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ISOLATED POINTS AND ESSENTIAL COMPONENTS OF COMPOSITION OPERATORS ON H^∞

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Abstract. We consider the topological space of all composition operators on the Banach algebra of bounded analytic functions on the unit disk. We obtain a function theoretic characterization of isolated points and show that each isolated composition operator is essentially isolated.

1. Introduction

Let H^∞ be the set of all bounded analytic functions on the open unit disk D. Then H^∞ is a Banach algebra under the supremum norm,

$$\|f\|_\infty = \sup\{|f(z)|; z \in D\}.$$

Every analytic self map φ of D induces through composition a linear composition operator C_φ on H^∞ defined by

$$C_\varphi(f) = f \circ \varphi$$

for $f \in H^\infty(D)$.

We consider here the set $\mathcal{C}(H^\infty)$ of composition operators on H^∞ as a subset of the bounded linear operators on H^∞, endowed with the operator norm. The basic problem we are interested in is the topological structure of $\mathcal{C}(H^\infty)$.

In [8], MacCluer, Ohno, and Zhao studied connected components and isolated points in $\mathcal{C}(H^\infty)$ and asked whether every isolated composition operator in $\mathcal{C}(H^\infty)$ is essentially isolated, that is, isolated in the space of composition operators with the topology induced by the essential semi-norm

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\|; K \text{ is compact on } H^\infty\}.$$

In this paper, we solve the above-mentioned problem affirmatively.

In [8, Corollary 9], it is proved that if

$$\int_0^{2\pi} \log(1 - |\varphi|)\, d\theta/2\pi > -\infty, \tag{1.1}$$

then C_φ is not isolated in $\mathcal{C}(H^\infty)$. By [2], it is known that φ satisfies condition (1.1) if and only if φ is not an extreme point of the closed unit ball of H^∞; see also [7, p. 138]. In Theorem 4.1, we prove that (1.1) holds if and only if C_φ is not isolated in $\mathcal{C}(H^\infty)$. In Lemma 4.2, we prove that if C_φ and C_ψ are not in the

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same connected component of $\mathcal{C}(H^\infty)$, then $1 \le \|C_\varphi - C_\psi\|_e \le 2$ for $\psi \neq \varphi$. As a consequence we have that C_φ and C_ψ are in the same connected component if and only if C_φ and C_ψ are in the same essentially connected component. This answers MacCluer, Ohno, and Zhao's problem posed in [8].

To prove our results, we need some preparation. A sequence $\{z_k\}_k$ in D is called *asymptotically interpolating* if for every sequence of complex numbers $\{a_k\}_k$ such that $|a_k| \le 1$ for every k, there exists $h \in H^\infty$ such that $\|h\|_\infty \le 1$ and $|h(z_k) - a_k| \to 0$. In Section 3, we prove that for a given sequence $\{w_n\}_n$ in D with $|w_n| \to 1$ there exists an asymptotically interpolating subsequence. This is a key in this paper.

There are many studies of composition operators on the Hardy space H^2; see [1, 7, 9, 11]. There are some differences in properties between H^∞ and H^2. For example, there exists φ such that C_φ is not isolated in $\mathcal{C}(H^2)$ but φ does not satisfy (1.1); see [10]. This is contrary to our Theorem 4.1.

2. Preliminaries

First we introduce some notation. Let $M(H^\infty)$ be the set of non-zero multiplicative linear functionals of H^∞. Then $M(H^\infty)$ is a compact Hausdorff space with the weak*-topology. For a subset E of $M(H^\infty)$, we denote by $cl\,E$ the closure of E in $M(H^\infty)$. We identify a function f in H^∞ with its Gelfand transform; $\hat{f}(m) = m(f)$, $m \in M(H^\infty)$.

For z and w in D, we define the pseudohyperbolic distance $\rho(z, w)$ by

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{z}w}\right|.$$

For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, there corresponds a Blaschke product

$$b(z) = \prod_{n=1}^\infty \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in D.$$

A sequence $\{z_n\}_n$ and an associated Blaschke product are called sparse or thin if

$$\lim_{n\to\infty} \prod_{k\neq n} \left|\frac{z_n - z_k}{1 - \overline{z}_k z_n}\right| = 1.$$

If b is a sparse Blaschke product with zeros $\{z_n\}_n$, then $|b(w_j)| \to 1$ for every sequence $\{w_j\}_j$ in D satisfying $\rho(w_j, \{z_n\}_n) \to 1$ as $j \to \infty$; see [5].

