The Horn conjecture for compact selfadjoint operators

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$N \times N$ complex Hermitian matrix A.

eigenvalues $\Lambda(A) = \{\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_N(A)\} \subset \mathbb{R}^N_\downarrow$

Question. *Characterize* $(\alpha, \beta, \gamma) \in (\mathbb{R}^N_\downarrow)^3$ *such that there exist Hermitian matrices* A, B*, and* C *such that* $\alpha = \Lambda(A), \beta = \Lambda(B)$*, and* $\gamma = \Lambda(C)$ *such that* $C = A + B$*.*

Notation:

$I = \{i_1 < i_2 < \ldots < i_r\}$.

$I^c = \mathbb{N} \setminus I$.

I^c_p = set consisting of the p smallest elements of I^c.

$|I| = |J| = |K| = r$

Theorem. *(conjectured by A. Horn, proved by Klyachko, Totaro, Knutson and Tao.)*

Let $(\alpha, \beta, \gamma) \in (\mathbb{R}^N_\downarrow)^3$. *The following are equivalents:*

(1) There exist Hermitian $N \times N$ *matrices* A, B, *and* $C = A + B$ *with* $\alpha = \Lambda(A)$, $\beta = \Lambda(B)$, $\gamma = \Lambda(C)$.

(2) For every Horn triple $(I, J, K) \in T^N_r$, $1 \leq r \leq N-1$, *the triple* (α, β, γ) *satisfies the Horn inequality*

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

and the trace equality

$$\sum_{i=1}^N \gamma_i = \sum_{i=1}^N \alpha_i + \sum_{i=1}^N \beta_i \, .$$

From Hermitian matrices to compact selfadjoint operators

$A =$ compact operator

$\Lambda_+(A) = \{\lambda_1(A) \geq \lambda_2(A) \geq \cdots\}$

Theorem. *Let $\alpha, \beta, \gamma \in \mathbb{R}^{\mathbb{N}}_{\downarrow}$, with limit zero. The following conditions are equivalent:*

(1) There exist positive compact operators A and B such that $\Lambda_+(A) = \alpha$, $\Lambda_+(B) = \beta$, $\Lambda_+(A+B) = \gamma$.

(2) For every Horn triple (I, J, K), and all positive integers p, q, we have the Horn inequality

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

and the extended reverse Horn inequality:

$$\sum_{k \in K^c_{p+q}} \gamma_k \geq \sum_{i \in I^c_p} \alpha_i + \sum_{j \in J^c_q} \beta_j .$$

‘Cut and interpolate’

$\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$
$\alpha^* =$ decreasing rearrangement of α.
α, α', $\alpha'' \in \mathbb{R}^N$.

Definition. *α is between α' and α'' if* $\min\{\alpha'_i, \alpha''_i\} \le \alpha_i \le \max\{\alpha'_i, \alpha''_i\}$.

Lemma. *If α', $\alpha'' \in \mathbb{R}^N$ are decreasing and α is between α' and α'', then α^* is between α' and α''.*

Proposition. *For $N \in \mathbb{N}$. Let α, α', β, β', γ, $\gamma' \in \mathbb{R}^N_\downarrow$, and satisfying all Horn and reverse Horn inequalities for all $r \le N$, i.e. for every $(I, J, K) \in T^N_r$,*

$$\sum_{k \in K} \gamma'_k \le \sum_{i \in I} \alpha'_i + \sum_{j \in J} \beta'_j$$

and

$$\sum_{k \notin K} \gamma''_k \ge \sum_{i \notin I} \alpha''_i + \sum_{i \notin J} \beta''_j \,.$$

Then, there exist Hermitian $N \times N$ matrices A, B, C such that $C = A + B$ and $\Lambda(A)$ (resp. $\Lambda(B)$, $\Lambda(C)$) is between α' and α'' (resp. β' and β'', γ' and γ'').

A selfadjoint compact operator on $\mathcal{H}$,

$Ah = \sum_k \mu_k(h, e_k)$, $h \in \mathcal{H}$, $\{e_k\}$ o.n. system, $\lim_{k\to\infty} \mu_k = 0$.

$\lambda_{\pm n}$

λ_n is the n-th largest non-negative term of (μ_k)

λ_{-n} is the n-th smallest non-positive term of (μ_k)

Denote $\Lambda_0(A)$ the sequence $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{-2} \geq \lambda_{-1}$.

Inserting a gap

(a technical lemma)

Lemma. *Fix* $(I, J, K) \in T_r^N$, $0 \leq p, q$, $p{+}q \leq r$ *and* $M \in \mathbb{N}$.

$$I'_\ell = \begin{cases} I_\ell & \textit{if } l \leq r - p \\ I_\ell + M & \textit{if } \ell > r - p \end{cases}$$

$$J'_\ell = \begin{cases} J_\ell & \textit{if } \ell \leq r - q \\ J_\ell + M & \textit{if } \ell > r - q \end{cases}$$

$$K'_\ell = \begin{cases} K_\ell & \textit{if } \ell \leq r - (p+q) \\ K_\ell + M & \textit{if } \ell > r - (p+q) \end{cases}$$

Then, $(I', J', K') \in T_r^{N+M}$.

