m-BEREZIN TRANSFORM AND COMPACT OPERATORS

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ABSTRACT. m-Berezin transforms are introduced for bounded operators on the Bergman space of the unit ball. The norm of the m-Berezin transform as a linear operator from the space of bounded operators to L^∞ is found. We show that the m-Berezin transforms are commuting with each other and Lipschitz with respect to the pseudo-hyperbolic distance on the unit ball. Using the m-Berezin transforms we show that a radial operator in the Toeplitz algebra is compact iff its Berezin transform vanishes on the boundary of the unit ball.

1. INTRODUCTION

Let B denote the unit ball in n-dimensional complex space $\mathbb{C}^n$ and dz be normalized Lebesgue volume measure on B. The Bergman space $L^2_a = L^2_a(B, dz)$ is the space of analytic functions h on B which are square-integrable with respect to Lebesgue volume measure. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $\langle z, w\rangle = \sum_{i=1}^n z_i \overline{w}_i$ and $|z|^2 = \langle z, z\rangle$.

For $z \in B$, let P_z be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[z]$ generated by z and let $Q_z = I - P_z$. Then

$$\phi_z(w) = \frac{z - P_z(w) - (1-|z|^2)^{1/2} Q_z(w)}{1 - \langle w, z\rangle}$$

is the automorphism of B that interchanges 0 and z. The pseudo-hyperbolic metric on B is defined as $\rho(z, w) = |\phi_z(w)|$.

The reproducing kernel in L^2_a is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1}},$$

for $z, w \in B$ and the normalized reproducing kernel k_z is $K_z(w)/\|K_z(\cdot)\|_2$. If $\langle \cdot, \cdot\rangle$ denotes the inner product in L^2, then $\langle h, K_z\rangle = h(z)$, for every $h \in L^2_a$ and $z \in B$. The fundamental property of the reproducing kernel $K_z(w)$ plays an important role in this paper:

$$K_z(w) = \overline{k_\lambda(z)} K_{\phi_\lambda(z)}(\phi_\lambda(w)) k_\lambda(w). \tag{1.1}$$

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Given $f \in L^\infty$, the Toeplitz operator T_f is defined on B by $T_f h = P(fh)$ where P denotes the orthogonal projection P of L^2 onto L^2_a.

Let $\mathfrak{L}(L^2_a)$ be the algebra of bounded operators on L^2_a. The Toeplitz algebra $\mathfrak{T}(L^\infty)$ is the closed subalgebra of $\mathfrak{L}(L^2_a)$ generated by $\{T_f : f \in L^\infty\}$.

For $z \in B$, let U_z be the unitary operator given by

$$U_z f = (f \circ \phi_z) \cdot J\phi_z$$

where $J\phi_z = (-1)^n k_z$. For $S \in \mathfrak{L}(L^2_a)$, set

$$S_z = U_z S U_z.$$

Since U_z is a selfadjoint unitary operator on L^2 and L^2_a, $U_z T_f U_z = T_{f\circ\phi_z}$ for every $f \in L^\infty$.

Let $\mathcal{T}$ denote the class of trace operators on L^2_a. For $T \in \mathcal{T}$, we will denote the trace of T by $tr[T]$ and let $\|T\|_{C_1}$ denote the C_1 norm of T given by ([12])

$$\|T\|_{C_1} = tr[\sqrt{T^*T}].$$

Suppose f and g are in L^2_a. Consider the operator $f \otimes g$ on L^2_a defined by

$$(f \otimes g)h = \langle h, g\rangle f,$$

for $h \in L^2_a$. It is easily proved that $f \otimes g$ is in $\mathcal{T}$ and with norm equal to $\|f \otimes g\|_{C_1} = \|f\|_2 \, \|g\|_2$ and

$$tr[f \otimes g] = \langle f, g\rangle.$$

For a nonnegative integer m, the m-Berezin transform of an operator $S \in \mathfrak{L}(L^2_a)$ is defined by

$$\begin{aligned} B_m S(z) &= C_n^{m+n} tr\left[S_z\left(\sum_{|k|=0}^{m} C_{m,k}\frac{n!k!}{(n+|k|)!}\frac{u^k}{\|u^k\|}\otimes\frac{u^k}{\|u^k\|}\right)\right] \qquad (1.2)\\ &= C_n^{m+n} tr\left[S_z\left(\sum_{|k|=0}^{m} C_{m,k} u^k \otimes u^k\right)\right] \end{aligned}$$

where $k = (k_1, \cdots, k_n) \in N^n$, N is the set of nonnegative integers, $|k| = \sum_{i=0}^{n} k_i$, $u^k = u_1^{k_1} \cdots u_n^{k_n}$, $k! = k_1! \cdots k_n!$,

$$C_n^{m+n} = \binom{m+n}{n} \quad \text{and} \quad C_{m,k} = C_{|k|}^{m}(-1)^{|k|}\frac{|k|!}{k_1! \cdots k_n!}.$$

Clearly, $B_m : \mathfrak{L}(L^2_a) \to L^\infty$ is a bounded linear operator, the norm of B_m will be given.

Given $f \in L^\infty$, define

$$B_m(f)(z) = B_m(T_f)(z).$$

$B_m(f)(z)$ equals the nice formula in [1]:

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u),$$

for $z \in B$ where $d\nu_m(u) = C_n^{m+n}(1 - |u|^2)^m du$.

Berezin first introduced the Berezin transform $B_0(S)$ of bounded operators S and the m-Berezin transform of functions in [5]. Because the Berezin transform encodes operator-theoretic information in function-theory in a striking but somewhat impenetrable way, the Berezin transform $B_0(S)$ has found useful applications in studying operators of "function-theoretic significance" on function spaces ([2], [3], [4], [6], [7], [11], and [15]). Suarez [16] introduced m-Berezin transforms of bounded operators on the Bergman space of the unit disk. We will show that our m-Berezin transform coincides with the one defined in [16] on the unit disk D by means of an integral representation of m-Berezin transform. The integral representation shows that many useful properties of the m-Berezin transforms inherit from the identity (1.1) of the reproducing kernel. On the unit ball, some useful properties of the m-Berezin transforms of functions were obtained by Ahern, Flores and Rudin [1]. Recently, Coburn [10] proved that $B_0(S)$ is Lipschitz with respect to the pseudo-hyperbolic distance $\rho(z, w)$. In this paper, we will show that $B_m S(z)$ is Lipschitz with respect to pseudo-hyperbolic distance $\rho(z, w)$. We will show that the m-Berezin transforms B_m are invariant under the Mobious transform,

$$B_m(S_z) = (B_m S) \circ \phi_z, \tag{1.3}$$

and commuting with each other,

$$B_j(B_m S)(z) = B_m(B_j S)(z) \tag{1.4}$$

for any nonnegative integers j and m. Properties (1.3) and (1.4) were obtained for $S = T_f$ in [1] and for operators S on the Bergman space of the unit disk [16].

A common intuition is that for operators on the Bergman space L_a^2 "closely associated with function theory", compactness is equivalent to having vanishing Berezin transform on the boundary of the unit ball B. On the unit disk, Axler and Zheng [2] showed that if the operator S equals the finite sum of finite products of Toeplitz operators with bounded symbols then S is compact if and only if $B_0(S)(z) \to 0$ as $z \to \partial D$. Englis extended this result to the unit ball even the bounded symmetric domains [11]. But the problem remains open whether the result is true if S is in the Toeplitz algebra. Recently, Suarez [17] solved the problem for radial operator S on the unit disk via the m-Berezin transform. Using the m-Berezin transform, we will show that for a radial operator S in the Toeplitz algebra on the unit ball, S is compact iff $B_0 S(z) \to 0$ as $|z| \to 1$.

Throughout the paper $C(m, n)$ will denote constant depending only on m and n, which may change at each occurrence.

