**m-BEREZIN TRANSFORM AND COMPACT OPERATORS**

KYEOOK NAM, DECHAO ZHENG, AND CHANGYONG ZHONG

**ABSTRACT.** $m$-Berezin transforms are introduced for bounded operators on the Bergman space of the unit ball. The norm of the $m$-Berezin transform as a linear operator from the space of bounded operators to $L^\infty$ is found. We show that the $m$-Berezin transforms are commuting with each other and Lipschitz with respect to the pseudo-hyperbolic distance on the unit ball. Using the $m$-Berezin transforms we show that a radial operator in the Toeplitz algebra is compact iff its Berezin transform vanishes on the boundary of the unit ball.

1. **INTRODUCTION**

Let $B$ denote the unit ball in $n$-dimensional complex space $\mathbb{C}^n$ and $dz$ be normalized Lebesgue volume measure on $B$. The Bergman space $L^2_a = L^2_a(B, dz)$ is the space of analytic functions $h$ on $B$ which are square-integrable with respect to Lebesgue volume measure. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $\langle z, w \rangle = \sum_{i=1}^n z_i \overline{w}_i$ and $|z|^2 = \langle z, z \rangle$.

For $z \in B$, let $P_z$ be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[z]$ generated by $z$ and let $Q_z = I - P_z$. Then

$$\phi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{1/2}Q_z(w)}{1 - \langle w, z \rangle}$$

is the automorphism of $B$ that interchanges 0 and $z$. The pseudo-hyperbolic metric on $B$ is defined as $\rho(z, w) = |\phi_z(w)|$.

The reproducing kernel in $L^2_a$ is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}},$$

for $z, w \in B$ and the normalized reproducing kernel $k_z$ is $K_z(w)/\|K_z(\cdot)\|_2$. If $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2$, then $\langle h, K_z \rangle = h(z)$, for every $h \in L^2_a$ and $z \in B$. The fundamental property of the reproducing kernel $K_z(w)$ plays an important role in this paper:

$$K_z(w) = k_\lambda(z)K_{\phi_\lambda(z)}(\phi_\lambda(w))k_\lambda(w). \quad (1.1)$$

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Given \( f \in L^\infty \), the Toeplitz operator \( T_f \) is defined on \( B \) by \( T_f h = P(fh) \) where \( P \) denotes the orthogonal projection \( P \) of \( L^2 \) onto \( L^2_a \).

Let \( \mathcal{L}(L^2_a) \) be the algebra of bounded operators on \( L^2_a \). The Toeplitz algebra \( \mathcal{T}(L^\infty) \) is the closed subalgebra of \( \mathcal{L}(L^2_a) \) generated by \( \{T_f : f \in L^\infty\} \).

For \( f \in B \), let \( U_z \) be the unitary operator given by
\[
U_z f = (f \circ \phi_z) \cdot J\phi_z
\]
where \( J\phi_z = (-1)^n k_z \). For \( S \in \mathcal{L}(L^2_a) \), set
\[
S_z = U_zSU_z.
\]

Since \( U_z \) is a selfadjoint unitary operator on \( L^2 \) and \( L^2_a \), \( U_zT_jU_z = T_{f_j}\phi_z \) for every \( f \in L^\infty \).

Let \( T \) denote the class of trace operators on \( L^2_a \). For \( T \in T \), we will denote the trace of \( T \) by \( tr[T] \) and let \( \|T\|_{C^1} \) denote the \( C^1 \) norm of \( T \) given by \((12)\)
\[
\|T\|_{C^1} = tr[\sqrt{T^*T}].
\]

Suppose \( f \) and \( g \) are in \( L^2_a \). Consider the operator \( f \otimes g \) on \( L^2_a \) defined by
\[
(f \otimes g)h = (h, g)f,
\]
for \( h \in L^2_a \). It is easily proved that \( f \otimes g \) is in \( T \) and with norm equal to \( \|f \otimes g\|_{C^1} = \|f\|_2 \|g\|_2 \) and
\[
tr[f \otimes g] = (f, g).
\]

For a nonnegative integer \( m \), the \( m \)-Berezin transform of an operator \( S \in \mathcal{L}(L^2_a) \) is defined by
\[
B_mS(z) = C_m^{m+n} tr \left[ S_z \left( \sum_{|k| = 0}^{m} C_{m,k} \frac{n!k!}{(n+|k|)!} \|u^k\| \|u^k\| \right) \right] \quad (1.2)
\]
\[
= C_m^{m+n} tr \left[ S_z \left( \sum_{|k| = 0}^{m} C_{m,k} u^k \otimes u^k \right) \right]
\]
where \( k = (k_1, \ldots, k_n) \in N^n \), \( N \) is the set of nonnegative integers, \( |k| = \sum_{i=0}^{n} k_i \), \( u^k = u_1^{k_1} \cdots u_n^{k_n} \), \( k! = k_1! \cdots k_n! \),
\[
C_m^{m+n} = \binom{m+n}{n} \quad \text{and} \quad C_{m,k} = C_{|k|}^{m}(-1)^{|k|} \frac{|k|!}{k_1! \cdots k_n!}.
\]

Clearly, \( B_m : \mathcal{L}(L^2_a) \to L^\infty \) is a bounded linear operator, the norm of \( B_m \) will be given.

Given \( f \in L^\infty \), define
\[
B_m(f)(z) = B_m(T_f)(z).
\]
$B_m(f)(z)$ equals the nice formula in [1]:

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u),$$

for $z \in B$ where $d\nu_m(u) = C_n^{m+n}(1-|u|^2)^mdu$.

Berezin first introduced the Berezin transform $B_0(S)$ of bounded operators $S$ and the $m$-Berezin transform of functions in [5]. Because the Berezin transform encodes operator-theoretic information in function-theory in a striking but somewhat impenetrable way, the Berezin transform $B_0(S)$ has found useful applications in studying operators of “function-theoretic significance” on function spaces ([2], [3], [4], [6], [7], [11], and [15]). Suarez [16] introduced $m$-Berezin transforms of bounded operators on the Bergman space of the unit disk. We will show that our $m$-Berezin transform coincides with the one defined in [16] on the unit disk $D$ by means of an integral representation of $m$-Berezin transform. The integral representation shows that many useful properties of the $m$-Berezin transforms inherit from the identity (1.1) of the reproducing kernel. On the unit ball, some useful properties of the $m$-Berezin transforms of functions were obtained by Ahern, Flores and Rudin [1]. Recently, Coburn [10] proved that $B_0(S)$ is Lipschitz with respect to the pseudo-hyperbolic distance $\rho(z,w)$. In this paper, we will show that $B_mS(z)$ is Lipschitz with respect to pseudo-hyperbolic distance $\rho(z,w)$. We will show that the $m$-Berezin transforms $B_m$ are invariant under the Mobious transform,

$$B_m(Sz) = (B_mS) \circ \phi_z,$$

and commuting with each other,

$$B_j(B_mS)(z) = B_m(B_jS)(z)$$

for any nonnegative integers $j$ and $m$. Properties (1.3) and (1.4) were obtained for $S = T_f$ in [1] and for operators $S$ on the Bergman space of the unit disk [16].

A common intuition is that for operators on the Bergman space $L^2_a$ “closely associated with function theory”, compactness is equivalent to having vanishing Berezin transform on the boundary of the unit ball $B$. On the unit disk, Axler and Zheng [2] showed that if the operator $S$ equals the finite sum of finite products of Toeplitz operators with bounded symbols then $S$ is compact if and only if $B_0(S)(z) \to 0$ as $z \to \partial D$. Englis extended this result to the unit ball even the bounded symmetric domains [11]. But the problem remains open whether the result is true if $S$ is in the Toeplitz algebra. Recently, Suarez [17] solved the problem for radial operator $S$ on the unit disk via the $m$-Berezin transform. Using the $m$-Berezin transform, we will show that for a radial operator $S$ in the Toeplitz algebra on the unit ball, $S$ is compact iff $B_0S(z) \to 0$ as $|z| \to 1$.