Extended Horn inequalities

Proposition. *Fix compact self-adjoint operators* A*,* B*,* C *on* $\mathcal{H}$*,* $C \leq A + B$*,* $(I, J, K) \in T_r^N$*,* $0 \leq p, q$*,* $p + q \leq r$*. Then the sequences* $\alpha = \Lambda_0(A)$*,* $\beta = \lambda_0(B)$*,* $\gamma = \Lambda_0(C)$ *satisfy the inequalities*

$$\sum_{\ell=1}^{r-(p+q)} \gamma_{K_\ell} + \sum_{\ell=r-(p+q)+1}^{r} \gamma_{K_\ell - N - 1}$$
$$\leq \sum_{\ell=1}^{r-p} \alpha_{I_\ell} + \sum_{\ell=r-p+1}^{r} \alpha_{I_\ell - N - 1} + \sum_{\ell=1}^{r-q} \beta_{J_\ell} + \sum_{\ell=e-q+1}^{r} \beta_{J_\ell - N - 1} \,.$$

Proof. Choose a projection P whose range contains all the eigenvectors of A, B, C corresponding with $\alpha_{\pm n}$, $\beta_{\pm n}$, $\gamma_{\pm n}$, $n \leq N$, rank of P equals $N + M$. Apply Lemma.

$C_{\downarrow 0 \uparrow}$

$\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq \ldots \geq \alpha_{-n} \geq \ldots \geq \alpha_{-1})$

$\lim_{n \to \infty} \alpha_{\pm n} = 0.$

$\alpha = (\alpha_{\pm n})$

$\overline{\alpha} = (-\alpha_{-1} \geq -\alpha_{-2} \geq \ldots \geq -\alpha_{+2} \geq -\alpha_{+1})$

Theorem. *Consider sequences* α', α'', β', β'', γ', $\gamma'' \in C_{\downarrow 0 \uparrow}$. *Assume both* $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ *satisfy all the extended Horn inequality. Then there exist compact self-adjoint operators* A, B, C *such that* $C = A + B$,

$$\Lambda_0(A) \text{ is between } \alpha' \text{ and } \alpha''$$

$$\Lambda_0(B) \text{ is between } \beta' \text{ and } \beta''$$

$$\Lambda_0(C) \text{ is between } \gamma' \text{ and } \gamma'' .$$

Corollary. *(Horn conjecture for compact self-adjoint operators). Let* α, β, $\gamma \in C_{\downarrow 0 \uparrow}$. *The following are equivalent:*

(i) There exist compact self-adjoint operators A, B, C *such that* $C = A + B$, $\Lambda_0(A) = \alpha$, $\Lambda_0(B) = \beta$, $\Lambda_0(C) = \gamma$.

(ii) (α, β, γ) *and* $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ *satisfy all the extended Horn inequalities.*

Partially specified eigenvalues

Under what conditions we can find operators A, B, C, $C = A + B$, such that $\Lambda_0(A), \Lambda_0(B)$, and $\Lambda_0(C)$ are only partially specified.

Matrix Case:

$\alpha \in \mathbb{R}^N_{\downarrow}$, with $\alpha_{i_1} \geq \alpha_{i_2} \geq \ldots \alpha_{i_p}$ are specified.

$$\alpha_i^{\min} = \begin{cases} \alpha_{i_1} & \text{if } i \leq i_1 \\ -\infty & \text{if } i_p < i \leq N \\ \alpha_{i_{j+1}} & \text{if } i_j < i \leq i_{j+1} \end{cases}$$

$$\alpha_i^{\max} = \begin{cases} +\infty & \text{if } i < i_1 \\ \alpha_{i_p} & \text{if } i_p \leq i \leq N \\ \alpha_{i_j} & \text{if } i_j \leq i < i_{j+1} \end{cases}$$

$\beta \in \mathbb{R}^N_{\downarrow}$ agrees with α on the specified indices iff $\alpha^{\min} \leq \beta \leq \alpha^{\max}$. Write $\beta \supset \alpha$.

Proposition. *$N \in \mathbb{N}$, partially specified decreasing vectors α, β, $\gamma \in \mathbb{R}^N_\downarrow$. TFAE:*

(i) $\exists$ A, B, C Hermitian such that $C = A + B$, $\Lambda(A) \supset \alpha$, $\Lambda(B) \supset \beta$, $\Lambda(C) \supset \gamma$;

(ii) $\forall (I, J, K) \in T^N_r$, $r \leq N$,

$$\sum_{k \in K} \gamma_k^{\min} \leq \sum_{i \in I} \alpha_i^{\max} + \sum_{j \in J} \beta_j^{\max}$$

and

$$\sum_{k \notin K} \gamma_k \max \geq \sum_{i \notin I} \alpha_i^{\min} + \sum_{j \notin J} \beta_j^{\min} .$$

Theorem. *Let α, β, $\gamma \in C_{\downarrow 0 \uparrow}$ be partially specified. TFAE*

(i) $\exists$ compact self-adjoint operators A, B, C with $C = A + B$ and $\Lambda_0(A) \supset \alpha$, $\Lambda_0(B) \supset \beta$, $\Lambda_0(C) \supset (\gamma)$;

(ii) both $(\alpha^{\max}, \beta^{\max}, \gamma^{\min})$ and $(\overline{\alpha^{\min}}, \overline{\beta^{\min}}, \overline{\gamma^{\max}})$ satisfy all the Horn inequalities.