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2. m-BEREZIN TRANSFORM

In this section we will show some useful properties of the m-Berezin transform. First we give an integral representation of the m-Berezin transform $B_m(S)$. For $z \in B$ and a nonnegative integer m, let

$$K_z^m(u) = \frac{1}{(1 - \langle u, z \rangle)^{m+n+1}}, \qquad u \in B.$$

For $u, \lambda \in B$, we can easily see that

$$\sum_{|k|=0}^{m} C_{m,k} u^k \overline{\lambda^k} = (1 - \langle u, \lambda \rangle)^m. \tag{2.1}$$

Proposition 2.1. *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 0$ and $z \in B$. Then*

$$B_m S(z) = C_n^{m+n}(1 - |z|^2)^{m+n+1} \times \int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda.$$

Proof. For $\lambda \in B$, the definition of B_m implies

$$\begin{aligned} B_m S(z) &= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \left\langle S_z \lambda^k, \lambda^k \right\rangle \\ &= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \int_B S(\phi_z^k k_z)(\lambda) \overline{\phi_z^k(\lambda) k_z(\lambda)} d\lambda \\ &= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \int_B \int_B \phi_z^k(u) k_z(u) \overline{\phi_z^k(\lambda) k_z(\lambda) S^* K_\lambda(u)} du d\lambda \end{aligned} \tag{2.2}$$

where the last equality holds by $S(\phi_z^k k_z)(\lambda) = \left\langle S(\phi_z^k k_z), K_\lambda \right\rangle = \left\langle \phi_z^k k_z, S^* K_\lambda \right\rangle$. Using (2.1) and (1.1), (2.2) equals

$$\begin{aligned} & C_n^{m+n} \int_B \int_B (1 - \langle \phi_z(u), \phi_z(\lambda) \rangle)^m k_z(u) \overline{k_z(\lambda) S^* K_\lambda(u)} du d\lambda \\ &= C_n^{m+n} \int_B \int_B \left(\frac{k_z(u) \overline{k_z(\lambda)}}{K_\lambda(u)} \right)^{m/(n+1)} k_z(u) \overline{k_z(\lambda) S^* K_\lambda(u)} du d\lambda \\ &= C_n^{m+n} (1 - |z|^2)^{m+n+1} \int_B \int_B (1 - \langle u, \lambda \rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda \end{aligned}$$

as desired. $\square$

Proposition 2.2 gives another form of B_m.

Proposition 2.2. *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 0$ and $z \in B$. Then*

$$B_m S(z) = C_n^{m+n}(1-|z|^2)^{m+n+1} \sum_{|k|=0}^{m} C_{m,k} \left\langle S(u^k K_z^m), u^k K_z^m \right\rangle . \tag{2.3}$$

Proof. Since

$$\begin{aligned}
&\int_B \int_B (1-\langle u, \lambda\rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda \\
&= \sum_{|k|=0}^{m} C_{m,k} \int_B \int_B u^k \overline{\lambda^k} K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda \\
&= \sum_{|k|=0}^{m} C_{m,k} \int_B S(u^k K_z^m)(\lambda) \overline{\lambda^k K_z^m(\lambda)} d\lambda,
\end{aligned}$$

Proposition 2.1 implies (2.3). □

For $n = 1$, the right hand side of (2.3) was used by Suarez in [16] to define the m-Berezin transforms on the unit disk.

Recall that given $f \in L^\infty$, define

$$B_m(f)(z) = B_m(T_f)(z).$$

The following proposition gives a nice formula of $B_m(f)(z)$. Let $d\nu_m(u) = C_n^{m+n}(1-|u|^2)^m du$.

Proposition 2.3. *Let $z \in B$ and $f \in L^\infty$. Then*

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u).$$

Proof. By the change of variables, Theorem 2.2.2 in [14] and (2.3), we have

$$\begin{aligned}
&\int_B f \circ \phi_z(u) d\nu_m(u) \\
&= C_n^{m+n} \int_B f(u) \left(\frac{(1-|z|^2)(1-|u|^2)}{|1-\langle u, z\rangle|^2} \right)^m \left(\frac{(1-|z|^2)}{|1-\langle u, z\rangle|^2} \right)^{n+1} du \\
&= C_n^{m+n}(1-|z|^2)^{m+n+1} \sum_{|k|=0}^{m} C_{m,k} \int_B f(u)|u^k|^2 |K_z^m(u)|^2 du \\
&= C_n^{m+n}(1-|z|^2)^{m+n+1} \sum_{|k|=0}^{m} C_{m,k} \left\langle T_f(u^k K_z^m), u^k K_z^m \right\rangle = B_m(T_f)(z).
\end{aligned}$$

The proof is complete. □

The formula in the above proposition was used in [1] to define the m-Berezin transform of functions f.

Clearly, (1.2) gives $\|B_m S\|_\infty \le C(m,n)\|S_z\| = C(m,n)\|S\|$ for $S \in \mathfrak{L}(L^2_a)$. Thus, $B_m : \mathfrak{L}(L^2_a) \to L^\infty$ is a bounded linear operator. The following theorem gives the norm of B_m.

Theorem 2.4. *Let $m \ge 0$. Then $\|B_m\| = C_n^{m+n} \sum_{|k|=0}^m |C_{m,k}| \frac{n!k!}{(n+|k|)!}$.*

Proof. From [8], we have the duality result $\mathfrak{L}(L^2_a) = \mathcal{T}^*$. So, the definition of B_m gives the norm of B_m. In fact,

$$\begin{aligned}\|B_m\| &= \left\| C_n^{m+n} \sum_{|k|=0}^m C_{m,k} \frac{n!k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right\|_{C_1} \\ &= C_n^{m+n} \sum_{|k|=0}^m |C_{m,k}| \frac{n!k!}{(n+|k|)!}\end{aligned}$$

as desired. □

The Mobius map $\phi_z(w)$ has the following property ([14]):

$$\phi_z'(0) = -(1-|z|^2)P_z - (1-|z|^2)^{1/2} Q_z. \tag{2.4}$$

To show that m-Berezin transforms are Lipschitz with respect to the pseudo-hyperbolic distance we need the following lemmas.

For $z, w \in \mathbb{C}^n$, $z \hat{\otimes} w$ on $\mathbb{C}^n$ is defined by $(z\hat{\otimes} w)\lambda = \langle \lambda, w\rangle z$.

Lemma 2.5. *Let $z, w \in B$ and $\lambda = \phi_z(w)$. Then*

$$\phi_z'(w) = (1 - \langle \lambda, z\rangle)(I - \lambda \hat{\otimes} z)[\phi_z'(0)]^{-1}.$$

Proof. Suppose that P_z and Q_z have the matrix representations as $((P_z)_{ij})$ and $((Q_z)_{ij})$ under the standard base of $\mathbb{C}^n$, respectively. In fact,

$$(P_z)_{ij} = \frac{z_i \bar{z}_j}{|z|^2} \qquad \text{if} \quad z \ne 0.$$

Let $(a_{ij}(w)) = \phi_z'(w)$. Write $\phi_z(w) = (f_1(w), \cdots, f_n(w))$. Then

$$a_{ij}(w) = \frac{\partial f_i}{\partial w_j}(w).$$

Noting that

$$f_i(w) = \frac{z_i - (P_z w)_i - (1-|z|^2)^{1/2}(Q_z w)_i}{1 - \langle w, z\rangle},$$

we have

$$\begin{aligned}a_{ij}(w) &= \frac{(z_i - (P_z w)_i - (1-|z|^2)^{1/2}(Q_z w)_i)\bar{z}_j}{(1-\langle w,z\rangle)^2} - \frac{(P_z)_{ij} + (1-|z|^2)^{1/2}(Q_z)_{ij}}{1-\langle w,z\rangle} \\ &= \frac{f_i(w)\bar{z}_j}{1-\langle w,z\rangle} - \frac{(P_z)_{ij} + (1-|z|^2)^{1/2}(Q_z)_{ij}}{1-\langle w,z\rangle}.\end{aligned}$$

Let $\lambda = \phi_z(w)$. The above equality becomes

$$a_{ij}(w) = \frac{\lambda_i \bar{z}_j - ((P_z)_{ij} + (1-|z|^2)^{1/2}(Q_z)_{ij})}{1 - \langle w, z\rangle}$$

Thus

$$\phi_z'(w) = \frac{\lambda \hat{\otimes} z - (P_z + (1-|z|^2)^{1/2} Q_z)}{1 - \langle w, z\rangle}.$$

From Theorem 2.2.5 in [14], we have

$$\frac{1}{1-\langle w, z\rangle} = \frac{1-\langle \lambda, z\rangle}{1-|z|^2}.$$

Thus (2.4) implies

$$\begin{aligned}
\phi_z'(w)\phi_z'(0) &= \frac{-(1-|z|^2)\lambda\hat{\otimes}z + (1-|z|^2)P_z + (1-|z|^2)Q_z}{1-\langle w, z\rangle} \\
&= \frac{(1-|z|^2)(-\lambda\hat{\otimes}z + I)}{1-\langle w, z\rangle} \\
&= (1-\langle \lambda, z\rangle)(I - \lambda\hat{\otimes}z)
\end{aligned}$$

where the first equality follows from $P_zQ_z = Q_zP_z = 0$, $P_z z = z$, and $Q_z z = 0$. The proof is complete. $\square$