For $z \in D$, and $0 < r$, let

$$\Delta(z, r) = \{w \in D; \rho(z, w) \le r\}$$

which is called the pseudo-hyperbolic disk. The pseudo-hyperbolic disk $\Delta(z, r)$ is also a euclidean disk.

Let $\mathcal{S}(D)$ denote the set of analytic self-mapping of the unit disk D. In [8, Theorems 1 and 2], MacCluer, Ohno, and Zhao proved the following.

Fact 2.1. Let $\varphi, \psi \in \mathcal{S}(D)$. Then the following hold:

(i) C_φ and C_ψ are in the same connected component in $\mathcal{C}(H^\infty)$ if and only if $\|C_\varphi - C_\psi\| < 1$ if and only if

$$\sup_{z\in D} \rho(\varphi(z), \psi(z)) < 1.$$

(ii) Every connected component of $\mathcal{C}(H^\infty)$ is open and closed.

(iii) C_φ is isolated in $\mathcal{C}(H^\infty)$ if and only if the connected component containing C_φ consists of only C_φ.

(iv) C_φ is isolated if and only if for all $\psi \neq \varphi$ one has $\|C_\varphi - C_\psi\| = 2$.

Theorem 3 in [8] is restated as follows.

Fact 2.2. Let $\varphi, \psi \in \mathcal{S}(D), \varphi \neq \psi$, and $\|\varphi\|_\infty = 1$. Then $C_\varphi - C_\psi$ is a compact operator on H^∞ if and only if $\limsup_{|\varphi(z)|\to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)|\to 1} \rho(\varphi(z), \psi(z)) = 0$.

Proof. By Theorem 3 in [8], $C_\varphi - C_\psi$ is compact if and only if

$$\partial\varphi(D) \cap \partial D = \partial\psi(D) \cap \partial D \neq \emptyset \tag{2.1}$$

and

$$\limsup_{|\varphi(z)|\to 1} \rho(\varphi(z), \psi(z)) = \limsup_{|\psi(z)|\to 1} \rho(\varphi(z), \psi(z)) = 0. \tag{2.2}$$

We need to show that (2.1) follows from (2.2). Suppose that $\max\{|\varphi(z_n)|, |\psi(z_n)|\} \to 1$. By (2.2), $\rho(\varphi(z_n), \psi(z_n)) \to 0$. Hence $|\varphi(z_n) - \psi(z_n)| \to 0$. Therefore (2.1) holds.

3. Asymptotically interpolating sequences

Let $\mathcal{A}$ be the disk algebra, that is, $\mathcal{A}$ is the space of continuous functions on the closed unit disk $\overline{D}$ and analytic in D.

Theorem 3.1. *For every sequence $\{w_n\}_n$ in D with $|w_n| \to 1$, there exists an asymptotically interpolating subsequence of $\{w_n\}_n$.*

Proof. We may assume that $|w_n - 1| \to 0$. Put $f(z) = (z+1)/2, z \in D$. Then $f \in \mathcal{A}$,

$$f(1) = 1 \quad \text{and} \quad |f| < 1 \ \text{ on } \overline{D} \setminus \{1\}. \tag{3.1}$$

Put $g(z) = (z-1)/2, z \in D$, and $g_n = g^{1/n}$ for every positive integer n. Then $g_n \in \mathcal{A}, \|g_n\|_\infty = 1, g_n(1) = 0$, and

$$|g_n(z)| \to 1 \quad \text{for each } z \in D. \tag{3.2}$$

By induction, we shall find two sequences of increasing positive integers $\{m_k\}_k$, $\{n_k\}_k$, a sequence of complex numbers $\{c_k\}_k$ with $|c_k| < 1$, and a subsequence $\{z_k\}_k$ in $\{w_n\}_n$ satisfying that

$$\sup_{z \in \overline{D}} \sum_{k=1}^{N} |(c_k f^{m_k} g_{n_k})(z)| < 1 \quad \text{for every } N, \tag{3.3}$$

$$\sum_{k=1}^{N-1} |(c_k f^{m_k} g_{n_k})(z_N)| < (1/2)^N \quad \text{for every } N \geq 2, \tag{3.4}$$

$$c_N (f^{m_N} g_{n_N})(z_N) > 1 - (1/2)^N \quad \text{for every } N, \tag{3.5}$$

and

$$|f^{m_N}(z_j)| < (1/2)^N \quad \text{for } 1 \leq j < N. \tag{3.6}$$

First, take $m_1 = 1$. By (3.1), there exists $z_1 \in \{w_n\}_n$ such that $|f(z_1)| > 1/2$. By (3.2), there exists n_1 such that $|(f^{m_1} g_{n_1})(z_1)| > 1/2$. Take a complex number c_1 such as

$$c_1(f^{m_1} g_{n_1})(z_1) = |(f^{m_1} g_{n_1})(z_1)|.$$

Then (3.3) and (3.5) hold for $N = 1$.

