.1.** Let \( \varphi \in S(D) \). Then \( C_\varphi \) is isolated in \( C(H^\infty) \) if and only if

\[
\int_0^{2\pi} \log(1 - |\varphi|) \, d\theta / 2\pi = -\infty.
\]

**Proof.** Suppose that \( \int_0^{2\pi} \log(1 - |\varphi|) \, d\theta / 2\pi = -\infty \). To prove that \( C_\varphi \) is isolated in \( C(H^\infty) \), suppose not. Then by Fact 2.1, there exists \( \psi \in S(D) \), \( \varphi \neq \psi \), such that

\[
\sup_{z \in D} \rho(\varphi(z), \psi(z)) < 1.
\]

Put

\[
\sigma = \sup_{\varphi(z) \in D} \rho(\varphi(z), \psi(z)).
\]

Then \( 0 < \sigma < 1 \). Put

\[
f = (\varphi + \psi) / 2.
\]

Then \( f \) is not an extreme point of the closed unit ball of \( H^\infty \). By de Leeuw and Rudin’s theorem [2],

\[
\int_0^{2\pi} \log(1 - |f|) \, d\theta / 2\pi > -\infty.
\]

By (4.1) and (4.2), the convexity of \( \Delta(\varphi(z), \sigma) \) gives that \( f(z) \in \Delta(\varphi(z), \sigma) \). By [3] p. 3], for \( z \in D \) we have

\[
\frac{|\varphi(z)| - \sigma}{1 - \sigma|\varphi(z)|} \leq |f(z)|.
\]

Hence

\[
1 - |f| \leq \frac{(1 + \sigma)(1 - |\varphi|)}{1 - \sigma|\varphi|} \leq \frac{1 + \sigma}{1 - \sigma}(1 - |\varphi|) \quad \text{on } D.
\]

Therefore

\[
\int_0^{2\pi} \log(1 - |f|) \, d\theta / 2\pi \leq \log \left( \frac{1 + \sigma}{1 - \sigma} \right) + \int_0^{2\pi} \log(1 - |\varphi|) \, d\theta / 2\pi.
\]
By our assumption, we get $\int_0^{2\pi} \log(1 - |f|) d\theta/2\pi = -\infty$. This contradicts (4.3).

The converse is proved in [8, Corollary 9].

In [8], MacCluer, Ohno, and Zhao showed that $C_\varphi$ and $C_\psi$ are in the same connected component if $C_\varphi - C_\psi$ is compact. They also gave an example of $\varphi \in \mathcal{S}(D)$ that $C_\varphi$ is not isolated but $C_\varphi - C_\psi$ is not compact for some $C_\psi$ in the same component of $C_\varphi$. Here we show that this occurs for every non-isolated connected component in $C(H^\infty)$, except the component consists of compact composition operators.

**Examples.** Let $\varphi \in \mathcal{S}(D)$. Suppose that $C_\varphi$ is not isolated and $\|\varphi\|_\infty = 1$. Then there exist $\psi_1, \psi_2 \in \mathcal{S}(D)$ satisfying the following conditions:

(i) $\varphi \neq \psi_1$ and $\varphi \neq \psi_2$.
(ii) $C_\varphi, C_{\psi_1}$ and $C_{\psi_2}$ are in the same component of $C(H^\infty)$.
(iii) $C_\varphi - C_{\psi_1}$ is compact.
(iv) $C_\varphi - C_{\psi_2}$ is not compact.