Throughout the paper $C(m,n)$ will denote constant depending only on $m$ and $n$, which may change at each occurrence.

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2. \( m \)-BEREZIN TRANSFORM

In this section we will show some useful properties of the \( m \)-Berezin transform. First we give an integral representation of the \( m \)-Berezin transform \( B_m(S) \).

For \( z \in B \) and a nonnegative integer \( m \), let

\[
K^m_z(u) = \frac{1}{(1 - \langle u, z \rangle)^{m+n+1}}, \quad u \in B.
\]

For \( u, \lambda \in B \), we can easily see that

\[
\sum_{|k|=0}^{m} C_{m,k} u^k \lambda^k = (1 - \langle u, \lambda \rangle)^m. \quad (2.1)
\]

**Proposition 2.1.** Let \( S \in \mathcal{L}(L^2_\alpha) \), \( m \geq 0 \) and \( z \in B \). Then

\[
B_m S(z) = C^{m+n}_n (1 - |z|^2)^{m+n+1} \times \int_B \int_B (1 - \langle u, \lambda \rangle)^m K^m_z(u) K^m_z(\lambda) S^* K_\lambda(u) dud\lambda.
\]

**Proof.** For \( \lambda \in B \), the definition of \( B_m \) implies

\[
B_m S(z) = C^{m+n}_n \sum_{|k|=0}^{m} C_{m,k} \langle S z^k, \lambda^k \rangle
\]

\[
= C^{m+n}_n \sum_{|k|=0}^{m} C_{m,k} \int_B S(\phi^k_z)(\lambda) \overline{\phi^k_z(\lambda)} d\lambda
\]

\[
= C^{m+n}_n \sum_{|k|=0}^{m} C_{m,k} \int_B \int_B \phi^k_z(u) K^m_z(u) \overline{K^m_z(\lambda)} S^* K_\lambda(u) dud\lambda \quad (2.2)
\]

where the last equality holds by \( S(\phi^k_z)(\lambda) = \langle S(\phi^k_z), K_\lambda \rangle = \langle \phi^k_z, S^* K_\lambda \rangle \). Using (2.1) and (1.1), (2.2) equals

\[
C^{m+n}_n \int_B \int_B (1 - \langle \phi_z(u), \phi_z(\lambda) \rangle)^m K^m_z(u) K^m_z(\lambda) S^* K_\lambda(u) dud\lambda
\]

\[
= C^{m+n}_n \int_B \int_B \left( \frac{k^m_z(u) k^m_z(\lambda)}{K^m_z(u) K^m_z(\lambda)} \right)^{m/(n+1)} K^m_z(u) K^m_z(\lambda) S^* K_\lambda(u) dud\lambda
\]

\[
= C^{m+n}_n (1 - |z|^2)^{m+n+1} \int_B \int_B (1 - \langle u, \lambda \rangle)^m K^m_z(u) K^m_z(\lambda) S^* K_\lambda(u) dud\lambda
\]

as desired. \( \square \)

Proposition 2.2 gives another form of \( B_m \).
Proposition 2.2. Let $S \in \mathcal{L}(L_a^2)$, $m \geq 0$ and $z \in B$. Then

$$B_m S(z) = C_n^{m+n}(1 - |z|^2)^m C_n^{m+n+1} \sum_{|k|=0}^m C_m,k \langle S(u^k K_z^m), u^k K_z^m \rangle. \quad (2.3)$$

Proof. Since

$$\int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) K_z^m(\lambda) S(\lambda)du d\lambda = \sum_{|k|=0}^m C_m,k \int_B \int_B u^k L_z^m(u) K_z^m(\lambda) S(\lambda) K_z^m(\lambda)du d\lambda,$$

Proposition 2.1 implies (2.3).

For $n = 1$, the right hand side of (2.3) was used by Suarez in [16] to define the $m$-Berezin transforms on the unit disk.

Recall that given $f \in L^\infty$, define

$$B_m(f)(z) = B_m(T_f)(z).$$

The following proposition gives a nice formula of $B_m(f)(z)$. Let $d\nu_m(u) = C_n^{m+n}(1 - |u|^2)^m du$.

Proposition 2.3. Let $z \in B$ and $f \in L^\infty$. Then

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u).$$

Proof. By the change of variables, Theorem 2.2.2 in [14] and (2.3), we have

$$\int_B f \circ \phi_z(u) d\nu_m(u) = C_n^{m+n} \int_B f(u) \left( \frac{(1 - |z|^2)(1 - |u|^2)}{|1 - \langle u, z \rangle|^2} \right)^m \left( \frac{(1 - |z|^2)}{|1 - \langle u, z \rangle|^2} \right)^{n+1} \frac{du}{d\nu_m(u)}$$

$$= C_n^{m+n}(1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_m,k \int_B f(u)|u^k|^2 K_z^m(u)du$$

$$= C_n^{m+n}(1 - |z|^2)^{m+n+1} \sum_{|k|=0}^m C_m,k \langle T_f(u^k K_z^m), u^k K_z^m \rangle = B_m(T_f)(z).$$

The proof is complete.

The formula in the above proposition was used in [1] to define the $m$-Berezin transform of functions $f$. 
Clearly, (1.2) gives \(|B_n S|_\infty \leq C(m, n)\|S\| = C(m, n)\|S\|\) for \(S \in \mathfrak{L}(L^2_n)\). Thus, \(B_m : \mathfrak{L}(L^2_n) \to L^\infty\) is a bounded linear operator. The following theorem gives the norm of \(B_m\).

**Theorem 2.4.** Let \(m \geq 0\). Then \(\|B_m\| = C_{n+m} \sum_{|k|=0}^{m} |C_{m,k}| \frac{n!k!}{(n+|k|)!}\).

**Proof.** From [8], we have the duality result \(\mathfrak{L}(L^2_n) = T^*\). So, the definition of \(B_m\) gives the norm of \(B_m\). In fact,

\[
\|B_m\| = \left\| C_{n+m} \sum_{|k|=0}^{m} C_{m,k} \frac{n!k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \frac{u^k}{\|u^k\|} \right\|_{C_1}
= C_{n+m} \sum_{|k|=0}^{m} |C_{m,k}| \frac{n!k!}{(n+|k|)!}
\]

as desired. \(\square\)

The Mobius map \(\phi_z(w)\) has the following property ([14]):

\[
\phi_z'(0) = -(1 - |z|^2)P_z - (1 - |z|^2)^{1/2}Q_z.
\] (2.4)

To show that \(m\)-Berezin transforms are Lipschitz with respect to the pseudo-hyperbolic distance we need the following lemmas.

For \(z, w \in \mathbb{C}^n\), \(z \hat{\otimes} w\) on \(\mathbb{C}^n\) is defined by \((z \hat{\otimes} w)\lambda = \langle \lambda, w \rangle z\).