Next, suppose that $\{m_k\}_{k=1}^N, \{n_k\}_{k=1}^N, \{c_k\}_{k=1}^N$, and $\{z_k\}_{k=1}^N$ are chosen satisfying our conditions. Put

$$F_N = \sum_{k=1}^{N} |c_k f^{m_k} g_{n_k}| \quad \text{on } \overline{D}.$$

Since $g_n(1) = 0$, $F_N(1) = 0$. Take an open subset U_N of $\overline{D}$ such that $1 \in U_N$,

$$\{z_1, z_2, \ldots, z_N\} \cap U_N = \emptyset, \tag{3.7}$$

and

$$F_N < (1/2)^{N+2} \quad \text{on } U_N. \tag{3.8}$$

By (3.1) and (3.3), there exists m_{N+1} such that $m_N < m_{N+1}$,

$$|f^{m_{N+1}}| < (1/2)^{N+1} \quad \text{on } \overline{D} \setminus U_N, \tag{3.9}$$

and

$$F_N + |f^{m_{N+1}}| < 1 \quad \text{on } \overline{D} \setminus U_N. \tag{3.10}$$

By (3.1) again, there is a point z_{N+1} in $\{w_n\}_n \cap U_N$ such that

$$|f^{m_{N+1}}(z_{N+1})| > \frac{1 - (1/2)^{N+1}}{1 - (1/2)^{N+2}}.$$

By (3.2), there exists n_{N+1} such that $n_N < n_{N+1}$ and

$$|(f^{m_{N+1}} g_{n_{N+1}})(z_{N+1})| > \frac{1 - (1/2)^{N+1}}{1 - (1/2)^{N+2}}. \tag{3.11}$$

By (3.10),

$$F_N + |f^{m_{N+1}} g_{n_{N+1}}| < 1 \quad \text{on } \overline{D} \setminus U_N. \tag{3.12}$$

Since $\|f^{m_{N+1}} g_{n_{N+1}}\|_\infty < 1$, by (3.8) and (3.12)

$$\sup_{z \in \overline{D}} \left[F_N(z) + (1 - (1/2)^{N+2}) |(f^{m_{N+1}} g_{n_{N+1}})(z)| \right] < 1. \tag{3.13}$$

Take a complex number b_{N+1} such that

$$b_{N+1}(1 - (1/2)^{N+2})(f^{m_{N+1}} g_{n_{N+1}})(z_{N+1}) = (1 - (1/2)^{N+2})|(f^{m_{N+1}} g_{n_{N+1}})(z_{N+1})|.$$

Put $c_{N+1} = b_{N+1}(1 - (1/2)^{N+2})$. Then $|c_{N+1}| = 1 - (1/2)^{N+2}$, and by (3.13) we get (3.3) for $N + 1$. Also, by (3.11)

$$\begin{aligned} c_{N+1}(f^{m_{N+1}} g_{n_{N+1}})(z_{N+1}) &= (1 - (1/2)^{N+2})|(f^{m_{N+1}} g_{n_{N+1}})(z_{N+1})| \\ &> 1 - (1/2)^{N+1}. \end{aligned}$$

Thus we get (3.5) for $N + 1$. Since $z_{N+1} \in U_N$, by (3.8) we have (3.4) for $N + 1$. By (3.7) and (3.9), (3.6) holds. This completes the induction.