**Proof.** By Theorem 4.1, \( \int_0^{2\pi} \log(1 - |\varphi|) d\theta/2\pi > -\infty \). There exists an outer function $\omega \in H^\infty$ such that $|\omega| = 1 - |\varphi|$ a.e. on $\partial D$; see [8]. For each $z \in D$, let $P_z(\theta)$ be the Possion kernel at $z$. The values of $\omega$ and $\varphi$ at $z$ are given by

$$\omega(z) = \int P_z(\theta) \omega(\theta) d\theta$$

and

$$\varphi(z) = \int P_z(\theta) \varphi(\theta) d\theta,$$

respectively. Thus

$$|\omega(z)| + |\varphi(z)| \leq \int P_z(\theta) (|\omega(\theta)| + |\varphi(\theta)|) d\theta \leq 1 \quad \text{on } D.$$  

Let $0 < t < 1$. Put $\psi_1 = \varphi + t \omega^2$. Then

$$\rho(\varphi(z), \psi_1(z)) \leq \frac{|t \omega^2(z)|}{1 - |\varphi(z)|^2 - |t \omega^2(z)\varphi(z)|} \leq \frac{|t \omega(z)|}{1 + |\varphi(z)| - |t \omega(z)\varphi(z)|}, \quad z \in D.$$  

The last inequality is obtained by dividing the denominator and nominator by $|\omega(z)|$ and using (4.4). Suppose that $|\varphi(z_n)| \to 1$. Then by (4.4), $\omega(z_n) \to 0$. Hence by (4.5), $\rho(\varphi(z_n), \psi_1(z_n)) \to 0$. Next suppose that $|\psi_1(z_n)| \to 1$. Since

$$|\psi_1(z_n)| \leq |\varphi(z_n)| + t|\omega(z_n)| \leq |\varphi(z_n)| + |\omega(z_n)| \leq 1,$$

we have

$$1 - (1 - t)|\omega(z_n)| \leq 1 - |\psi_1(z_n)|.$$  

Thus $(1 - t)|\omega(z_n)| \to 0$ and $\omega(z_n) \to 0$. So $\rho(\varphi(z_n), \psi_1(z_n)) \to 0$. By Fact 2.2, $C_\varphi - C_{\psi_1}$ is compact.

Since $1 - |\varphi(e^{i\theta})| = |\omega(e^{i\theta})|$ and $\omega(e^{i\theta}) \neq 0$ for almost everywhere, $1 - |\varphi(e^{i\theta})| < \frac{|\omega(e^{i\theta})|}{t}$ for almost everywhere. Also by our assumption, the Lebesgue measure of the set $\{e^{i\theta}; r < |\varphi(e^{i\theta})| < 1\}$ is positive for every $r, 0 < r < 1$. Therefore there exists a sequence $\{z_n\}_n$ in $D$ such that

$$1 - \frac{1 - |\varphi(z_n)|}{|\omega(z_n)|} < \frac{1}{t} \quad \text{and} \quad |\varphi(z_n)| \to 1.$$

Therefore there exists a sequence $\{z_n\}_n$ in $D$ such that

$$1 - \frac{1 - |\varphi(z_n)|}{|\omega(z_n)|} < \frac{1}{t} \quad \text{and} \quad |\varphi(z_n)| \to 1.$$
Moreover we may assume that
\[
(4.6) \quad \frac{1 - |\varphi(z_n)|}{\omega(z_n)} \rightarrow Re^{i\theta_1}, \quad 1 \leq R \leq 1/t, \quad \text{and} \quad \varphi(z_n) \rightarrow e^{i\theta_2}.
\]

Put \( \theta_3 = \theta_1 + \theta_2 \) and \( \psi_2 = \varphi + te^{i\theta_3} \omega \). Then in the same way as above,
\[
\rho(\varphi(z), \psi_2(z)) \leq \frac{t}{1 + |\varphi(z)| - |t\varphi(z)|} \leq 1, \quad z \in D,
\]
so that \( C_\varphi \) and \( C_{\psi_2} \) are in the same component. To prove that \( C_\varphi - C_{\psi_2} \) is not compact, by Fact 2.2 it is sufficient to prove \( \limsup \rho(\varphi(z), \psi_2(z)) > 0 \). We have
\[
\rho(\varphi(z_n), \psi_2(z_n)) = \left| \frac{te^{i\theta_3} \omega(z_n)}{1 - |\varphi(z_n)|^2 - te^{i\theta_3} \omega(z_n) \varphi(z_n)} \right|
\]
\[
\geq \frac{t}{1 - |\varphi(z_n)|^2 - te^{i\theta_3} \varphi(z_n)} \rightarrow \frac{t}{2Re^{i\theta_1} - te^{i(\theta_2 - \theta_3)}} \quad \text{by (4.6)}
\]
\[
= \frac{t}{2R - t} \geq \frac{t^2}{2 - t^2} \quad \text{by (4.6)}.
\]

Hence by Fact 2.2, \( C_\varphi - C_{\psi_1} \) is not compact.