**Lemma 2.5.** Let \(z, w \in B\) and \(\lambda = \phi_z(w)\). Then

\[
\phi'_z(w) = (1 - \langle \lambda, z \rangle)(I - \lambda \hat{\otimes} z)[\phi'_z(0)]^{-1}.
\]

**Proof.** Suppose that \(P_z\) and \(Q_z\) have the matrix representations as \(((P_z)_{ij})\) and \(((Q_z)_{ij})\) under the standard base of \(\mathbb{C}^n\), respectively. In fact,

\[
(P_z)_{ij} = \frac{z_i \bar{z}_j}{|z|^2} \quad \text{if} \quad z \neq 0.
\]

Let \((a_{ij}(w)) = \phi'_z(w)\). Write \(\phi_z(w) = (f_1(w), \cdots, f_n(w))\). Then

\[
a_{ij}(w) = \frac{\partial f_i}{\partial w_j}(w).
\]

Noting that

\[
f_i(w) = \frac{z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i}{1 - \langle w, z \rangle},
\]

we have

\[
a_{ij}(w) = \frac{(z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i) \bar{z}_j}{(1 - \langle w, z \rangle)^2} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z \rangle} \frac{f_i(w) \bar{z}_j}{1 - \langle w, z \rangle} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z \rangle} \frac{f_i(w) \bar{z}_j}{1 - \langle w, z \rangle}.
\]
Let \( \lambda = \phi_z(w) \). The above equality becomes

\[
a_{ij}(w) = \frac{\lambda_i z_j - ((P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij})}{1 - \langle w, z \rangle}
\]

Thus

\[
\phi'_z(w) = \frac{\lambda \hat{\otimes} z - (P_z + (1 - |z|^2)^{1/2}Q_z)}{1 - \langle w, z \rangle}
\]

From Theorem 2.2.5 in [14], we have

\[
1 - \langle w, z \rangle = 1 - \langle \lambda, z \rangle
\]

Thus (2.4) implies

\[
\phi'_z(w) \phi'_z(0) = \frac{-((1 - |z|^2)\lambda \hat{\otimes} z + (1 - |z|^2)P_z + (1 - |z|^2)Q_z)}{1 - \langle w, z \rangle}
\]

\[
= \frac{(1 - |z|^2)(-\lambda \hat{\otimes} z + I)}{1 - \langle w, z \rangle}
\]

\[
= (1 - \langle \lambda, z \rangle)(I - \lambda \hat{\otimes} z)
\]

where the first equality follows from \( P_zQ_z = Q_zP_z = 0 \), \( P_zz = z \), and \( Q_zz = 0 \). The proof is complete. \( \square \)

**Lemma 2.6.** Suppose \( |z| > 1/2 \) and \( |w| > 1/2 \). If \( |\phi_z(w)| \leq \epsilon < 1/2 \), then

\[
\|P_z - P_w\| \leq 50\epsilon(1 - |z|^2)^{1/2}.
\]

**Proof.** First we will get the estimate of the distance between \( z \) and \( w \). Since \( |\phi_z(w)| \leq \epsilon < 1/2 \), \( w \) is in the ellipsoid:

\[
\phi_z(\epsilon B) = \{ w \in B : \frac{|P_zw - c|^2}{\epsilon^2 \rho^2} + \frac{|Q_zw|^2}{\epsilon^2 \rho} < 1 \}
\]

with center \( c = \frac{(1 - \epsilon^2)z}{1 - \epsilon^2 |z|^2} \) and \( \rho = \frac{1 - |z|^2}{1 - \epsilon^2 |z|^2} \). Noting that \( |z| > 1/2 \) and \( \epsilon < 1/2 \), we have \( \rho \leq 2(1 - |z|^2) \). Thus

\[
|Q_zw|^2 \leq \epsilon^2 \rho \leq 2 \epsilon^2 (1 - |z|^2), \quad |P_zw - c| \leq \epsilon \rho \leq 2 \epsilon (1 - |z|^2)
\]

and

\[
|z - c| \leq \frac{\epsilon^2 (1 - |z|^2)}{1 - \epsilon^2 |z|^2} \leq 2 \epsilon^2 (1 - |z|^2).
\]

So, we have

\[
|P_zw - z| \leq |P_zw - c| + |z - c| \leq 3 \epsilon (1 - |z|^2).
\]

Because \( I = P_z + Q_z \) and \( P_zQ_z = 0 \), writing

\[
(z - w) = P_z(z - w) + Q_z(z - w),
\]
we have
\[ |z - w|^2 = |P_z(z - w)|^2 + |Q_z(z - w)|^2 \]
\[ = |P_z w - z|^2 + |Q_z w|^2 \]
\[ \leq 11\epsilon^2(1 - |z|^2). \] (2.5)

Noting that
\[ \begin{aligned}
\frac{z}{|z|} \otimes z &= \frac{(z - w)}{|z|} \otimes z + \frac{w}{|z|} \otimes \left( \frac{z - w}{|z|} \right) + \\
&\quad \left[ \left( \frac{1}{|z|^2} - \frac{1}{|w|^2} \right) w \right] \otimes w + \frac{w}{|w|} \otimes \frac{w}{|w|},
\end{aligned} \]
we have
\[ P_z - P_w = \frac{(z - w)}{|z|} \otimes z + \frac{w}{|z|} \otimes \left( \frac{z - w}{|z|} \right) + \left[ \left( \frac{1}{|z|^2} - \frac{1}{|w|^2} \right) w \right] \otimes w, \]
to obtain
\[ \|P_z - P_w\| \leq \frac{|z - w|}{|z|} + 2\frac{|z - w|}{|z|} + \frac{||z|^2 - |w|^2|}{|z|^2} \]
\[ \leq 2|z - w| + 4|z - w| + 8|z - w| \]
\[ \leq 4\sqrt{11}\epsilon(1 - |z|^2)^{1/2} \]
\[ \leq 50\epsilon(1 - |z|^2)^{1/2} \]
where the last inequality holds by (2.5).

For given \( z, w \in B \), set \( A(z, w) = -(1 - |z|^2)P_w - (1 - |z|^2)^{1/2}Q_w \).

**Lemma 2.7.** Suppose \( |z| > 1/2 \) and \( |w| > 1/2 \). If \( |\phi_z(w)| \leq \epsilon < 1/2 \), then
\[ \|\phi'_z(0) - A(z, w)\| \leq 150\epsilon(1 - |z|^2). \]

**Proof.** Using (2.4), we have
\[ \|\phi'_z(0) - A(z, w)\| = \|(1 - |z|^2)(P_w - P_z) + (1 - |z|^2)^{1/2}(P_z - P_w)\| \]
\[ \leq 3(1 - |z|^2)^{1/2}\|P_z - P_w\| \]
\[ \leq 150\epsilon(1 - |z|^2) \]
as desired. The last inequality follows from Lemma 2.6.

Let \( U(n) \) be the group of \( n \times n \) complex unitary matrices.

**Lemma 2.8.** Let \( z, w \in B \). Then \( U_z U_w = V_{U\phi_w(z)} \) where
\[ (V_{U\phi})(u) = f(U\phi)\)d\text{e}U \]
for \( f \in L_a^2 \) and \( U = \phi_{\phi_w(z)} \circ \phi_w \circ \phi_z \) satisfying
\[ \|I + U\| \leq C(n)\rho(z, w). \]
Proof. The map \( \phi_{\phi(w(z))} \circ \phi_w \circ \phi_z \) is an automorphism of \( B \) that fixes 0, hence it is unitary by the Cartan theorem in [14]. Thus \( \phi_w \circ \phi_z = \phi_{\phi(w(z))} \circ \mathcal{U} \) for some \( \mathcal{U} \in \mathfrak{U}(n) \). Since \( \phi_w \) is an involution, we have
\[
U_z U_w f(u) = (f \circ \phi_w \circ \phi_z)(u) J \phi_w(\phi_z(u)) J \phi_z(u)
\]
\[
= (f \circ \phi_{\phi(w(z))}(\mathcal{U} u) J \phi_w(\phi_w(\phi_{\phi(w(z))}(\mathcal{U} u))) J \phi_w(\phi_{\phi(w(z))}(\mathcal{U} u)) \det \mathcal{U}
\]
\[
= (f \circ \phi_{\phi(w(z))}(\mathcal{U} u) J \phi_{\phi(w(z))}(\mathcal{U} u)) \det \mathcal{U}
\]
\[
= V \mathcal{U} U_{\phi(w(z))} f(u)
\]
as desired.