By (3.6),

$$\sum_{k=N+1}^{\infty} |(c_k f^{m_k} g_{n_k})(z_N)| < \sum_{k=N+1}^{\infty} (1/2)^k = 1/2^N. \tag{3.14}$$

Let $\{a_k\}_k$ be a sequence of complex numbers such that $|a_k| \le 1$ for every k. Put

$$h(z) = \sum_{k=1}^{\infty} a_k (c_k f^{m_k} g_{n_k})(z), \quad z \in D.$$

By (3.3), $h \in B(H^\infty)$, and

$$\begin{aligned} |h(z_N) - a_N| &\le \big(|1 - (c_N f^{m_N} g_{n_N})(z_N)|\big) + \sum_{k=1}^{N-1} |(c_k f^{m_k} g_{n_k})(z_N)| \\ &\quad + \sum_{k=N+1}^{\infty} |(c_k f^{m_k} g_{n_k})(z_N)| \\ &< 3(1/2)^N \qquad \text{by (3.4), (3.5), and (3.14)} \\ &\to 0 \qquad \text{as } N \to \infty. \end{aligned}$$

This completes the proof.

4. Main results

By Fact 2.1(iii), a composition operator C_φ is an isolated point if and only if the connected component containing C_φ in $\mathcal{C}(H^\infty)$ consists of only C_φ. Our first main result is the following theorem which gives a function theoretic characterization of isolated points in $\mathcal{C}(H^\infty)$.

Theorem 4.1. *Let $\varphi \in \mathcal{S}(D)$. Then C_φ is isolated in $\mathcal{C}(H^\infty)$ if and only if $\int_0^{2\pi} \log(1 - |\varphi|)\, d\theta/2\pi = -\infty$.*

Proof. Suppose that $\int_0^{2\pi} \log(1 - |\varphi|)\, d\theta/2\pi = -\infty$. To prove that C_φ is isolated in $\mathcal{C}(H^\infty)$, suppose not. Then by Fact 2.1, there exists $\psi \in \mathcal{S}(D), \varphi \ne \psi$, such that $\sup_{z\in D} \rho(\varphi(z), \psi(z)) < 1$. Put

$$\sigma = \sup_{z\in D} \rho(\varphi(z), \psi(z)). \tag{4.1}$$

Then $0 < \sigma < 1$. Put

$$f = (\varphi + \psi)/2. \tag{4.2}$$

Then f is not an extreme point of the closed unit ball of H^∞. By de Leeuw and Rudin's theorem [2],

$$\int_0^{2\pi} \log(1 - |f|)\, d\theta/2\pi > -\infty. \tag{4.3}$$

By (4.1) and (4.2), the convexity of $\Delta(\varphi(z), \sigma)$ gives that $f(z) \in \Delta(\varphi(z), \sigma)$. By [3, p. 3], for $z \in D$ we have

$$\frac{|\varphi(z)| - \sigma}{1 - \sigma|\varphi(z)|} \le |f(z)|.$$

Hence

$$1 - |f| \le \frac{(1+\sigma)(1 - |\varphi|)}{1 - \sigma|\varphi|} \le \frac{1+\sigma}{1-\sigma}(1 - |\varphi|) \quad \text{on } D.$$

Therefore

$$\int_0^{2\pi} \log(1 - |f|)\, d\theta/2\pi \le \log\Big(\frac{1+\sigma}{1-\sigma}\Big) + \int_0^{2\pi} \log(1 - |\varphi|)\, d\theta/2\pi.$$

By our assumption, we get $\int_0^{2\pi} \log(1-|f|)\, d\theta/2\pi = -\infty$. This contradicts (4.3).

The converse is proved in [8, Corollary 9].

In [8], MacCluer, Ohno, and Zhao showed that C_φ and C_ψ are in the same connected component if $C_\varphi - C_\psi$ is compact. They also gave an example of $\varphi \in \mathcal{S}(D)$ that C_φ is not isolated but $C_\varphi - C_\psi$ is not compact for some C_ψ in the same component of C_φ. Here we show that this occurs for every non-isolated connected component in $\mathcal{C}(H^\infty)$, except the component consists of compact composition operators.

Examples. Let $\varphi \in \mathcal{S}(D)$. Suppose that C_φ is not isolated and $\|\varphi\|_\infty = 1$. Then there exist $\psi_1, \psi_2 \in \mathcal{S}(D)$ satisfying the following conditions:

(i) $\varphi \neq \psi_1$ *and* $\varphi \neq \psi_2$.
(ii) C_φ, C_{ψ_1} *and* C_{ψ_2} *are in the same component of* $\mathcal{C}(H^\infty)$.
(iii) $C_\varphi - C_{\psi_1}$ *is compact.*
(iv) $C_\varphi - C_{\psi_2}$ *is not compact.*