Lemma 2.6. *Suppose* $|z| > 1/2$ *and* $|w| > 1/2$. *If* $|\phi_z(w)| \le \epsilon < 1/2$, *then*

$$\|P_z - P_w\| \le 50\epsilon(1-|z|^2)^{1/2}.$$

Proof. First we will get the estimate of the distance between z and w. Since $|\phi_z(w)| \le \epsilon < 1/2$, w is in the ellipsoid:

$$\phi_z(\epsilon B) = \{w \in B : \frac{|P_z w - c|^2}{\epsilon^2\rho^2} + \frac{|Q_z w|^2}{\epsilon^2\rho} < 1\}$$

with center $c = \frac{(1-\epsilon^2)z}{(1-\epsilon^2|z|^2)}$ and $\rho = \frac{1-|z|^2}{1-\epsilon^2|z|^2}$. Noting that $|z| > 1/2$ and $\epsilon < 1/2$, we have $\rho \le 2(1-|z|^2)$. Thus

$$|Q_z w|^2 \le \epsilon^2\rho \le 2\epsilon^2(1-|z|^2), \qquad |P_z w - c| \le \epsilon\rho \le 2\epsilon(1-|z|^2)$$

and

$$|z - c| \le \frac{\epsilon^2(1-|z|^2)}{(1-\epsilon^2|z|^2)} \le 2\epsilon^2(1-|z|^2).$$

So, we have

$$|P_z w - z| \le |P_z w - c| + |z - c| \le 3\epsilon(1-|z|^2).$$

Because $I = P_z + Q_z$ and $P_zQ_z = 0$, writing

$$(z - w) = P_z(z-w) + Q_z(z-w),$$

we have

$$\begin{aligned} |z-w|^2 &= |P_z(z-w)|^2 + |Q_z(z-w)|^2 \\ &= |P_z w - z|^2 + |Q_z w|^2 \\ &\le 11\epsilon^2(1-|z|^2). \end{aligned} \tag{2.5}$$

Noting that

$$\frac{z}{|z|}\hat{\otimes}\frac{z}{|z|} = \frac{(z-w)}{|z|}\hat{\otimes}\frac{z}{|z|} + \frac{w}{|z|}\hat{\otimes}\frac{(z-w)}{|z|} + \left[\left(\frac{1}{|z|^2} - \frac{1}{|w|^2}\right)w\right]\hat{\otimes}w + \frac{w}{|w|}\hat{\otimes}\frac{w}{|w|},$$

we have

$$P_z - P_w = \frac{(z-w)}{|z|}\hat{\otimes}\frac{z}{|z|} + \frac{w}{|z|}\hat{\otimes}\frac{(z-w)}{|z|} + \left[\left(\frac{1}{|z|^2} - \frac{1}{|w|^2}\right)w\right]\hat{\otimes}w,$$

to obtain

$$\begin{aligned} \|P_z - P_w\| &\le \frac{|z-w|}{|z|} + \frac{2|z-w|}{|z|} + \frac{||z|^2 - |w|^2|}{|z|^2} \\ &\le 2|z-w| + 4|z-w| + 8|z-w| \\ &\le 14\sqrt{11}\epsilon(1-|z|^2)^{1/2} \\ &\le 50\epsilon(1-|z|^2)^{1/2} \end{aligned}$$

where the last inequality holds by (2.5). □

For given $z, w \in B$, set $A(z,w) = -(1-|z|^2)P_w - (1-|z|^2)^{1/2}Q_w$.

Lemma 2.7. *Suppose* $|z| > 1/2$ *and* $|w| > 1/2$*. If* $|\phi_z(w)| \le \epsilon < 1/2$*, then*

$$\|\phi_z'(0) - A(z,w)\| \le 150\epsilon(1-|z|^2).$$

Proof. Using (2.4), we have

$$\begin{aligned} \|\phi_z'(0) - A(z,w)\| &= \|(1-|z|^2)(P_w - P_z) + (1-|z|^2)^{1/2}(P_z - P_w)\| \\ &\le 3(1-|z|^2)^{1/2}\|P_z - P_w\| \\ &\le 150\epsilon(1-|z|^2) \end{aligned}$$

as desired. The last inequality follows from Lemma 2.6. □

Let $\mathfrak{U}(n)$ be the group of $n \times n$ complex unitary matrices.

Lemma 2.8. *Let* $z, w \in B$*. Then* $U_z U_w = V_{\mathcal{U}} U_{\phi_w(z)}$ *where*

$$(V_{\mathcal{U}} f)(u) = f(\mathcal{U}u) det\mathcal{U}$$

for $f \in L_a^2$ *and* $\mathcal{U} = \phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$ *satisfying*

$$\|I + \mathcal{U}\| \le C(n)\rho(z,w).$$

Proof. The map $\phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$ is an automorphism of B that fixes 0, hence it is unitary by the Cartan theorem in [14]. Thus $\phi_w \circ \phi_z = \phi_{\phi_w(z)} \circ \mathcal{U}$ for some $\mathcal{U} \in \mathfrak{U}(n)$. Since ϕ_w is an involution, we have

$$\begin{aligned}
&U_z U_w f(u) = (f \circ \phi_w \circ \phi_z)(u) J\phi_w(\phi_z(u)) J\phi_z(u) \\
&= (f \circ \phi_{\phi_w(z)})(\mathcal{U}u) J\phi_w(\phi_w \circ \phi_{\phi_w(z)}(\mathcal{U}u)) J\phi_w(\phi_{\phi_w(z)}(\mathcal{U}u)) J\phi_{\phi_w(z)}(\mathcal{U}u) det\mathcal{U} \\
&= (f \circ \phi_{\phi_w(z)})(\mathcal{U}u) J\phi_{\phi_w(z)}(\mathcal{U}u) det\mathcal{U} \\
&= V_{\mathcal{U}} U_{\phi_w(z)} f(u)
\end{aligned}$$

as desired.

Now we will show that

$$\|I + \mathcal{U}\| \le C(n)\rho(z, w).$$

Noting that $\mathcal{U}$ is continuous for $|z| \le 1/2$ and $|w| \le 1/2$, we need only to prove

$$\|I + \mathcal{U}\| \le 20000\rho(z, w),$$

for $|z| > 1/2$, $|w| > 1/2$ and $|\phi_w(z)| < 1/2$. Let $\lambda = \phi_w(z)$. Then $|\lambda| = \rho(z, w)$ and $z = \phi_w(\lambda)$. Since

$$\phi_w \circ \phi_z(u) = \phi_\lambda(\mathcal{U}u),$$

taking derivatives both sides of the above equations and using the chain rule give

$$\phi_w'(\phi_z(u))\, \phi_z'(u) = \phi_\lambda'(\mathcal{U}u)\mathcal{U}.$$

Letting $u = 0$, the above equality gives

$$\mathcal{U} = [\phi_\lambda'(0)]^{-1} \phi_w'(z) \phi_z'(0).$$

By Lemma 2.5, write

$$\begin{aligned}
\mathcal{U} + I &= [\phi_\lambda'(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda \hat{\otimes} w)[\phi_w'(0)]^{-1}\phi_z'(0) + I \\
&= [\phi_\lambda'(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda \hat{\otimes} w)[\phi_w'(0)]^{-1}[\phi_z'(0) - A(z, w)] \\
&\quad + \big([\phi_\lambda'(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda \hat{\otimes} w)[\phi_w'(0)]^{-1}A(z, w) + I\big) \\
&:= I_1 + I_2.
\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}
\|I_1\| &\le \|[\phi_\lambda'(0)]^{-1}\| |1 - \langle \lambda, w\rangle| \|I - \lambda \hat{\otimes} w\| \|[\phi_w'(0)]^{-1}\| \|\phi_z'(0) - A(z, w)\| \\
&\le 4 \times 2 \times 2 \times \frac{3}{(1 - |w|^2)} \left[150|\lambda|(1 - |z|^2)\right].
\end{aligned}$$

Theorem 2.2.2 in [14] leads to

$$\frac{1 - |z|^2}{1 - |w|^2} = \frac{1 - |\lambda|^2}{|1 - \langle \lambda, w\rangle|^2}.$$