Now we will show that
\[
\| I + \mathcal{U} \| \leq C(n) \rho(z, w).
\]
Noting that \( \mathcal{U} \) is continuous for \( |z| \leq 1/2 \) and \( |w| \leq 1/2 \), we need only to prove
\[
\| I + \mathcal{U} \| \leq 20000 \rho(z, w),
\]
for \( |z| > 1/2, |w| > 1/2 \) and \( |\phi_w(z)| < 1/2 \). Let \( \lambda = \phi_w(z) \). Then \( |\lambda| = \rho(z, w) \) and \( z = \phi_w(\lambda) \). Since
\[
\phi_w \circ \phi_z(u) = \phi_{\lambda}(\mathcal{U} u),
\]
taking derivatives both sides of the above equations and using the chain rule give
\[
\phi_w'(\phi_z(u)) \phi_z'(u) = \phi_{\lambda}'(\mathcal{U} u) \mathcal{U}.
\]
Letting \( u = 0 \), the above equality gives
\[
\mathcal{U} = [\phi_{\lambda}'(0)]^{-1} \phi_w'(z) \phi_z'(0).
\]
By Lemma 2.5, write
\[
\mathcal{U} + I = [\phi_{\lambda}'(0)]^{-1} (1 - \langle \lambda, w \rangle) (I - \lambda \hat{\otimes} w) [\phi_w'(0)]^{-1} \phi_z'(0) + I
\]
\[
= [\phi_{\lambda}'(0)]^{-1} (1 - \langle \lambda, w \rangle) (I - \lambda \hat{\otimes} w) [\phi_w'(0)]^{-1} [\phi_z'(0) - A(z, w)]
\]
\[
+ ([\phi_{\lambda}'(0)]^{-1} (1 - \langle \lambda, w \rangle) (I - \lambda \hat{\otimes} w) [\phi_w'(0)]^{-1} A(z, w) + I)
\]
\[
:= I_1 + I_2.
\]
By Lemma 2.7, we have
\[
\| I_1 \| \leq \|[\phi_{\lambda}'(0)]^{-1} (1 - \langle \lambda, w \rangle) (I - \lambda \hat{\otimes} w) [\phi_w'(0)]^{-1} [\phi_z'(0) - A(z, w)\|
\]
\[
\leq 4 \times 2 \times 2 \times \frac{3}{(1 - |w|^2)} [150|\lambda| (1 - |z|^2)].
\]
Theorem 2.2.2 in [14] leads to
\[
\frac{1 - |z|^2}{1 - |w|^2} = \frac{1 - |\lambda|^2}{|1 - \langle \lambda, w \rangle|^2}.
\]
Thus
\[
\| I_1 \| \leq 4 \times 2 \times 3 \times 2 \times 150|\lambda| = 14400|\lambda|.
\]
Also, we have
\[
\left| 1 - \frac{(1 - |z|^2)^{1/2}}{(1 - |w|^2)^{1/2}} \right| \leq \left| 1 - \frac{1 - |z|^2}{1 - |w|^2} \right| \leq 32|\lambda|.
\]
Hence, we get
\[
\left\| I - \frac{1 - |z|^2}{1 - |w|^2} P_w - \frac{(1 - |z|^2)^{1/2}}{(1 - |w|^2)^{1/2}} Q_w \right\| \leq 32|\lambda|.
\]
On the other hand, clearly,
\[
\| \phi'_\lambda(0)^{-1} + I \| \leq 4|\lambda|, \quad |(1 - \langle \lambda, w \rangle) - 1| \leq |\lambda|
\]
and
\[
\| (I - \lambda \hat{\otimes} w) - I \| \leq |\lambda|.
\]
These give
\[
\| I + \phi'_\lambda(0)^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w) \| \leq 16|\lambda|.
\]
Hence, we have
\[
\| I_2 \| \leq \| \phi'_\lambda(0)^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\phi'_w(0)^{-1} A(z, w)
\]
\[
- [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| + \| \phi'_\lambda(0)^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w) + I \|
\]
\[
\leq \| \phi'_\lambda(0)^{-1}(1 - \langle \lambda, w \rangle)(I - \lambda \hat{\otimes} w)\| \left\| I - \frac{1 - |z|^2}{1 - |w|^2} P_w - \frac{(1 - |z|^2)^{1/2}}{(1 - |w|^2)^{1/2}} Q_w \right\|
\]
\[
+ 16|\lambda|
\]
\[
\leq 4 \times 2 \times 2 \times 32|\lambda| + 16|\lambda| < 600|\lambda|.
\]
Combining the above estimates we conclude that
\[
\| U + I \| \leq 14400|\lambda| + 600|\lambda| < 20000|\lambda|.
\]
\[\square\]

**Theorem 2.9.** Let $S \in \mathcal{L}(L^2_a)$, $m \geq 0$ and $z \in B$. Then $B_m S_z = (B_m S) \circ \phi_z$.

**Proof.** Proposition 2.2 and (1.2) give
\[
B_m S_z(0) = C^{m+n}_{n} \sum_{|k|=0}^{m} C_{m,k} \langle S_z u^k, u^k \rangle = B_m S(z) = (B_m S) \circ \phi_z(0).
\]
For any \( w \in B \), Proposition 2.1 and Lemma 2.8 imply
\[
(B_m S_z) \circ \phi_w(0) = B_m((S_z)_w)(0) = C_m + n \int_B \int_B (1 - \langle u, \lambda \rangle)^m S^*U_z^*U_w K(\lambda(u))dud\lambda = B_m S_z(0) - C_m + n \sum_{i=1}^n \sum_{|k|=0}^m (ST_{u_i}(u^k), T_{u_i}(u^k))d\lambda
\]
where \( V_{\phi} \) is in Lemma 2.8. Thus, \( B_m S_z = (B_m S) \circ \phi_z \).

Lemma 2.10. Let \( S \in L(L_2^2) \), \( m \geq 1 \) and \( z \in B \). Then
\[
B_m S(z) = \frac{m + n}{m} B_{m-1} \left( S - \sum_{i=1}^n T_{(\phi_z),i}ST_{(\phi_z),i} \right)(z)
\]
where \( (\phi_z)_i \) is \( i \)-th variable of \( \phi_z \).

Proof. By Theorem 2.9, we just need to show that
\[
B_m S(0) = \frac{m + n}{m} B_{m-1} \left( S - \sum_{i=1}^n T_{u_i}ST_{u_i} \right)(0).
\]
Using Proposition 2.1 and (2.1), we get
\[
B_m S(0) = C_m + n \int_B \int_B (1 - \langle u, \lambda \rangle)^m S^*K\lambda(u)dud\lambda = \frac{m + n}{m} B_{m-1} S(0) - C_m + n \sum_{i=1}^n \sum_{|k|=0}^m (S_{u_i}(u^k), \lambda^k \lambda_i)d\lambda
\]
as desired.

For \( m = 0 \), the following result was obtained in [10].

Theorem 2.11. Let \( S \in L(L_2^2) \) and \( m \geq 0 \). Then there exists a constant \( C(m, n) > 0 \) such that
\[
|B_m S(z) - B_m S(w)| < C(m, n)\|S\|\rho(z, w).
\]
Proof. We will prove this theorem by induction on \( m \). If \( m = 0 \), (1.2) gives

\[
|B_0S(z) - B_0S(w)| = |tr[S_z(1 \otimes 1)] - tr[S_w(1 \otimes 1)]|
= |tr[S_z(1 \otimes 1) - SU_w(1 \otimes 1)U_w]|
= |tr[S_z(1 \otimes 1) - SU_z(U_zU_w1 \otimes U_w1)U_w]|
\]

From Lemma 2.8, the last term equals

\[
|tr[S_z(1 \otimes 1 - U_{\phi_w}(z)1 \otimes U_{\phi_w}(z))]| \leq S_z\|1 \otimes 1 - U_{\phi_w}(z)1 \otimes U_{\phi_w}(z)\|^{1/2}
\leq \sqrt{2}\|S_z(2 - |(1, k_{\phi_w}(z))|^2)^{1/2}
= 2\|S\|[1 - \|\phi_w(z)\|^2]^{n+1})^{1/2}
\leq C(n)\|S\|\|\phi_w(z)\|
\]

where the second equality holds by \( \|T\|^{1/2} \leq \sqrt{l}(tr[T^*T])^{1/2} \) where \( l \) is the rank of \( T \).