Proof. By Theorem 4.1, $\int_0^{2\pi} \log(1-|\varphi|)\, d\theta/2\pi > -\infty$. There exists an outer function $\omega \in H^\infty$ such that $|\omega| = 1 - |\varphi|$ a.e. on ∂D; see [6]. For each $z \in D$, let $P_z(\theta)$ be the Possion kernel at z. The values of ω and φ at z are given by

$$\omega(z) = \int P_z(\theta)\omega(\theta) d\theta$$

and

$$\varphi(z) = \int P_z(\theta)\varphi(\theta) d\theta,$$

respectively. Thus

$$|\omega(z)| + |\varphi(z)| \le \int P_z(\theta)[|\omega(\theta)| + |\varphi(\theta)|] d\theta \le 1 \quad \text{on } D. \tag{4.4}$$

Let $0 < t < 1$. Put $\psi_1 = \varphi + t\omega^2$. Then

(4.5)

$$\rho(\varphi(z), \psi_1(z)) \le \frac{|t\omega^2(z)|}{1 - |\varphi(z)|^2 - |t\omega^2(z)\overline{\varphi(z)}|} \le \frac{|t\omega(z)|}{1 + |\varphi(z)| - |t\omega(z)\overline{\varphi(z)}|}, \quad z \in D.$$

The last inequality is obtained by dividing the denominator and nominator by $|\omega(z)|$ and using (4.4). Suppose that $|\varphi(z_n)| \to 1$. Then by (4.4), $\omega(z_n) \to 0$. Hence by (4.5), $\rho(\varphi(z_n), \psi_1(z_n)) \to 0$. Next suppose that $|\psi_1(z_n)| \to 1$. Since

$$|\psi_1(z_n)| \le |\varphi(z_n)| + t|\omega(z_n)| \le |\varphi(z_n)| + |\omega(z_n)| \le 1,$$

we have

$$(1-t)|\omega(z_n)| \le 1 - |\psi_1(z_n)|.$$

Thus $(1-t)|\omega(z_n)| \to 0$ and $\omega(z_n) \to 0$. So $\rho(\varphi(z_n), \psi_1(z_n)) \to 0$. By Fact 2.2, $C_\varphi - C_{\psi_1}$ is compact.

Since $1 - |\varphi(e^{i\theta})| = |\omega(e^{i\theta})|$ and $\omega(e^{i\theta}) \neq 0$ for almost everywhere, $1 - |\varphi(e^{i\theta})| < \frac{|\omega(e^{i\theta})|}{t}$ for almost everywhere. Also by our assumption, the Lebesgue measure of the set $\{e^{i\theta}; r < |\varphi(e^{i\theta})| < 1\}$ is positive for every $r, 0 < r < 1$. Therefore there exists a sequence $\{z_n\}_n$ in D such that

$$1 \le \frac{1 - |\varphi(z_n)|}{|\omega(z_n)|} < \frac{1}{t} \quad \text{and} \quad |\varphi(z_n)| \to 1.$$

Moreover we may assume that

$$\frac{1-|\varphi(z_n)|}{\omega(z_n)} \to Re^{i\theta_1}, 1 \leq R \leq 1/t, \quad \text{and} \quad \varphi(z_n) \to e^{i\theta_2}. \tag{4.6}$$

Put $\theta_3 = \theta_1 + \theta_2$ and $\psi_2 = \varphi + te^{i\theta_3}\omega$. Then in the same way as above,

$$\rho(\varphi(z), \psi_2(z)) \leq \frac{t}{1 + |\varphi(z)| - |t\overline{\varphi(z)}|} \leq t < 1, \quad z \in D,$$

so that C_φ and C_{ψ_2} are in the same component. To prove that $C_\varphi - C_{\psi_2}$ is not compact, by Fact 2.2 it is sufficient to prove $\limsup_{|\varphi(z)|\to 1} \rho(\varphi(z), \psi_2(z)) > 0$. We have

$$\begin{aligned}
\rho(\varphi(z_n), \psi_2(z_n)) &= \Big|\frac{te^{i\theta_3}\omega(z_n)}{1 - |\varphi(z_n)|^2 - te^{i\theta_3}\omega(z_n)\overline{\varphi(z_n)}}\Big| \\
&\geq \Big|\frac{t}{\frac{1-|\varphi(z_n)|^2}{\omega(z_n)} - te^{i\theta_3}\overline{\varphi(z_n)}}\Big| \\
&\to \frac{t}{|2Re^{i\theta_1} - te^{i(\theta_3-\theta_2)}|} \qquad \text{by (4.6)} \\
&= \frac{t}{2R - t} \\
&\geq \frac{t^2}{2 - t^2} \qquad \text{by (4.6).}
\end{aligned}$$

Hence by Fact 2.2, $C_\varphi - C_{\psi_t}$ is not compact.