Thus

$$\|I_1\| \le 4 \times 2 \times 2 \times 3 \times 2 \times 150|\lambda| = 14400|\lambda|.$$

Also, we have

$$\left|1 - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}\right| \le \left|1 - \frac{1-|z|^2}{1-|w|^2}\right| \le 32|\lambda|.$$

Hence, we get

$$\left\|I - \frac{1-|z|^2}{1-|w|^2}P_w - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}Q_w\right\| \le 32|\lambda|.$$

On the other hand, clearly,

$$\|[\phi'_\lambda(0)]^{-1} + I\| \le 4|\lambda|, \quad |(1 - \langle \lambda, w\rangle) - 1| \le |\lambda|$$

and

$$\|(I - \lambda\hat{\otimes}w) - I\| \le |\lambda|.$$

These give

$$\|I + [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda\hat{\otimes}w)\| \le 16|\lambda|.$$

Hence, we have

$$\begin{aligned}
\|I_2\| &\le \|[\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda\hat{\otimes}w)[\phi'_w(0)]^{-1}A(z,w) \\
&\quad - [\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda\hat{\otimes}w)\| \\
&\quad + \|[\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda\hat{\otimes}w) + I\| \\
&\le \left\|[\phi'_\lambda(0)]^{-1}(1 - \langle \lambda, w\rangle)(I - \lambda\hat{\otimes}w)\right\| \left\|I - \frac{1-|z|^2}{1-|w|^2}P_w - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}Q_w\right\| \\
&\quad + 16|\lambda| \\
&\le 4 \times 2 \times 2 \times 32|\lambda| + 16|\lambda| < 600|\lambda|.
\end{aligned}$$

Combining the above estimates we conclude that

$$\|\mathcal{U} + I\| \le 14400|\lambda| + 600|\lambda| < 20000|\lambda|.$$

□

Theorem 2.9. *Let $S \in \mathfrak{L}(L_a^2)$, $m \ge 0$ and $z \in B$. Then $B_m S_z = (B_m S) \circ \phi_z$.*

Proof. Proposition 2.2 and (1.2) give

$$B_m S_z(0) = C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \left\langle S_z u^k, u^k \right\rangle = B_m S(z) = (B_m S) \circ \phi_z(0).$$

For any $w \in B$, Proposition 2.1 and Lemma 2.8 imply

$$\begin{aligned}(B_m S_z) \circ \phi_w(0) &= B_m((S_z)_w)(0) \\ &= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{U_w U_z S^* U_z U_w K_\lambda(u)} du d\lambda \\ &= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{V_{\mathcal{U}} U_{\phi_z(w)} S^* U_{\phi_z(w)} V_{\mathcal{U}}^* K_\lambda(u)} du d\lambda \\ &= B_m S_{\phi_z(w)}(0)\end{aligned}$$

where $V_{\mathcal{U}}$ is in Lemma 2.8. Thus, $B_m S_z(w) = (B_m S) \circ \phi_z(w)$. □

Lemma 2.10. *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 1$ and $z \in B$. Then*

$$B_m S(z) = \frac{m+n}{m} B_{m-1} \left(S - \sum_{i=1}^n T_{\overline{(\phi_z)_i}} S T_{(\phi_z)_i} \right)(z)$$

where $(\phi_z)_i$ is i-th variable of ϕ_z.

Proof. By Theorem 2.9, we just need to show that

$$B_m S(0) = \frac{m+n}{m} B_{m-1} \left(S - \sum_{i=1}^n T_{\overline{u_i}} S T_{u_i} \right)(0).$$

Using Proposition 2.1 and (2.1), we get

$$\begin{aligned}&B_m S(0) = C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{S^* K_\lambda(u)} du d\lambda \\ &= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^n \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B \int_B u_i \overline{\lambda_i} u^k \overline{\lambda^k S^* K_\lambda(u)} du d\lambda \\ &= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^n \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B S(u^k u_i)(\lambda) \overline{\lambda^k \lambda_i} d\lambda \\ &= \frac{m+n}{m} B_{m-1} S(0) - C_n^{m+n} \sum_{i=1}^n \sum_{|k|=0}^{m-1} C_{m-1,k} \left\langle S T_{u_i}(u^k), T_{u_i}(u^k) \right\rangle\end{aligned}$$

as desired. □

For $m = 0$, the following result was obtained in [10].

Theorem 2.11. *Let $S \in \mathfrak{L}(L_a^2)$ and $m \geq 0$. Then there exists a constant $C(m,n) > 0$ such that*

$$|B_m S(z) - B_m S(w)| < C(m,n) \|S\| \rho(z,w).$$

Proof. We will prove this theorem by induction on m. If $m = 0$, (1.2) gives

$$\begin{aligned}|B_0S(z) - B_0S(w)| &= |tr[S_z(1\otimes 1)] - tr[S_w(1\otimes 1)]| \\ &= |tr[S_z(1\otimes 1) - SU_w(1\otimes 1)U_w]| \\ &= |tr[S_z(1\otimes 1) - SU_z(U_zU_w1\otimes U_zU_w1)U_z]|\end{aligned}$$

From Lemma 2.8, the last term equals

$$\begin{aligned}|tr[S_z(1\otimes 1 - U_{\phi_w(z)}1\otimes U_{\phi_w(z)}1)]| &\le \|S_z\|\|1\otimes 1 - U_{\phi_w(z)}1\otimes U_{\phi_w(z)}1\|_{C^1} \\ &\le \sqrt{2}\|S_z\|(2-2|\langle 1, k_{\phi_w(z)}\rangle|^2)^{1/2} \\ &= 2\|S\|[1-(1-|\phi_w(z)|^2)^{n+1}]^{1/2} \\ &\le C(n)\|S\||\phi_w(z)|\end{aligned}$$

where the second equality holds by $\|T\|_{C^1} \le \sqrt{l}(tr[T^*T])^{1/2}$ where l is the rank of T.

Suppose $|B_{m-1}S(z) - B_{m-1}S(w)| < C(m,n)\|S\|\rho(z,w)$. By Lemma 2.10, we have

$$\begin{aligned}&|B_mS(z) - B_mS(w)| \\ &\quad\le \frac{m+n}{m}|B_{m-1}S(z) - B_{m-1}S(w)| \\ &\qquad + \frac{m+n}{m}\sum_{i=1}^{n}\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(w)\right|.\end{aligned}$$

Since the term in the summation is less than or equals

$$\begin{aligned}&\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)\right| \\ &\quad + \left|B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(z)\right| \\ &\quad + \left|B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(w)\right|,\end{aligned}$$

it is sufficient to show that

$$\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z) - B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)\right| < C(m,n)\|S\|\rho(z,w).$$

Lemma 2.8 gives

$$\begin{aligned}
&\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}-\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)\right| \\
&= C_n^{m+n-1}\left|tr\left[\left(T_{\overline{(\phi_z)_i}-\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)_z\sum_{|k|=0}^{m-1}C_{m-1,k}\frac{n!k!}{(n+|k|)!}\frac{u^k}{\|u^k\|}\otimes\frac{u^k}{\|u^k\|}\right]\right| \\
&\leq C_n^{m+n-1}\sum_{|k|=0}^{m-1}|C_{m-1,k}|\frac{n!k!}{(n+|k|)!}\left|\left\langle S_zT_{(\phi_z)_i\circ\phi_z}\frac{u^k}{\|u^k\|},T_{((\phi_z)_i-(\phi_w)_i)\circ\phi_z}\frac{u^k}{\|u^k\|}\right\rangle\right| \\
&\leq C(m,n)\|S_z\|\left\|T_{((\phi_z)_i-(\phi_w)_i)\circ\phi_z}\frac{u^k}{\|u^k\|}\right\|_2. \qquad (2.6)
\end{aligned}$$

Let $\lambda=\phi_w(z)$. Then

$$\begin{aligned}
\left\|T_{((\phi_z)_i-(\phi_w)_i)\circ\phi_z}\frac{u^k}{\|u^k\|}\right\|_2^2 &\leq \int_B|(\phi_z\circ\phi_z)_i(u)-(\phi_w\circ\phi_z)_i(u)|^2du \\
&= \int_B|(\mathcal{U}u)_i-(\phi_\lambda(u))_i|^2du \\
&\leq 2\int_B|(\mathcal{U}u)_i+u_i|^2+|u_i+(\phi_\lambda(u))_i|^2du
\end{aligned}$$

where $\phi_w\circ\phi_z=\phi_\lambda\circ\mathcal{U}$ for some $\mathcal{U}\in\mathfrak{U}(n)$.