Suppose \( |B_{m-1}S(z) - B_{m-1}S(w)| < C(m, n)\|S\|\rho(z, w) \). By Lemma 2.10, we have

\[
|B_{m}S(z) - B_{m}S(w)|
\leq \frac{m+n}{m} |B_{m-1}S(z) - B_{m-1}S(w)|
+ \frac{m+n}{m} \sum_{i=1}^{n} \left| B_{m-1} \left( \frac{T_{(\phi_z)_{1,i}} ST_{(\phi_z)_{1,i}}}{(\phi_z)_{1,i}} \right) (z) - B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (w) \right|
\]

Since the term in the summation is less than or equals

\[
\left| B_{m-1} \left( \frac{T_{(\phi_z)_{1,i}} ST_{(\phi_z)_{1,i}}}{(\phi_z)_{1,i}} \right) (z) - B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (z) \right|
+ \left| B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (z) - B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (z) \right|
+ \left| B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (z) - B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (w) \right|
\]

it is sufficient to show that

\[
\left| B_{m-1} \left( \frac{T_{(\phi_z)_{1,i}} ST_{(\phi_z)_{1,i}}}{(\phi_z)_{1,i}} \right) (z) - B_{m-1} \left( \frac{T_{(\phi_w)_{1,i}} ST_{(\phi_w)_{1,i}}}{(\phi_w)_{1,i}} \right) (z) \right| < C(m, n)\|S\|\rho(z, w).
\]
Lemma 2.8 gives
\[
\left| B_{m-1} \left( T_{\phi_w, \phi_w \circ \phi_z} ST_{\phi_z} \right) (z) \right|
\]
\[
= C_n^{m+n-1} \left| tr \left( \left( T_{\phi_w, \phi_w \circ \phi_z} ST_{\phi_z} \right) \right) z \sum_{|k|=0}^{m-1} C_{m-1,k} \left( \frac{n!k!}{(n+|k|)!} \left\| u^k \right\| \otimes \left\| u^k \right\| \right) \right|
\]
\[
\leq C_n^{m+n-1} \sum_{|k|=0}^{m-1} \left| C_{m-1,k} \frac{n!k!}{(n+|k|)!} \right| \left| \left\langle S_z T_{\phi_z} \phi_z \left\| u^k \right\| \otimes \left\| u^k \right\| \right\rangle \right|
\]
\[
\leq C(m, n) \left\| S_z \right\| \left\| T_{\phi_w, \phi_w \circ \phi_z} \left\| u^k \right\| \right\| _2 \|. \quad (2.6)
\]

Let \( \lambda = \phi_w(z) \). Then
\[
\left\| T_{\phi_w, \phi_w \circ \phi_z} \left\| u^k \right\| \right\| _2^2 \leq \int_B \left| (\phi_z \circ \phi_w)_{\lambda} (u) - (\phi_w \circ \phi_z)_{\lambda} (u) \right|^2 du
\]
\[
= \int_B \left| (U)_{\lambda} (u) - (\phi_{\lambda})_{\lambda} (u) \right|^2 du
\]
\[
\leq 2 \int_B \left| (U)_{\lambda} + u \right|^2 + \left| u + (\phi_{\lambda})_{\lambda} (u) \right|^2 du
\]
where \( \phi_w \circ \phi_z = \phi_{\lambda} \circ U \) for some \( U \in U(n) \).

Noting that
\[
\phi_{\lambda} (u) + u = \frac{\lambda - \langle u, \lambda \rangle u + |1 - (1 - |\lambda|^2)^{1/2} |Q_{\lambda} (u)}{1 - \langle u, \lambda \rangle},
\]
we have that for \( |\lambda| \leq 1/2 \),
\[
|\phi_{\lambda} (u) + u| \leq 2(|\lambda| + |\lambda| + |\lambda|^2) \leq 6|\lambda|.
\]

By Lemma 2.8 we also have
\[
\int_B \left| (U)_{\lambda} + u \right|^2 du = \int_B \left| ((U + I)u)_{\lambda} \right|^2 du \leq C \|U + I\|^2 \leq C|\lambda|^2.
\]

Thus (2.6) is less than or equal to
\[
C(m, n) \left\| S_z \right\| |\lambda|^2 \leq C(m, n) \left\| S \right\| |\lambda|.
\]
The proof is complete.

\[
\square
\]

**Lemma 2.12.** Let \( S \in \mathcal{L}(L^n) \) and \( m, j \geq 0 \). If \( |S^* K_{\lambda} (z)| \leq C \) for any \( z \in B \) then \( (B_mB_j)(S) = (B_mB_j)(S) \).
From Proposition 2.3, Proposition 2.1 and Fubini’s Theorem, we have

\[ B_m(B_j S)(0) = B_m(T_{B_j S})(0) \]

By Theorem 2.9, it is enough to show that

\[ \text{F} \]

Then

\[ l \]

as desired. \( \square \)

**Lemma 2.13.** For any \( S \in \mathcal{L}(L^2_a) \), there exists sequences \( \{S_\alpha\} \) satisfying

\[ |S^*_\alpha K_\lambda(u)| \leq C(\alpha) \]

such that \( B_m(S_\alpha) \) converges to \( B_m(S) \) pointwise.

**Proof.** Since \( H^\infty \) is dense in \( L^2_a \) and the set of finite rank operators is dense in the ideal \( K \) of compact operators on \( L^2 \), the set \( \{\sum_{i=1}^l f_i \otimes g_i : f_i, g_i \in H^\infty\} \) is dense in the ideal \( K \) in the norm topology. Since \( K \) is dense in the space of bounded operators on \( L^2_a \) in strong operator topology, (2.3) gives that for any \( S \in \mathcal{L}(L^2_a) \), there exists a finite rank operator sequences \( S_\alpha = \sum_{i=1}^l f_i \otimes g_i \) such that \( B_m(S_\alpha) \) converges to \( B_m(S) \) pointwise for some \( f_i, g_i \) in \( H^\infty \). Also, for \( l \geq 0 \), for such
Proposition 2.14. Let $S \in \mathcal{L}(L^2_a)$ and $m, j \geq 0$. Then

$$(B_mB_j)(S) = (B_jB_m)(S).$$

Proof. Let $S \in \mathcal{L}(L^2_a)$. Then Lemma 2.13 implies that there exists a sequence $\{S_\alpha\}$ satisfying $|S_\alpha^*K_\alpha(u)| \leq C(\alpha)$, hence $B_m(B_jS_\alpha)(z) = B_j(B_mS_\alpha)(z)$ by Lemma 2.12. From Proposition 2.3, we know

$$B_m(B_jS_\alpha)(z) = \int_B (B_jS_\alpha) \circ \phi_z(u)d\nu_m(u)$$

and $\|(B_jS_\alpha) \circ \phi_z\|_\infty \leq C(j, n)\|S\|$. Also, $(B_jS_\alpha) \circ \phi_z(u)$ converges to $(B_jS) \circ \phi_z(u)$. Therefore $B_m(B_jS_\alpha)(z)$ converges to $B_m(B_jS)(z)$. By the uniqueness of the limit, we have $(B_mB_j)(S) = (B_jB_m)(S)$.

Proposition 2.15. Let $S \in \mathcal{L}(L^2_a)$ and $m \geq 0$. If $B_0S(z) \to 0$ as $z \to \partial B$ then $B_mB_0S(z) \to 0$ as $z \to \partial B$.