**Lemma 4.2.** Let \( \varphi, \psi \in S(D) \) and \( \varphi \neq \psi \). If \( C_\varphi \) and \( C_\psi \) are not contained in the same connected component in \( \mathcal{C}(H^\infty) \), then \( \|C_\varphi - C_\psi\|_e \geq 1 \).

**Proof.** By Fact 2.1(i), \( \sup_{z \in D} \rho(\varphi(z), \psi(z)) = 1 \). Then we may assume that there exists a sequence \( \{z_n\}_n \) in \( D \) such that \( |\varphi(z_n)| < |\varphi(z_{n+1})| \rightarrow 1 \) and
\[
(4.7) \quad \rho(\varphi(z_n), \psi(z_n)) \rightarrow 1.
\]

Then \( |z_n| \rightarrow 1 \). By Theorem 3.1, we may assume that \( \{\varphi(z_n)\}_n \) is asymptotically interpolating.

To prove our assertion, suppose that \( \|C_\varphi - C_\psi\|_e < 1 \). Take a positive number \( \sigma \) such that \( \|C_\varphi - C_\psi\|_e < \sigma < 1 \) and take a compact operator \( K \) on \( H^\infty \) such that
\[
(4.8) \quad \|C_\varphi - C_\psi + K\| < \sigma < 1.
\]

We claim that there are a Blaschke product \( b_0 \) and a subsequence \( \{w_n\}_n \) of \( \{z_n\} \) such that
\[
(4.9) \quad b_0(\psi(w_n)) \rightarrow 0
\]
and
\[
(4.10) \quad |b_0(\varphi(w_n))| \rightarrow 1.
\]

Assume the claim first. Put \( E = \{w_n\}_n \) and take a sequence of subsets \( \{E_k\}_k \) of \( E \) such that
\[
(4.11) \quad \text{for each positive integer } k, \text{ the sets } E_{k+1} \subset E_k, \text{ and } E_k \setminus E_{k+1} \text{ is an infinite set for every } k.
\]

Fix a positive integer \( k \). Since \( \{\varphi(w_n)\}_n \) is asymptotically interpolating, there exists \( h_k \in H^\infty \) such that \( \|h_k\|_\infty \leq 1 \) and
\[
(4.12) \quad |h_k(\varphi(w_n)) - b_0(\varphi(w_n))| \rightarrow 0 \quad \text{as } |w_n| \rightarrow 1 \text{ and } w_n \in E_k.
\]
and
\[ |h_k(\varphi(w_n)) + b_0(\varphi(w_n))| \to 0 \] as \( |w_n| \to 1 \) and \( w_n \notin E_k \).

Since \( h_kb_0 \in H^\infty \) and \( \|h_kb_0\|_\infty \leq 1 \), by (4.8)
\[ |h_k(\varphi(w_n))b_0(\varphi(w_n)) - h_k(\psi(w_n))b_0(\psi(w_n)) + K(h_kb_0)(w_n)| < \sigma < 1. \]

Hence by (4.9), (4.10), (4.12), and (4.13),
\[ |1 + K(h_kb_0)| \leq \sigma < 1 \] on \( cl E_k \setminus E_k \)
and
\[ | -1 + K(h_kb_0)| \leq \sigma < 1 \] on \( cl (E \setminus E_k) \setminus (E \setminus E_k). \)

By (4.11), we have \( cl (E_k \setminus E_{k+1}) \setminus (E_k \setminus E_{k+1}) \neq \emptyset \) for every \( k \). Take a point \( \zeta_k \) in \( cl (E_k \setminus E_{k+1}) \setminus (E_k \setminus E_{k+1}) \). By (4.11), \( \zeta_n \in cl E_k \setminus E_k \) for every \( n \geq k \). Hence by (4.14), \( |1 + K(h_kb_0)(\zeta_n)| \leq \sigma < 1 \) for \( n \geq k \). Let \( \zeta_0 \) be a cluster point of \( \{ \zeta_k \}_k \).