Noting that

$$\phi_\lambda(u)+u=\frac{\lambda-\langle u,\lambda\rangle u+[1-(1-|\lambda|^2)^{1/2}]Q_\lambda(u)}{1-\langle u,\lambda\rangle},$$

we have that for $|\lambda|\leq 1/2$,

$$|\phi_\lambda(u)+u|\leq 2(|\lambda|+|\lambda|+|\lambda|^2)\leq 6|\lambda|.$$

By Lemma 2.8 we also have

$$\int_B|(\mathcal{U}u)_i+u_i|^2du=\int_B|((\mathcal{U}+I)u)_i|^2du\leq C\|\mathcal{U}+I\|^2\leq C|\lambda|^2.$$

Thus (2.6) is less than or equal to

$$C(m,n)\|S_z\|[36|\lambda|^2+C|\lambda|^2]^{1/2}\leq C(m,n)\|S\||\lambda|.$$

The proof is complete. □

Lemma 2.12. *Let $S\in\mathfrak{L}(L_a^2)$ and $m,j\geq 0$. If $|S^*K_\lambda(z)|\leq C$ for any $z\in B$ then $(B_mB_j)(S)=(B_jB_m)(S)$.*

Proof. By Theorem 2.9, it is enough to show that $(B_m B_j)S(0) = (B_j B_m)S(0)$. From Proposition 2.3, Proposition 2.1 and Fubini's Theorem, we have

$$\begin{aligned}
B_m(B_jS)(0) &= B_m(T_{B_jS})(0)\\
&= C_n^{m+n}\int_B B_jS(z)(1-|z|^2)^m dz\\
&= C_n^{m+n}C_n^{j+n}\int_B\int_B\int_B (1-|z|^2)^{m+j+n+1}(1-\langle u,\lambda\rangle)^j\times\\
&\qquad K_z^j(u)\overline{K_z^j(\lambda)}\,\overline{S^*K_\lambda(u)}\,dud\lambda dz\\
&= C_n^{m+n}C_n^{j+n}\int_B\int_B (1-\langle u,\lambda\rangle)^j\int_B (1-|z|^2)^{m+j+n+1}\times\\
&\qquad K_z^j(u)\overline{K_z^j(\lambda)}dz\,\overline{S^*K_\lambda(u)}\,dud\lambda.
\end{aligned}$$

Let

$$F_{m,j}(u,\lambda) = (1-\langle u,\lambda\rangle)^j\int_B (1-|z|^2)^{m+j+n+1}K_z^j(u)\overline{K_z^j(\lambda)}dz.$$

Then $F_{m,j}(u,\lambda) = \sum_{i=1}^l H_i(u)\overline{G_i}(\lambda)$ where H_i and G_i are holomorphic functions and for some $l \geq 0$. Thus, from Lemma 9 in [9], we just need to show $F_{m,j}(\lambda,\lambda) = F_{j,m}(\lambda,\lambda)$ for $\lambda \in B$. The change of variables implies

$$\begin{aligned}
F_{m,j}(\lambda,\lambda) &= (1-|\lambda|^2)^j\int_B (1-|z|^2)^{m+j+n+1}|K_\lambda^j(z)|^2 dz\\
&= (1-|\lambda|^2)^j\int_B (1-|\phi_\lambda(z)|^2)^{m+j+n+1}|K_\lambda^j(\phi_\lambda(z))|^2|k_\lambda(z)|^2 dz\\
&= (1-|\lambda|^2)^m\int_B (1-|z|^2)^{m+j+n+1}|K_\lambda^m(z)|^2 dz\\
&= F_{j,m}(\lambda,\lambda)
\end{aligned}$$

as desired. □

Lemma 2.13. *For any $S \in \mathfrak{L}(L_a^2)$, there exists sequences $\{S_\alpha\}$ satisfying*

$$|S_\alpha^* K_\lambda(u)| \leq C(\alpha)$$

such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwise.

Proof. Since H^∞ is dense in L_a^2 and the set of finite rank operators is dense in the ideal $\mathcal{K}$ of compact operators on L^2, the set $\{\sum_{i=1}^l f_i\otimes g_i : f_i,\ g_i \in H^\infty\}$ is dense in the ideal $\mathcal{K}$ in the norm topology. Since $\mathcal{K}$ is dense in the space of bounded operators on L_a^2 in strong operator topology, (2.3) gives that for any $S \in \mathfrak{L}(L_a^2)$, there exists a finite rank operator sequences $S_\alpha = \sum_{i=1}^l f_i \otimes g_i$ such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwise for some $f_i,\ g_i$ in H^∞. Also, for $l \geq 0$, for such

$S_\alpha = \sum_{i=1}^{l} f_i \otimes g_i$, we have

$$\begin{aligned} |S_\alpha^* K_\lambda(u)| &= \left| \sum_{i=1}^{l} (g_i \otimes f_i) K_\lambda(u) \right| \\ &= \left| \sum_{i=1}^{l} \langle K_\lambda(u), f_i(u) \rangle \, g_i(u) \right| \\ &\le \sum_{i=1}^{l} |f_i(\lambda)| |g_i(u)| \\ &\le \sum_{i=1}^{l} \|f_i\|_\infty \|g_i\|_\infty < C. \end{aligned}$$

The proof is complete. □

Proposition 2.14. *Let $S \in \mathfrak{L}(L_a^2)$ and $m, j \ge 0$. Then*

$$(B_m B_j)(S) = (B_j B_m)(S).$$

Proof. Let $S \in \mathfrak{L}(L_a^2)$. Then Lemma 2.13 implies that there exists a sequence $\{S_\alpha\}$ satisfying $|S_\alpha^* K_\lambda(u)| \le C(\alpha)$, hence $B_m(B_j S_\alpha)(z) = B_j(B_m S_\alpha)(z)$ by Lemma 2.12. From Proposition 2.3, we know

$$B_m(B_j S_\alpha)(z) = \int_B (B_j S_\alpha) \circ \phi_z(u) d\nu_m(u)$$

and $\|(B_j S_\alpha) \circ \phi_z\|_\infty \le C(j, n)\|S\|$. Also, $(B_j S_\alpha) \circ \phi_z(u)$ converges to $(B_j S) \circ \phi_z(u)$. Therefore $B_m(B_j S_\alpha)(z)$ converges to $B_m(B_j S)(z)$. By the uniqueness of the limit, we have $(B_m B_j)(S) = (B_j B_m)(S)$. □

Proposition 2.15. *Let $S \in \mathfrak{L}(L_a^2)$ and $m \ge 0$. If $B_0 S(z) \to 0$ as $z \to \partial B$ then $B_m S(z) \to 0$ as $z \to \partial B$.*

Proof. Suppose $B_0 S(z) \to 0$ as $z \to \partial B$. Then we will prove that $S_z \to 0$ in the $\mathcal{T}^*$-norm as $z \to \partial B$. Suppose it is not true. Then for some net $\{w_\alpha\} \in B$ and an operator $V \ne 0$ in $\mathfrak{L}(L_a^2)$, there exists a sequence $\{S_{w_\alpha}\}$ such that $S_{w_\alpha} \to V$ in the $\mathcal{T}^*$-norm as $w_\alpha \to \partial B$, hence $tr[S_{w_\alpha} T] \to tr[VT]$ for any $T \in \mathcal{T}$. Let $T = k_\lambda \otimes k_\lambda$ for fixed $\lambda \in B$. Then Theorem 2.9 implies

$$\begin{aligned} tr[S_{w_\alpha} T] &= tr[S_{w_\alpha}(k_\lambda \otimes k_\lambda)] \\ &= \langle S_{w_\alpha} k_\lambda, k_\lambda \rangle \\ &= B_0 S_{w_\alpha}(\lambda) \\ &= (B_0 S) \circ \phi_{w_\alpha}(\lambda) \to 0 \end{aligned}$$

as $w_\alpha \to \partial B$. Since $tr[VT] = B_0 V(\lambda)$ and B_0 is one-to-one mapping, $V = 0$. This is the contradiction. Thus $S_z \to 0$ as $z \to \partial B$ in the $\mathcal{T}^*$-norm. (1.2) finishes the proof of this proposition. □

3. OPERATORS S APPROXIMATED BY TOEPLITZ OPERATORS $T_{B_m(S)}$

In this section we will give a criterion for operators approximated by Toeplitz operators with symbol equal to their m-Berezin transforms. The main result in this section is Theorem 3.7. It extends and improves Theorem 2.4 in [17]. Even on the unit disk, we will show an example that the result in the theorem is sharp on the unit disk.