Proof. Suppose $B_0S(z) \to 0$ as $z \to \partial B$. Then we will prove that $S_z \to 0$ in the $T^*$-norm as $z \to \partial B$. Suppose it is not true. Then for some net $\{w_\alpha\} \in B$ and an operator $V \neq 0$ in $\mathcal{L}(L^2_a)$, there exists a sequence $\{S_{w_\alpha}\}$ such that $S_{w_\alpha} \to V$ in the $T^*$-norm as $w_\alpha \to \partial B$, hence $tr[S_{w_\alpha}T] \to tr[VT]$ for any $T \in T$. Let $T = k_\lambda \otimes k_\lambda$ for fixed $\lambda \in B$. Then Theorem 2.9 implies

$$tr[S_{w_\alpha}T] = tr[S_{w_\alpha}(k_\lambda \otimes k_\lambda)] = \langle S_{w_\alpha}k_\lambda, k_\lambda \rangle = B_0S_{w_\alpha}(\lambda) = (B_0S) \circ \phi_{w_\alpha}(\lambda) \to 0$$

as $w_\alpha \to \partial B$. Since $tr[VT] = B_0V(\lambda)$ and $B_0$ is one-to-one mapping, $V = 0$. This is the contradiction. Thus $S_z \to 0$ as $z \to \partial B$ in the $T^*$-norm. (1.2) finishes the proof of this proposition.
3. OPERATORS S APPROXIMATED BY TOEPLITZ OPERATORS $T_{B_m}(S)$

In this section we will give a criterion for operators approximated by Toeplitz operators with symbol equal to their $m$-Berezin transforms. The main result in this section is Theorem 3.7. It extends and improves Theorem 2.4 in [17]. Even on the unit disk, we will show an example that the result in the theorem is sharp on the unit disk.

From Proposition 1.4.10 in [14], we have the following lemma

**Lemma 3.1.** Suppose $a < 1$ and $a + b < n + 1$. Then

$$\sup_{z \in B} \int_B \frac{d\lambda}{(1 - |\lambda|^2)^a |1 - \langle \lambda, z \rangle|^b} < \infty.$$  

This lemma gives the following lemma which extends Lemma 4.2 in [13].

**Lemma 3.2.** Let $S \in \mathcal{L}(L^2_a)$ and $p > n + 2$. Then there exists $C(n, p) > 0$ such that $h(z) = (1 - |z|^2)^{-a}$ where $a = (n + 1)/(n + 2)$ satisfies

$$\int_B |(SK_z)(w)| |h(w)| dw \leq C(n, p) \|S_z^1\|_p h(z)$$  

(3.1)

for all $z \in B$ and

$$\int_B |(SK_z)(w)| |h(z)| dz \leq C(n, p) \|S_w^a\|_p h(w)$$  

(3.2)

for all $w \in B$.

**Proof.** Fix $z \in B$. Since

$$U_z 1 = (-1)^n (1 - |z|^2)^{(n+1)/2} K_z,$$

we have

$$SK_z = (-1)^n (1 - |z|^2)^{-(n+1)/2} SU_z 1$$
$$= (-1)^n (1 - |z|^2)^{-(n+1)/2} U_z S_z 1$$
$$= (1 - |z|^2)^{-(n+1)/2} (S_z 1 \circ \phi_z) k_z.$$

Thus, letting $\lambda = \phi_z(w)$, the change of variables implies

$$\int_B \frac{|(SK_z)(w)|}{(1 - |w|^2)^a} dw = \frac{1}{(1 - |z|^2)^{(n+1)/2}} \int_B \frac{|(S_z 1 \circ \phi_z)(w)||k_z(w)|}{(1 - |w|^2)^a} dw$$
$$= \frac{1}{(1 - |z|^2)^a} \int_B \frac{|S_z 1(\lambda)|}{(1 - |\lambda|^2)^a |1 - \langle \lambda, z \rangle|^{n+1-2a}} d\lambda$$
$$\leq \|S_z 1\|_p \left( \int_B \frac{1}{(1 - |\lambda|^2)^{aq} |1 - \langle \lambda, z \rangle|^{(n+1-2a)q}} d\lambda \right)^{1/q}.$$
The last inequality comes from Holder’s inequality. Since $aq < 1$ and $aq + (n + 1 - 2a)q < n + 1$, Lemma 3.1 implies (3.1).

To prove (3.2), replace $S$ by $S^*$ in (3.1), interchange $w$ and $z$ in (3.1) and then use the equation

$$
(S^* K_w)(z) = \langle S^* K_w, K_z \rangle = \langle K_w, S K_z \rangle = S \overline{K_z}(w)
$$

(3.3)

to obtain the desired result. □

**Lemma 3.3.** Let $S \in \mathcal{L}(L^2_a)$ and $p > n + 2$. Then

$$
\|S\| \leq C(n, p) \left( \sup_{z \in B} \|S_z 1\|_p \right)^{1/2} \left( \sup_{z \in B} \|S^*_z 1\|_p \right)^{1/2}
$$

where $C(n, p)$ is the constant of Lemma 3.2.

**Proof.** (3.3) implies

$$
(Sf)(w) = \langle Sf, K_w \rangle = \int_B f(z) \overline{(S^* K_w)(z)} \, dz
$$

$$
= \int_B f(z) \overline{(SK_z)(w)} \, dz
$$

for $f \in L^2_a$ and $w \in B$. Thus, Lemma 3.2 and the classical Schur’s theorem finish the proof. □

**Lemma 3.4.** Let $S_m$ be a bounded sequence in $\mathcal{L}(L^2_a)$ such that $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$. Then

$$
\sup_{z \in B} |\langle (S_m)_z 1, f \rangle| \to 0
$$

(3.4)

as $m \to \infty$ for any $f \in L^2_a$ and

$$
\sup_{z \in B} |\langle S_m \rangle_z 1| \to 0
$$

(3.5)

uniformly on compact subsets of $B$ as $m \to \infty$.

**Proof.** To prove (3.4), we only need to have

$$
\sup_{z \in B} |\langle (S_m)_{z} 1, w^k \rangle| \to 0
$$

(3.6)

as $m \to \infty$ for any multi-index $k$.

Since

$$
K_z(w) = \sum_{|\alpha|=0}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \xi^\alpha w^\alpha,
$$

(3.7)
we have

$$B_0 S_m (\phi_z (\lambda)) = B_0 (S_m)_z (\lambda)$$

$$= (1 - |\lambda|^2)^{n+1} \sum_{|\alpha| = 0}^{\infty} \sum_{|\beta| = 0}^{\infty} \frac{(n + |\alpha|)! (n + |\beta|)!}{n! \alpha! n! \beta!} \langle (S_m)_z w^\alpha, w^\beta \rangle \bar{\lambda}^\alpha \lambda^\beta$$

where $\alpha, \beta$ are multi-indices.

Then for any fixed $k$ and $0 < r < 1$,

$$\int_{rB} \frac{B_0 S_m (\phi_z (\lambda)) \bar{\lambda}^k}{(1 - |\lambda|^2)^{n+1}} d\lambda$$

$$= \sum_{|\alpha| = 0}^{\infty} \sum_{|\beta| = 0}^{\infty} \frac{(n + |\alpha|)! (n + |\beta|)!}{n! \alpha! n! \beta!} \langle (S_m)_z w^\alpha, w^\beta \rangle \int_{rB} \bar{\lambda}^{n+|\alpha|} \lambda^{n+|\beta|} d\lambda$$

$$= r^{2n+2|k|} \left( \langle (S_m)_z 1, w^k \rangle + \sum_{|\alpha| = 1}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \langle (S_m)_z w^\alpha, (w^{\alpha+k})_r \rangle r^{2|\alpha|} \right).$$

Since $S_m$ is bounded sequence, we have

$$\left| \langle (S_m)_z 1, w^k \rangle \right|$$

$$\leq r^{-2n-2|k|} \left| \int_{rB} \frac{B_0 S_m (\phi_z (\lambda)) \bar{\lambda}^k}{(1 - |\lambda|^2)^{n+1}} d\lambda \right| +$$

$$\sum_{|\alpha| = 1}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \| (S_m)_z \| \| w^\alpha \| \| w^{\alpha+k} \| r^{2|\alpha|}$$

$$\leq r^{-2n-2|k|} \| B_0 S_m \| \| \int_{rB} \frac{|\lambda|^k}{(1 - |\lambda|^2)^{n+1}} d\lambda \| + C \sum_{|\alpha| = 1}^{\infty} r^{2|\alpha|},$$

hence, by assumption

$$\limsup_{m \to \infty} \sup_{z \in B} \left| \langle (S_m)_z 1, w^k \rangle \right| \leq C \sum_{|\alpha| = 1}^{\infty} r^{2|\alpha|}.$$

Letting $r \to 0$, we have (3.6).