Lemma 4.2. *Let $\varphi, \psi \in \mathcal{S}(D)$ and $\varphi \neq \psi$. If C_φ and C_ψ are not contained in the same connected component in $\mathcal{C}(H^\infty)$, then $\|C_\varphi - C_\psi\|_e \geq 1$.*

Proof. By Fact 2.1(i), $\sup_{z\in D} \rho(\varphi(z), \psi(z)) = 1$. Then we may assume that there exists a sequence $\{z_n\}_n$ in D such that $|\varphi(z_n)| < |\varphi(z_{n+1})| \to 1$ and

$$\rho(\varphi(z_n), \psi(z_n)) \to 1. \tag{4.7}$$

Then $|z_n| \to 1$. By Theorem 3.1, we may assume that $\{\varphi(z_n)\}_n$ is asymptotically interpolating.

To prove our assertion, suppose that $\|C_\varphi - C_\psi\|_e < 1$. Take a positive number σ such that $\|C_\varphi - C_\psi\|_e < \sigma < 1$ and take a compact operator K on H^∞ such that

$$\|C_\varphi - C_\psi + K\| < \sigma < 1. \tag{4.8}$$

We claim that there are a Blaschke product b_0 and a subsequence $\{w_n\}_n$ of $\{z_n\}$ such that

$$b_0(\psi(w_n)) \to 0 \tag{4.9}$$

and

$$|b_0(\varphi(w_n))| \to 1. \tag{4.10}$$

Assume the claim first. Put $E = \{w_n\}_n$ and take a sequence of subsets $\{E_k\}_k$ of E such that

$$E_{k+1} \subset E_k \ \text{and} \ E_k \setminus E_{k+1} \ \text{is an infinite set for every } k. \tag{4.11}$$

Fix a positive integer k. Since $\{\varphi(w_n)\}_n$ is asymptotically interpolating, there exists $h_k \in H^\infty$ such that $\|h_k\|_\infty \leq 1$ and

$$|h_k(\varphi(w_n)) - \overline{b_0(\varphi(w_n))}| \to 0 \quad \text{as } |w_n| \to 1 \text{ and } w_n \in E_k \tag{4.12}$$

and

$$|h_k(\varphi(w_n)) + \overline{b_0(\varphi(w_n))}| \to 0 \quad \text{as } |w_n| \to 1 \text{ and } w_n \notin E_k. \tag{4.13}$$

Since $h_k b_0 \in H^\infty$ and $\|h_k b_0\|_\infty \le 1$, by (4.8)

$$|h_k(\varphi(w_n))b_0(\varphi(w_n)) - h_k(\psi(w_n))b_0(\psi(w_n)) + K(h_k b_0)(w_n)| < \sigma < 1.$$

Hence by (4.9), (4.10), (4.12), and (4.13),

$$|1 + K(h_k b_0)| \le \sigma < 1 \quad \text{on } cl\, E_k \setminus E_k \tag{4.14}$$

and

$$|-1 + K(h_k b_0)| \le \sigma < 1 \quad \text{on } cl\,(E \setminus E_k) \setminus (E \setminus E_k). \tag{4.15}$$

By (4.11), we have $cl\,(E_k \setminus E_{k+1}) \setminus (E_k \setminus E_{k+1}) \ne \emptyset$ for every k. Take a point ζ_k in $cl\,(E_k \setminus E_{k+1}) \setminus (E_k \setminus E_{k+1})$. By (4.11), $\zeta_n \in cl\, E_k \setminus E_k$ for every $n \ge k$. Hence by (4.14), $|1 + K(h_k b_0)(\zeta_n)| \le \sigma < 1$ for $n \ge k$. Let ζ_0 be a cluster point of $\{\zeta_k\}_k$. Then

$$|1 + K(h_k b_0)(\zeta_0)| \le \sigma < 1. \tag{4.16}$$

Since K is a compact operator on H^∞, considering a subsequence of $\{h_k\}_k$ we may assume that $\|K(h_k b_0) - F\|_\infty \to 0$ for some $F \in H^\infty$. By (4.16),

$$|1 + F(\zeta_0)| \le \sigma < 1. \tag{4.17}$$

By (4.11) again, $\zeta_n \in cl\,(E \setminus E_k) \setminus (E \setminus E_k)$ for $k > n$. Hence by (4.15),

$$|-1 + K(h_k b_0)(\zeta_n)| \le \sigma < 1 \quad \text{for } k > n.$$

Thus $|-1 + F(\zeta_n)| \le \sigma < 1$ for every n, so that $|-1 + F(\zeta_0)| \le \sigma < 1$. This contradicts (4.17).