From Proposition 1.4.10 in [14], we have the following lemma

Lemma 3.1. *Suppose* $a < 1$ *and* $a + b < n + 1$. *Then*

$$\sup_{z\in B} \int_B \frac{d\lambda}{(1-|\lambda|^2)^a |1-\langle \lambda, z\rangle|^b} < \infty.$$

This lemma gives the following lemma which extends Lemma 4.2 in [13].

Let $1 < q < \infty$ and p be the conjugate exponent of q. If we take $p > n + 2$, then $q < (n + 2)/(n + 1)$.

Lemma 3.2. *Let* $S \in \mathfrak{L}(L_a^2)$ *and* $p > n + 2$. *Then there exists* $C(n, p) > 0$ *such that* $h(z) = (1 - |z|^2)^{-a}$ *where* $a = (n + 1)/(n + 2)$ *satisfies*

$$\int_B |(SK_z)(w)| h(w) dw \le C(n,p) \|S_z 1\|_p h(z) \tag{3.1}$$

for all $z \in B$ *and*

$$\int_B |(SK_z)(w)| h(z) dz \le C(n,p) \|S_w^* 1\|_p h(w) \tag{3.2}$$

for all $w \in B$.

Proof. Fix $z \in B$. Since

$$U_z 1 = (-1)^n (1-|z|^2)^{(n+1)/2} K_z,$$

we have

$$\begin{aligned} SK_z &= (-1)^n (1-|z|^2)^{-(n+1)/2} S U_z 1 \\ &= (-1)^n (1-|z|^2)^{-(n+1)/2} U_z S_z 1 \\ &= (1-|z|^2)^{-(n+1)/2} (S_z 1 \circ \phi_z) k_z. \end{aligned}$$

Thus, letting $\lambda = \phi_z(w)$, the change of variables implies

$$\begin{aligned} \int_B \frac{|(SK_z)(w)|}{(1-|w|^2)^a} dw &= \frac{1}{(1-|z|^2)^{(n+1)/2}} \int_B \frac{|(S_z 1 \circ \phi_z)(w)||k_z(w)|}{(1-|w|^2)^a} dw \\ &= \frac{1}{(1-|z|^2)^a} \int_B \frac{|S_z 1(\lambda)|}{(1-|\lambda|^2)^a |1-\langle \lambda, z\rangle|^{n+1-2a}} d\lambda \\ &\le \frac{\|S_z 1\|_p}{(1-|z|^2)^a} \left(\int_B \frac{1}{(1-|\lambda|^2)^{aq} |1-\langle \lambda, z\rangle|^{(n+1-2a)q}} d\lambda \right)^{\frac{1}{q}}. \end{aligned}$$

The last inequality comes from Holder's inequality. Since $aq < 1$ and $aq + (n + 1 - 2a)q < n + 1$, Lemma 3.1 implies (3.1).

To prove (3.2), replace S by S^* in (3.1), interchange w and z in (3.1) and then use the equation

$$(S^*K_w)(z) = \langle S^*K_w, K_z\rangle = \langle K_w, SK_z\rangle = \overline{SK_z}(w) \tag{3.3}$$

to obtain the desired result. □

Lemma 3.3. *Let $S \in \mathfrak{L}(L_a^2)$ and $p > n + 2$. Then*

$$\|S\| \leq C(n,p)\left(\sup_{z\in B}\|S_z 1\|_p\right)^{1/2}\left(\sup_{z\in B}\|S_z^* 1\|_p\right)^{1/2}$$

where $C(n,p)$ is the constant of Lemma 3.2.

Proof. (3.3) implies

$$\begin{aligned}(Sf)(w) &= \langle Sf, K_w\rangle \\ &= \int_B f(z)\overline{(S^*K_w)}(z)dz \\ &= \int_B f(z)(SK_z)(w)dz\end{aligned}$$

for $f \in L_a^2$ and $w \in B$. Thus, Lemma 3.2 and the classical Schur's theorem finish the proof. □

Lemma 3.4. *Let S_m be a bounded sequence in $\mathfrak{L}(L_a^2)$ such that $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$. Then*

$$\sup_{z\in B}|\langle (S_m)_z 1, f\rangle| \to 0 \tag{3.4}$$

as $m \to \infty$ for any $f \in L_a^2$ and

$$\sup_{z\in B}|(S_m)_z 1| \to 0 \tag{3.5}$$

uniformly on compact subsets of B as $m \to \infty$.

Proof. To prove (3.4), we only need to have

$$\sup_{z\in B}\left|\left\langle (S_m)_z 1, w^k\right\rangle\right| \to 0 \tag{3.6}$$

as $m \to \infty$ for any multi-index k.

Since

$$K_z(w) = \sum_{|\alpha|=0}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}\overline{z}^\alpha w^\alpha, \tag{3.7}$$

we have

$$\begin{aligned}&B_0S_m(\phi_z(\lambda)) = B_0(S_m)_z(\lambda)\\&= (1-|\lambda|^2)^{n+1}\sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=0}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}\frac{(n+|\beta|)!}{n!\beta!}\left\langle (S_m)_z w^\alpha, w^\beta\right\rangle \overline{\lambda}^\alpha\lambda^\beta\end{aligned}$$

where α, β are multi-indices.

Then for any fixed k and $0<r<1$,

$$\begin{aligned}&\int_{rB}\frac{B_0S_m(\phi_z(\lambda))\overline{\lambda}^k}{(1-|\lambda|^2)^{n+1}}d\lambda\\&= \sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=0}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}\frac{(n+|\beta|)!}{n!\beta!}\left\langle (S_m)_z w^\alpha, w^\beta\right\rangle \int_{rB}\overline{\lambda}^{\alpha+k}\lambda^\beta d\lambda\\&= r^{2n+2|k|}\left(\left\langle (S_m)_z 1, w^k\right\rangle + \sum_{|\alpha|=1}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle r^{2|\alpha|}\right).\end{aligned}$$

Since S_m is bounded sequence, we have

$$\begin{aligned}&\left|\left\langle (S_m)_z 1, w^k\right\rangle\right|\\&\le r^{-2n-2|k|}\left|\int_{rB}\frac{B_0S_m(\phi_z(\lambda))\overline{\lambda}^k}{(1-|\lambda|^2)^{n+1}}d\lambda\right| +\\&\qquad\qquad \sum_{|\alpha|=1}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}\|(S_m)_z\|\|w^\alpha\|\|w^{\alpha+k}\|r^{2|\alpha|}\\&\le r^{-2n-2|k|}\|B_0S_m\|_\infty\int_{rB}\frac{|\lambda^k|}{(1-|\lambda|^2)^{n+1}}d\lambda + C\sum_{|\alpha|=1}^{\infty}r^{2|\alpha|},\end{aligned}$$

hence, by assumption

$$\limsup_{m\to\infty}\sup_{z\in B}|\left\langle (S_m)_z 1, w^k\right\rangle| \le C\sum_{|\alpha|=1}^{\infty}r^{2|\alpha|}.$$

Letting $r\to 0$, we have (3.6).

Now we prove (3.5). From (3.7), we get

$$\begin{aligned}|(S_m)_z1(\lambda)| &= |\langle (S_m)_z 1, K_\lambda\rangle|\\&\le \sum_{|\alpha|=0}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}|\langle (S_m)_z1, w^\alpha\rangle|\,|\lambda^\alpha|\\&\le \sum_{|\alpha|=0}^{l-1}\frac{(n+|\alpha|)!}{n!\alpha!}|\langle (S_m)_z1, w^\alpha\rangle| + \sum_{|\alpha|=l}^{\infty}\frac{(n+|\alpha|)!}{n!\alpha!}\|S_m\|\|w^\alpha\||\lambda^\alpha|\end{aligned}$$

for $z \in B$, $\lambda \in rB$ and $l \geq 1$. Since the second summation is less than or equals to

$$\sum_{j=l}^{\infty} \left(\frac{(n+j)!}{n!j!}\right)^{1/2} \sum_{|\alpha|=j} \left(\frac{j!}{\alpha!}\right)^{1/2} |\lambda^{\alpha}| \leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} \left[\sum_{|\alpha|=j} \frac{j!}{\alpha!}|\lambda^{\alpha}|^2\right]^{1/2}$$
$$\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} r^j,$$

for any $\epsilon > 0$, we can find sufficiently large l such that the second summation is less than ϵ. Thus, (3.6) imply $\sup_{z\in B} |(S_m)_z 1| \to 0$ uniformly on compact subsets of B as $m \to \infty$. $\square$

Lemma 3.5. *Let $\{S_m\}$ be a sequence in $\mathfrak{L}(L^2_a)$ such that for some $p > n+2$, $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$,*

$$\sup_{z\in B} \|(S_m)_z 1\|_p \leq C \qquad \text{and} \qquad \sup_{z\in B} \|(S^*_m)_z 1\|_p \leq C$$

where $C > 0$ is independent of m, then $S_m \to 0$ as $m \to \infty$ in $\mathfrak{L}(L^2_a)$-norm.