Now we prove (3.5). From (3.7), we get

$$\left| (S_m)_z 1 (\lambda) \right| = \left| \langle (S_m)_z 1, K_\lambda \rangle \right|$$

$$\leq \sum_{|\alpha| = 0}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \left| \langle (S_m)_z 1, w^\alpha \rangle \right| |\lambda^\alpha|$$

$$\leq \sum_{|\alpha| = 0}^{l-1} \frac{(n + |\alpha|)!}{n! \alpha!} \left| \langle (S_m)_z 1, w^\alpha \rangle \right| + \sum_{|\alpha| = l}^{\infty} \frac{(n + |\alpha|)!}{n! \alpha!} \| S_m \| \| w^\alpha \| |\lambda^\alpha|.$$
for \( z \in B, \lambda \in rB \) and \( l \geq 1 \). Since the second summation is less than or equals to
\[
\sum_{j=l}^{\infty} \left( \frac{(n+j)!}{n!j!} \right)^{\frac{1}{2}} \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\lambda^\alpha| \leq \sum_{j=l}^{\infty} \left( \frac{(n+j)!}{n!j!} \right)^{\frac{1}{2}} \left[ \sum_{|\alpha|=j} \frac{j!}{\alpha!} |\lambda^\alpha|^2 \right]^{\frac{1}{2}}
\]
\[
\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} r^j,
\]
for any \( \epsilon > 0 \), we can find sufficiently large \( l \) such that the second summation
is less than \( \epsilon \). Thus, (3.6) imply
\[
\sup_{z \in B} \| (S_m)z1 \| \to 0
\]
uniformly on compact subsets of \( B \) as \( m \to \infty \). \( \square \)

Lemma 3.5. Let \( \{S_m\} \) be a sequence in \( \mathcal{L}(L^2_a) \) such that for some \( p > n + 2 \),
\[
\|B_0S_m\|_{\infty} \to 0 \quad \text{as} \quad m \to \infty,
\]
\[
\sup_{z \in B} \| (S_m)z1 \|_p \leq C \quad \text{and} \quad \sup_{z \in B} \| (S^*_m)z1 \|_p \leq C
\]
where \( C > 0 \) is independent of \( m \), then \( S_m \to 0 \) as \( m \to \infty \) in \( \mathcal{L}(L^2_a) \)-norm.

Proof. Lemma 3.3 implies
\[
\|S_m\| \leq C(n, p) \left( \sup_{z \in B} \| (S_m)z1 \|_p \right)^{\frac{1}{2}} \left( \sup_{z \in B} \| (S^*_m)z1 \|_p \right)^{\frac{1}{2}} \leq C(n, p),
\]
hence, Lemma 3.4 gives
\[
\sup_{z \in B} \| (S_m)z1 \| \to 0 \quad (3.8)
\]
uniformly on compact subsets of \( B \) as \( m \to \infty \).

Here, for \( n + 2 < s < p \), Holder’s inequality gives
\[
\sup_{z \in B} \| (S_m)z1 \|^s \leq \sup_{z \in B} \int_{B \setminus rB} \| (S_m)z1(w) \|^s dw + \sup_{z \in B} \int_{rB} \| (S_m)z1(w) \|^s dw
\]
\[
\leq C \sup_{z \in B} \| (S_m)z1 \|^p (1 - r)^{1-s/p} + \sup_{z \in B} \int_{rB} \| (S_m)z1(w) \|^s dw
\]
and (3.8) implies the second term tends to 0 as \( m \to \infty \). Also, the first term is
less than or equals to \( C^s(1 - r)^{1-s/p} \) which can be small by taking \( r \) close to 1.
Consequently, Lemma 3.3 gives
\[
\|S_m\| \leq C(n, s) \left( \sup_{z \in B} \| (S_m)z1 \|_s \right)^{\frac{1}{2}} \left( \sup_{z \in B} \| (S^*_m)z1 \|_s \right)^{\frac{1}{2}} .
\]
\[
\leq C(n, s) \left( \sup_{z \in B} \| (S_m)z1 \|_s \right)^{\frac{1}{2}} \to 0
\]
\( \square \)
Corollary 3.6. Let $S \in \mathcal{L}(L_a^2)$ such that for some $p > n + 2$,
\[
\sup_{z \in B} \|S_z 1 - (T_{B_m} S) z 1\|_p \leq C \quad \text{and} \quad \sup_{z \in B} \|S^*_z 1 - (T_{B_m} S^*) z 1\|_p \leq C,
\]
(3.9)
where $C > 0$ is independent of $m$. Then $T_{B_m} S \to S$ as $m \to \infty$ in $\mathcal{L}(L_a^2)$-norm.

Proof. Let $S_m = S - T_{B_m} S$. Then Proposition 2.14 and Theorem 2.11 imply
\[
B_0(S_m) = B_0 S - B_0(T_{B_m} S) = B_0 S - B_0(B_m S) = B_0 S - B_m(B_0 S)
\]
which tends uniformly to 0 as $m \to \infty$, hence $\|B_0(S_m)\|_\infty \to 0$. Consequently, by Lemma 3.5 we complete the proof. \qed

Theorem 3.7. Let $S \in \mathcal{L}(L_a^2)$. If there is $p > n + 2$ such that
\[
\sup_{z \in B} \|T_{(B_m S) \circ \phi_z} z 1\|_p < C \quad \text{and} \quad \sup_{z \in B} \|T^*_{(B_m S) \circ \phi_z} z 1\|_p < C
\]
(3.10)
where $C > 0$ is independent of $m$, then $T_{B_m} S \to S$ as $m \to \infty$ in $\mathcal{L}(L_a^2)$-norm.

Proof. By Corollary 3.6, we only need to show that (3.10) implies (3.9). Since
\[
T^*_{(B_m S) \circ \phi_z} = T_{B_m(S^*)} \circ \phi_z = T_{(B_m(S^*)) \circ \phi_z},
\]
it is sufficient to show that
\[
\sup_{z \in B} \|S_z 1\|_p < \infty.
\]
By Lemma 3.3, we get
\[
\|T_{B_m} S\| \leq C(n, p) \left( \sup_{z \in B} \|T_{B_m S \circ \phi_z} z 1\|_p \right)^{1/2} \left( \sup_{z \in B} \|T^*_{B_m S \circ \phi_z} z 1\|_p \right)^{1/2} < C
\]
where $C$ is independent of $m$, hence writing $S_m = S - T_{B_m} S$, we have $\|S_m\| \leq C$ where $C$ is independent of $m$. Also, the proof of Corollary 3.6 implies
\[
\|B_0 S_m\|_\infty \to 0
\]
as $m \to \infty$.

Let $f$ be a polynomial with $\|f\|_q = 1$. Then Lemma 3.4 implies
\[
\sup_{z \in B} |\langle (S_m) z 1, f \rangle| \to 0
\]
as $m \to \infty$. Thus, for any $\epsilon > 0$ and $z_0 \in B$, we have
\[
|\langle S_{z_0} 1, f \rangle| \leq \sup_{z \in B} |\langle (S_m) z 1, f \rangle| + |\langle (T_{B_m} S) z_0 1, f \rangle| \leq \epsilon + C
\]
for sufficiently large \( m \), where \( C \) is independent of \( m \). Since \( \epsilon \) is arbitrary, we get

\[
\sup_{z \in B} \| S_z 1 \|_p < \infty
\]
as desired. \( \square \)

4. Compact Radial Operator

Given \( U \in \mathcal{U}(n) \), define \( V_U f(w) = f(Uw) \det U \) for \( f \in L^2_\alpha \). Then \( V_U \) is a unitary operator on \( L^2_\alpha \). We say that \( S \in \mathcal{L}(L^2_\alpha) \) is a radial operator if \( SV_U = V_U S \) for any \( U \in \mathcal{U}(n) \).