Then
\[ |1 + K(h_kb_0)(\zeta_0)| \leq \sigma < 1. \]

Since \( K \) is a compact operator on \( H^\infty \), considering a subsequence of \( \{ h_k \}_k \) we may assume that \( \|K(h_kb_0) - F\|_\infty \to 0 \) for some \( F \in H^\infty \). By (4.16),
\[ |1 + F(\zeta_0)| \leq \sigma < 1. \]

By (4.11) again, \( \zeta_n \in cl E_k \setminus E_k \) for \( k > n \). Hence by (4.15),
\[ | -1 + K(h_kb_0)(\zeta_n)| \leq \sigma < 1 \] for \( k > n \).

Thus \( | -1 + F(\zeta_n)| \leq \sigma < 1 \) for every \( n \), so that \( | -1 + F(\zeta_0)| \leq \sigma < 1 \). This contradicts (4.17).

In order to prove our claim we divide the proof into two cases.

\textit{Case 1.} \( \liminf_{n \to \infty} \|\psi(z_n)\| < 1 \).

In this case, considering a subsequence of \( \{ z_n \}_n \) we may further assume that \( \psi(z_n) \to a \) and \( |a| < 1 \). Let \( b_0(z) = (z - a)/(1 - \overline{a}z), z \in D \). Then
\[ b_0(\psi(z_n)) \to 0. \]

Since \( \|\varphi(z_n)\| \to 1 \),
\[ |b_0(\varphi(z_n))| \to 1. \]

This proves the claim desired.

\textit{Case 2.} \( \psi(z_n) \to 1 \).

Considering a subsequence of \( \{ z_n \}_n \), we may assume that \( \{ \psi(z_n) \}_n \) is a sparse sequence; see page 42 in [4]. Since \( \|\varphi(z_n)\| \to 1 \) and (4.7), we may further assume that
\[ \rho(\varphi(z_n), \psi(z_n)) > 1 - 1/n \] and \( \rho(\varphi(z_j), \psi(z_n)) > 1 - 1/n \) for \( 1 \leq j \leq n \).

Then \( \rho(\varphi(z_k), \{ \psi(z_n) \}_n) \to 1 \) as \( k \to \infty \). Let \( b_0 \) be the sparse Blaschke product with zeros \( \{ \psi(z_n) \}_n \). Hence \( |b_0(\varphi(z_k))| \to 1 \); see [5]. Then the claim is true, too.

As pointed out in Section 1, we may introduce the essential norm topology on \( C(H^\infty) \). With this topology, we consider essentially connected components of \( C(H^\infty) \).
Theorem 4.3. Let \( \varphi, \psi \in S(D) \). Then we have the following:

(i) Every connected component of \( \mathcal{C}(H^\infty) \) is open and closed in the essential norm topology.

(ii) \( C_\varphi \) and \( C_\psi \) are in the same connected component if and only if \( C_\varphi \) and \( C_\psi \) are in the same essentially connected component.

Proof. By Lemma 4.2, each connected component of \( \mathcal{C}(H^\infty) \) is open and hence closed in the essential norm topology. Since the essential norm topology is weaker than the norm topology, we get our assertion.

In [8], MacCluer, Ohno, and Zhao posed the problem of whether every isolated composition operator in \( \mathcal{C}(H^\infty) \) is essentially isolated. The following theorem answers this problem affirmatively.

Theorem 4.4. \( C_\varphi \) is isolated in \( \mathcal{C}(H^\infty) \) if and only if \( C_\varphi \) is essentially isolated.

Proof. This follows from Theorem 4.3(i).

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