Proof. Lemma 3.3 implies

$$\|S_m\| \leq C(n,p) \left(\sup_{z\in B} \|(S_m)_z 1\|_p\right)^{1/2} \left(\sup_{z\in B} \|(S^*_m)_z 1\|_p\right)^{1/2} \leq C(n,p),$$

hence, Lemma 3.4 gives

$$\sup_{z\in B} |(S_m)_z 1| \to 0 \tag{3.8}$$

uniformly on compact subsets of B as $m \to \infty$.

Here, for $n+2 < s < p$, Holder's inequality gives

$$\sup_{z\in B} \|(S_m)_z 1\|_s^s \leq \sup_{z\in B} \int_{B\setminus r\overline{B}} |(S_m)_z 1(w)|^s dw + \sup_{z\in B} \int_{r\overline{B}} |(S_m)_z 1(w)|^s dw$$
$$\leq C \sup_{z\in B} \|(S_m)_z 1\|_p^s (1-r)^{1-s/p} + \sup_{z\in B} \int_{r\overline{B}} |(S_m)_z 1(w)|^s dw$$

and (3.8) implies the second term tends to 0 as $m \to \infty$. Also, the first term is less than or equals to $C^s(1-r)^{1-s/p}$ which can be small by taking r close to 1. Consequently, Lemma 3.3 gives

$$\|S_m\| \leq C(n,s) \left(\sup_{z\in B} \|(S_m)_z 1\|_s\right)^{1/2} \left(\sup_{z\in B} \|(S^*_m)_z 1\|_s\right)^{1/2}.$$
$$\leq C(n,s) \left(\sup_{z\in B} \|(S_m)_z 1\|_s\right)^{1/2} \to 0$$

$\square$

Corollary 3.6. *Let $S \in \mathfrak{L}(L_a^2)$ such that for some $p > n+2$,*

$$\sup_{z\in B} \|S_z 1 - (T_{B_m S})_z 1\|_p \le C \qquad \text{and} \qquad \sup_{z\in B} \|S_z^* 1 - (T_{B_m(S^*)})_z 1\|_p \le C, \tag{3.9}$$

where $C > 0$ is independent of m. Then $T_{B_m S} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. Let $S_m = S - T_{B_m S}$. Then Proposition 2.14 and Theorem 2.11 imply

$$\begin{aligned} B_0(S_m) &= B_0 S - B_0(T_{B_m S}) \\ &= B_0 S - B_0(B_m S) \\ &= B_0 S - B_m(B_0 S) \end{aligned}$$

which tends uniformly to 0 as $m \to \infty$, hence $\|B_0(S_m)\|_\infty \to 0$. Consequently, by Lemma 3.5 we complete the proof. □

Theorem 3.7. *Let $S \in \mathfrak{L}(L_a^2)$. If there is $p > n+2$ such that*

$$\sup_{z\in B} \|T_{(B_m S)\circ\phi_z} 1\|_p < C \quad \textit{and} \quad \sup_{z\in B} \|T^*_{(B_m S)\circ\phi_z} 1\|_p < C \tag{3.10}$$

where $C > 0$ is independent of m, then $T_{B_m S} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. By Corollary 3.6, we only need to show that (3.10) implies (3.9). Since $T_{(B_m S)\circ\phi_z} = (T_{B_m S})_z$ and

$$T^*_{(B_m S)\circ\phi_z} = T_{\overline{B_m S_z}} = T_{B_m(S_z^*)} = T_{(B_m(S^*))\circ\phi_z},$$

it is sufficient to show that

$$\sup_{z\in B} \|S_z 1\|_p < \infty.$$

By Lemma 3.3, we get

$$\begin{aligned} \|T_{B_m S}\| &\le C(n,p) \left(\sup_{z\in B} \|T_{B_m S\circ\phi_z} 1\|_p\right)^{1/2} \left(\sup_{z\in B} \|T^*_{B_m S\circ\phi_z} 1\|_p\right)^{1/2} \\ &< C \end{aligned}$$

where C is independent of m, hence writing $S_m = S - T_{B_m S}$, we have $\|S_m\| \le C$ where C is independent of m. Also, the proof of Corollary 3.6 implies

$$\|B_0 S_m\|_\infty \to 0$$

as $m \to \infty$.

Let f be a polynomial with $\|f\|_q = 1$. Then Lemma 3.4 implies

$$\sup_{z\in B} |\langle (S_m)_z 1, f\rangle| \to 0$$

as $m \to \infty$. Thus, for any $\epsilon > 0$ and $z_0 \in B$, we have

$$|\langle S_{z_0} 1, f\rangle| \le \sup_{z\in B} |\langle (S_m)_z 1, f\rangle| + |\langle (T_{B_m S})_{z_0} 1, f\rangle| \le \epsilon + C$$

for sufficiently large m, where C is independent of m. Since ϵ is arbitrary, we get

$$\sup_{z\in B} \|S_z 1\|_p < \infty$$

as desired. □

4. Compact Radial operator

Given $\mathcal{U} \in \mathfrak{U}(n)$, define $V_{\mathcal{U}} f(w) = f(\mathcal{U}w) det\mathcal{U}$ for $f \in L_a^2$. Then $V_{\mathcal{U}}$ is a unitary operator on L_a^2. We say that $S \in \mathfrak{L}(L_a^2)$ is a radial operator if $SV_{\mathcal{U}} = V_{\mathcal{U}} S$ for any $\mathcal{U} \in \mathfrak{U}(n)$.

If $S \in \mathfrak{L}(L_a^2)$, the radialization of S is defined by

$$S^{\sharp} = \int_{\mathfrak{U}} V_{\mathcal{U}}{}^{*} S V_{\mathcal{U}} d\mathcal{U}$$

where $d\mathcal{U}$ is the Haar measure on the compact group $\mathfrak{U}(n)$ and the integral is taken in the weak sense. Then $S^{\sharp} = S$ if S is radial and $\mathfrak{U}$-invariance of $d\mathcal{U}$ shows that $S^{\sharp}$ is indeed a radial operator.

If $f \in L^{\infty}$ and $g, h \in L_a^2$ then

$$\begin{aligned} \langle V_{\mathcal{U}}{}^{*} T_f V_{\mathcal{U}} g, h\rangle &= \int_B f(w) V_{\mathcal{U}} g(w) \overline{V_{\mathcal{U}} h(w)} dw \\ &= \int_B f(\mathcal{U}^{*} w) g(w) \overline{h(w)} dw. \end{aligned}$$

Thus $V_{\mathcal{U}}{}^{*} T_f V_{\mathcal{U}} = T_{f\circ\mathcal{U}^{*}}$ and

$$V_{\mathcal{U}}{}^{*} T_{f_1} \cdots T_{f_l} V_{\mathcal{U}} = T_{f_1\circ\mathcal{U}^{*}} \cdots T_{f_l\circ\mathcal{U}^{*}}$$

for $f_1, \ldots, f_l \in L^{\infty}$, $l \geq 0$.