In order to prove our claim we divide the proof into two cases.

Case 1. $\liminf_{n\to\infty} |\psi(z_n)| < 1$.

In this case, considering a subsequence of $\{z_n\}_n$ we may further assume that $\psi(z_n) \to a$ and $|a| < 1$. Let $b_0(z) = (z - a)/(1 - \overline{a}z), z \in D$. Then

$$b_0(\psi(z_n)) \to 0.$$

Since $|\varphi(z_n)| \to 1$,

$$|b_0(\varphi(z_n))| \to 1.$$

This proves the claim desired.

Case 2. $|\psi(z_n)| \to 1$.

Considering a subsequence of $\{z_n\}_n$, we may assume that $\{\psi(z_n)\}_n$ is a sparse sequence; see page 42 in [4]. Since $|\varphi(z_n)| \to 1$ and (4.7), we may further assume that

$$\rho(\varphi(z_n), \psi(z_j)) > 1 - 1/n \quad \text{and} \quad \rho(\varphi(z_j), \psi(z_n)) > 1 - 1/n \quad \text{for } 1 \le j \le n.$$

Then $\rho(\varphi(z_k), \{\psi(z_n)\}_n) \to 1$ as $k \to \infty$. Let b_0 be the sparse Blaschke product with zeros $\{\psi(z_n)\}_n$. Hence $|b_0(\varphi(z_k))| \to 1$; see [5]. Then the claim is true, too.

As pointed out in Section 1, we may introduce the essential norm topology on $\mathcal{C}(H^\infty)$. With this topology, we consider essentially connected components of $\mathcal{C}(H^\infty)$.

Theorem 4.3. *Let $\varphi, \psi \in \mathcal{S}(D)$. Then we have the following*:

(i) *Every connected component of $\mathcal{C}(H^\infty)$ is open and closed in the essential norm topology.*

(ii) *C_φ and C_ψ are in the same connected component if and only if C_φ and C_ψ are in the same essentially connected component.*

Proof. By Lemma 4.2, each connected component of $\mathcal{C}(H^\infty)$ is open and hence closed in the essential norm topology. Since the essential norm topology is weaker than the norm topology, we get our assertion.

In [8], MacCluer, Ohno, and Zhao posed the problem of whether every isolated composition operator in $\mathcal{C}(H^\infty)$ is essentially isolated. The following theorem answers this problem affirmatively.

Theorem 4.4. *C_φ is isolated in $\mathcal{C}(H^\infty)$ if and only if C_φ is essentially isolated.*

Proof. This follows from Theorem 4.3(i).

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References

1. C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995. MR **97i:**47056
2. K. deLeeuw and W. Rudin, *Extreme points and extremum problems in H^1*, Pacific J. Math. **8**(1958), 467-485. MR **20:**5426
3. J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981. MR **83g:**30037
4. P. Gorkin, *Decompositions of the maximal ideal space of L^∞,* Trans. Amer. Math. Soc. **282** (1984), no. 1, 33–44. MR **85a:**46028
5. P. Gorkin, H.-M. Lingenberg, and R. Mortini, *Homeomorphic disks in the spectrum of H^∞*, Indiana Univ. Math. J. **39**(1990), 961-983. MR **92b:**46082
6. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, N.J., 1962. MR **24:**A2844
7. B. MacCluer, *Components in the space of composition operators*, Integral Equation Operator Theory **12**(1989), 725-738. MR **91b:**47070
8. B. MacCluer, S. Ohno, R. Zhao, *Topological structure of the space of composition operators on H^∞*, Integral Equation Operator Theory **40** (2000), 481–494.
9. J. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993. MR **94k:**47049
10. J. Shapiro and C. Sundberg, *Isolation amongst the composition operators*, Pacific J. Math. **145**(1990), 117-152. MR **92g:**47041
11. K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990. MR **92c:**47031

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