If \( S \in \mathcal{L}(L^2_\alpha) \), the radialization of \( S \) is defined by

\[
S^\flat = \int_{U(n)} V_U^* S V_U dU
\]
where \( dU \) is the Haar measure on the compact group \( U(n) \) and the integral is taken in the weak sense. Then \( S^\flat = S \) if \( S \) is radial and \( U(n) \)-invariance of \( dU \) shows that \( S^\flat \) is indeed a radial operator.

If \( f \in L^\infty \) and \( g, h \in L^2_\alpha \), then

\[
\langle V_U^* T_f V_U g, h \rangle = \int_B f(w) V_U g(w) \overline{V_U h(w)} dw
\]
\[
= \int_B f(U^* w) g(w) \overline{h(w)} dw.
\]
Thus \( V_U^* T_f V_U = T_{f U^*} \) and

\[
V_U^* T_{f_1} \cdots T_{f_l} V_U = T_{f_1 U^*} \cdots T_{f_l U^*}
\]
for \( f_1, \ldots, f_l \in L^\infty, l \geq 0 \).

**Lemma 4.1.** Let \( S \in \mathcal{L}(L^2_\alpha) \) be a radial operator. Then \( T_{B_m(S)} = \int_B S_w d\nu_m(w) \).

**Proof.** Let \( z \in B \). By (2.3) and Lemma 2.8, we obtain

\[
B_0 \left( \int_B S_w d\nu_m(w) \right)(z) = \left\langle \left( \int_B S_w d\nu_m(w) \right) z, 1, 1 \right\rangle
\]
\[
= \int_B \langle U_z U_w S U_z 1, 1 \rangle d\nu_m(w)
\]
\[
= \int_B \langle U_{\phi_z(w)} V_U^* S V_U U_{\phi_z(w)} 1, 1 \rangle d\nu_m(w)
\]

where $V_U$ is in Lemma 2.8. Since $S$ is a radial operator, Theorem 2.9, Proposition 2.3 and Proposition 2.14 imply that the last integral equals
\[
\int_B \langle U_{\phi_z(w)} S U_{\phi_z(w)} 1, 1 \rangle \, d\nu_m(w) = \int_B B_0 S \circ \phi_z(w) d\nu_m(w)
\]
\[
= B_m B_0 S(z)
\]
\[
= B_0 B_m S(z)
\]
\[
= B_0(T_{B_m S})(z).
\]
Since $B_0$ is one-to-one mapping, the proof is complete.

\[\square\]

Theorem 4.2. Let $S \in \mathfrak{S}(L^\infty)$ be a radial operator. Then $S$ is compact if and only if $B_0 S \equiv 0$ on $\partial B$.

Proof. Suppose $B_0 S \equiv 0$ on $\partial B$. Then $B_m S \equiv 0$ on $\partial B$ by Proposition 2.15, hence $T_{B_m S}$ is compact for all $m \geq 0$.

Let
\[
Q = \int B T_{f_1 \circ U^*} \cdots T_{f_l \circ U^*} dU
\]
with $f_1, \ldots, f_l \in L^\infty$ for some $l \geq 0$. Then $Q \in \mathfrak{L}(L^2_a)$. By Lemma 4.1, for any $z \in B$, we have
\[
T_{(B_m Q) \circ \phi_z} = \int_B ((Q)_{z})_w d\nu_m(w)
\]
\[
= \int_B \int \mathfrak{U} T_{f_1 \circ U^* \circ \phi_z \circ \phi_w} \cdots T_{f_l \circ U^* \circ \phi_z \circ \phi_w} dU d\nu_m(w).
\]
Consequently,
\[
\|T_{(B_m Q) \circ \phi_z}\| \leq C(l) \|f_1 \circ U^* \circ \phi_z \circ \phi_w\|_\infty \cdots \|f_l \circ U^* \circ \phi_z \circ \phi_w\|_\infty
\]
\[
= C(l) \|f_1\|_\infty \cdots \|f_l\|_\infty.
\]
Similarly, we have
\[
\|T_{(B_m Q)\circ \phi_z}^*\| \leq C(l) \|f_1\|_\infty \cdots \|f_l\|_\infty.
\]
Thus, Theorem 3.7 gives that
\[
T_{B_m(Q)} \to Q \quad (4.1)
\]
in $\mathfrak{L}(L^2_a)$-norm.

Since $S \in \mathfrak{S}(L^\infty)$, there exists a sequence $\{S_k\}$ such that $S_k \to S$ in $\mathfrak{L}(L^2_a)$-norm where each $S_k$ is a finite sum of finite products of Toeplitz operators. Since the radialization is continuous and $S$ is radial, $S_k^\# \to S^\# = S$. From Lemma 4.1, we have
\[
\|T_{B_m S}\| = \left\| \int_B S_w d\nu_m(w) \right\| \leq \int_B \|S_w\| d\nu_m(w) = \|S\|.
Thus
$$\|S - T_{B_m}S\| \leq \|S - S_k^2\| + \|S_k^2 - T_{B_m}(S_k^2)\| + \|T_{B_m}(S_k^2) - T_{B_m}S\| \leq 2\|S - S_k^2\| + \|S_k^2 - T_{B_m}(S_k^2)\|$$

and (4.1) imply $T_{B_m}(S) \to S$ as $m \to \infty$ in $L(L_a^2)$-norm, hence $S$ is compact.

The other direction is trivial. □

**Example.** This example shows that for $n = 1$, the number $n + 2 = 3$ in Theorem 3.7 is sharp. We show that there is a bounded operator $S$ on $L_a^2$ such that
$$\sup_{z \in D} \max\{\|T_{(B_mS)\circ \phi_z}1\|_3, \|T_{(B_mS)\circ \phi_z}^*1\|_3\} < \infty,$$

and for each $m \geq 0$, $B_m(S)(z) \to 0$ as $z \to \partial D$, but $S$ is not compact on $L_a^2$.

The following operator $S$ was constructed in [3] to show that $B_0(S)(z) \to 0$ as $z \to \partial D$, but $S$ is not compact on $L_a^2$. Let $S$ be defined on $L_a^2$ by
$$S \left(\sum_{l=0}^{\infty} a_l w^l\right) = \sum_{l=0}^{\infty} a_2^l w^{2^l}.$$  

It is clear that $S$ is a self-adjoint projection with infinite-dimensional range. Thus $S$ is not compact on $L_a^2$. From
$$B_0(S)(z) = (Sk_z, k_z) = \|Sk_z\|_2^2 = (1 - |z|^2)^2 \sum_{l=0}^{\infty} (2^l + 1)(|z|^2)^{2^l},$$

it is easy to see that $B_0(S)(z) \to 0$ as $z \to \partial D$. By Proposition 2.15, we see that $B_m(S)(z) \to 0$ as $z \to \partial D$. This gives that $T_{B_m}(S)$ is compact. Hence $T_{B_m}(S)$ does not converge to $S$ in the norm topology.

By means of the Zygmund theorem on gap series [18], it was proved in [13] that
$$C = \sup_{z \in D} \max\{\|S_z1\|_3, \|S_z^*1\|_3\} < \infty.$$  

Clearly, $S$ is a radial operator. By Lemma 4.1, we have
$$T_{(B_mS)\circ \phi_z}1 = \int_D \langle S w, 1 \rangle \nu_m(w) = \int_D S \phi_z(w) \nu_m(w) = \int_D S \phi_z \nu_m \circ \phi_z.$$
Noting that for each $z \in D$, $d\nu_m \circ \phi_z$ is a probability measure on $D$, we have

$$\|T_{(B_mS)\circ \phi_z}1\|_3 \leq \int_D \|S_\lambda 1\|_3 d\nu_m \circ \phi_z(\lambda) \leq C.$$ 

Similarly, we also have

$$\|T^*_{(B_mS)\circ \phi_z}1\|_3 \leq C.$$ 

REFERENCES

DEPARTMENT OF MATHEMATICS, HANSIN UNIVERSITY, GYEONGGI 447-791, KOREA
E-mail address: ksnam@hanshin.ac.kr

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37240, USA
E-mail address: zheng@math.vanderbilt.edu

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37240, USA
E-mail address: zhong@math.vanderbilt.edu