Lemma 4.1. *Let $S \in \mathfrak{L}(L_a^2)$ be a radial operator. Then $T_{B_m(S)} = \int_B S_w d\nu_m(w)$.*

Proof. Let $z \in B$. By (2.3) and Lemma 2.8, we obtain

$$\begin{aligned} B_0\left(\int_B S_w d\nu_m(w)\right)(z) &= \left\langle \left(\int_B S_w d\nu_m(w)\right)_z 1, 1\right\rangle \\ &= \int_B \langle U_z U_w S U_w U_z 1, 1\rangle \, d\nu_m(w) \\ &= \int_B \left\langle U_{\phi_z(w)} V_{\mathcal{U}}^{*} S V_{\mathcal{U}} U_{\phi_z(w)} 1, 1\right\rangle d\nu_m(w) \end{aligned}$$

where $V_{\mathcal{U}}$ is in Lemma 2.8. Since S is a radial operator, Theorem 2.9, Proposition 2.3 and Proposition 2.14 imply that the last integral equals

$$\begin{aligned}\int_B \left\langle U_{\phi_z(w)} S U_{\phi_z(w)} 1, 1\right\rangle d\nu_m(w) &= \int_B B_0 S \circ \phi_z(w) d\nu_m(w)\\ &= B_m B_0 S(z)\\ &= B_0 B_m S(z)\\ &= B_0(T_{B_m(S)})(z).\end{aligned}$$

Since B_0 is one-to-one mapping, the proof is complete. □

Theorem 4.2. *Let $S \in \mathfrak{T}(L^\infty)$ be a radial operator. Then S is compact if and only if $B_0 S \equiv 0$ on ∂B.*

Proof. Suppose $B_0 S \equiv 0$ on ∂B. Then $B_m S \equiv 0$ on ∂B by Proposition 2.15, hence $T_{B_m S}$ is compact for all $m \geq 0$.

Let

$$Q = \int_{\mathfrak{U}} T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*} d\mathcal{U}$$

with $f_1, \ldots, f_l \in L^\infty$ for some $l \geq 0$. Then $Q \in \mathfrak{L}(L^2_a)$. By Lemma 4.1, for any $z \in B$, we have

$$\begin{aligned}T_{(B_m(Q))\circ\phi_z} &= \int_B ((Q)_z)_w d\nu_m(w)\\ &= \int_B \int_{\mathfrak{U}} T_{f_1\circ\mathcal{U}^*\circ\phi_z\circ\phi_w} \cdots T_{f_l\circ\mathcal{U}^*\circ\phi_z\circ\phi_w} d\mathcal{U} d\nu_m(w).\end{aligned}$$

Consequently,

$$\begin{aligned}\|T_{(B_m(Q))\circ\phi_z}\| &\leq C(l)\|f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \cdots \|f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty\\ &= C(l)\|f_1\|_\infty \cdots \|f_l\|_\infty.\end{aligned}$$

Similarly, we have

$$\|T^*_{(B_m(Q))\circ\phi_z}\| \leq C(l)\|f_1\|_\infty \cdots \|f_l\|_\infty.$$

Thus, Theorem 3.7 gives that

$$T_{B_m(Q)} \to Q \tag{4.1}$$

in $\mathfrak{L}(L^2_a)$-norm.

Since $S \in \mathfrak{T}(L^\infty)$, there exists a sequence $\{S_k\}$ such that $S_k \to S$ in $\mathfrak{L}(L^2_a)$-norm where each S_k is a finite sum of finite products of Toeplitz operators. Since the radialization is continuous and S is radial, $S_k^\sharp \to S^\sharp = S$. From Lemma 4.1, we have

$$\|T_{B_m S}\| = \left\|\int_B S_w d\nu_m(w)\right\| \leq \int_B \|S_w\| d\nu_m(w) = \|S\|.$$

Thus

$$\begin{aligned}\|S - T_{B_m S}\| &\le \|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| + \|T_{B_m(S_k^\sharp)} - T_{B_m S}\| \\ &\le 2\|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\|\end{aligned}$$

and (4.1) imply $T_{B_m(S)} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm, hence S is compact.

The other direction is trivial. □

Example. This example shows that for $n = 1$, the number $n + 2 = 3$ in Theorem 3.7 is sharp. We show that there is a bounded operator S on L_a^2 such that

$$\sup_{z\in D} \max\{\|T_{(B_m S)\circ\phi_z} 1\|_3, \|T^*_{(B_m S)\circ\phi_z} 1\|_3\} < \infty,$$

and for each $m \ge 0$, $B_m(S)(z) \to 0$ as $z \to \partial D$, but S is not compact on L_a^2.

The following operator S was constructed in [3] to show that $B_0(S)(z) \to 0$ as $z \to \partial D$, but S is not compact on L_a^2. Let S be defined on L_a^2 by

$$S\left(\sum_{l=0}^\infty a_l w^l\right) = \sum_{l=0}^\infty a_{2^l} w^{2^l}.$$

It is clear that S is a self-adjoint projection with infinite-dimensional range. Thus S is not compact on L_a^2. From

$$\begin{aligned}B_0(S)(z) &= \langle S k_z, k_z\rangle \\ &= \|S k_z\|_2^2 \\ &= (1 - |z|^2)^2 \sum_{l=0}^\infty (2^l + 1)(|z|^2)^{2^l},\end{aligned}$$

it is easy to see that $B_0(S)(z) \to 0$ as $z \to \partial D$. By Proposition 2.15, we see that $B_m(S)(z) \to 0$ as $z \to \partial D$. This gives that $T_{B_m(S)}$ is compact. Hence $T_{B_m(S)}$ does not converge to S in the norm topology.

By means of the Zygmund theorem on gap series [18], it was proved in [13] that

$$C = \sup_{z\in D} \max\{\|S_z 1\|_3, \|S_z^* 1\|_3\} < \infty.$$

Clearly, S is a radial operator. By Lemma 4.1, we have

$$\begin{aligned}T_{(B_m S)\circ\phi_z} 1 &= \int_D (S_w)_z 1 d\nu_m(w) \\ &= \int_D S_{\phi_z(w)} 1 d\nu_m(w) \\ &= \int_D S_\lambda 1 d\nu_m \circ \phi_z(\lambda).\end{aligned}$$

Noting that for each $z \in D$, $d\nu_m \circ \phi_z$ is a probability measure on D, we have

$$\|T_{(B_m S)\circ\phi_z} 1\|_3 \leq \int_D \|S_\lambda 1\|_3 d\nu_m \circ \phi_z(\lambda) \leq C.$$

Similarly, we also have

$$\|T^*_{(B_m S)\circ\phi_z} 1\|_3 \leq C.$$

REFERENCES

[1] P. Ahern, M. Flores, and W. Rudin, An invariant volume-mean-value property, *J. Funct. Anal.*, 111 (1993), 380–397.

[2] S. Axler and D. Zheng, The Berezin transform on the Toeplitz algebra, *Studia Math.*, 127 (1998), 113–136.

[3] S. Axler and D. Zheng, Compact operators via the Berezin transform, *Indiana Univ. Math. J.* 47 (1998), 387–399.

[4] D. Bekolle, C. Berger, L. Coburn, and K. Zhu, BMO in the Bergman metric on bounded symmetric domains, *J. Funct. Anal.*, 93(1990), 310–350.

[5] F. A. Berezin, Covariant and contravariant symbols of operators, *Math. USSR Izv.*, 6 (1972), 1117–1151.

[6] C. Berger and L. Coburn, Toeplitz operators on the Segal-Bargman space, *Tran. AMS*, 301(1987), 813–829.

[7] C. Berger and L. Coburn, Heat flow and Berezin-Toeplitz estimates, *Amer. J. Math.*, 116(1994), 563–590.

[8] O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, I, II, *Springer Verlag, New York*, 1979, 1981.

[9] B. R. Choe and Y. J. Lee, Pluriharmonic symbols of commuting Toeplitz operators, *Illinois J. Math.*, **37**, No. 3 (1993), 424–436.

[10] L. Coburn, A Lipschitz estimate for Berezin's operator calculus, *Proc. Amer. Math. Soc.*, 133 (2005), no. 1, 127–131.

[11] M. Engliš, Compact Toeplitz operators via the Berezin transform on bounded symmetric domains , *Integral Equations Operator Theory*, **33**, No. 4 (1999), 426–455.

[12] I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators, *Translations of Mathematical Monographs* 18, *Amer. Math. Soc., Providence*, 1969.

[13] J. Miao and D. Zheng, Compact operators on Bergman spaces, *Integral Equations and Operator Theory*, 48(2004), 61–79.

[14] W. Rudin, Function Theory in the unit Ball of $\mathbb{C}^n$, *Springer-Verlag, Berlin, Heidelberg, New york*, 1980.

[15] K. Stroethoff, The Berezin transform and operators on spaces of analytic functions, *Banach Center Publ.*, 38 (1997), 361–380.

[16] D. Suárez, Approximation and symbolic calculus for Toeplitz algebras on the Bergman space, *Rev. Mat. Iberoamericana*, 20 (2004), no. 2, 563–610..

[17] D. Suárez, Approximation and the n-Berezin transform of operators on the Bergman space, preprint.

[18] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London, 1959.

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