Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 243 (2007) 67–86
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Toeplitz algebras on the disk ☆

Sheldon Axler [a,*], Dechao Zheng [b]

[a] *Department of Mathematics, San Francisco State University, San Francisco, CA 94132, USA*
[b] *Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA*

Received 11 July 2005; accepted 29 September 2006

Available online 15 November 2006

Communicated by D. Voiculescu

Abstract

Let B be a Douglas algebra and let $\mathcal{B}$ be the algebra on the disk generated by the harmonic extensions of the functions in B. In this paper we show that $\mathcal{B}$ is generated by $H^\infty(D)$ and the complex conjugates of the harmonic extensions of the interpolating Blaschke products invertible in B. Every element S in the Toeplitz algebra $\mathcal{T}_{\mathcal{B}}$ generated by Toeplitz operators (on the Bergman space) with symbols in $\mathcal{B}$ has a canonical decomposition $S = T_{\tilde{S}} + R$ for some R in the commutator ideal $\mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$; and S is in $\mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ iff the Berezin transform $\tilde{S}$ vanishes identically on the union of the maximal ideal space of the Douglas algebra B and the set $\mathcal{M}_1$ of trivial Gleason parts.

Keywords: Toeplitz operator; Berezin transform; Bergman space; Douglas algebra; Commutator ideal

1. Introduction

Let dA denote Lebesgue area measure on the open unit disk D, normalized so that the measure of D equals 1. The *Bergman space* L_a^2 is the Hilbert space consisting of the analytic functions on D that are also in $L^2(D, dA)$. For $z \in D$, the *Bergman reproducing kernel* is the function $K_z \in L_a^2$ such that

$$h(z) = \langle h, K_z \rangle$$

☆ Both authors were partially supported by the National Science Foundation.
* Corresponding author.
E-mail addresses: axler@sfsu.edu (S. Axler), dechao.zheng@vanderbilt.edu (D. Zheng).
URLs: http://www.axler.net (S. Axler), http://www.math.vanderbilt.edu~zheng (D. Zheng).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.09.014

for every $h \in L_a^2$. The *normalized Bergman reproducing kernel* k_z is the function $K_z/\|K_z\|_2$. Here, as elsewhere in this paper, the norm $\| \ \|_2$ and the inner product $\langle\,,\rangle$ are taken in the space $L^2(D, dA)$. The set of bounded linear operators on L_a^2 is denoted by $\mathcal{L}(L_a^2)$.

For $S \in \mathcal{L}(L_a^2)$, the *Berezin transform* of S is the function $\tilde{S}$ on D defined by

$$\tilde{S}(z) = \langle Sk_z, k_z\rangle.$$

Often the behavior of the Berezin transform of an operator provides important information about the operator.

For $u \in L^\infty(D, dA)$, the *Toeplitz operator* T_u with symbol u is the operator on L_a^2 defined by $T_u h = P(uh)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L_a^2. Note that if $u \in H^\infty(D)$ (the set of bounded analytic functions on D), then T_u is just the operator of multiplication by u on L_a^2.

The Berezin transform $\tilde{u}$ of a function $u \in L^\infty(D, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_u. In other words, $\tilde{u} = \widetilde{T_u}$.

For $\mathcal{V}$ a family of bounded harmonic functions on D, let $H^\infty(D)[\mathcal{V}]$ denote the closed subalgebra of $L^\infty(D, dA)$ generated by $H^\infty(D)$ and $\mathcal{V}$. We will show that the Berezin transform maps $H^\infty(D)[\mathcal{V}]$ into $H^\infty(D)[\mathcal{V}]$.

We let $\mathcal{U}$ denote the closed subalgebra of $L^\infty(D, dA)$ generated by $H^\infty(D)$ and the complex conjugates of the functions in $H^\infty(D)$. As is well known (see [1, Proposition 4.5]), $\mathcal{U}$ equals the closed subalgebra of $L^\infty(D, dA)$ generated by the set of bounded harmonic functions on D. C. Bishop [4] has given a more geometric characterization of the functions in $\mathcal{U}$.

Let $|dz|$ denote arc length measure on the unit circle ∂D. Each function $f \in L^\infty(\partial D, |dz|)$ extends to a bounded harmonic function $\hat{f}$ on D, via the Poisson integral

$$\hat{f}(w) = \frac{1}{2\pi}\int\limits_{\partial D} f(z)\frac{1-|w|^2}{|1-\bar{w}z|^2}\,|dz|$$

for $w \in D$. The algebra $H^\infty(\partial D)$ is defined to be the set of functions $f \in L^\infty(\partial D, |dz|)$ such that $\hat{f}$ is analytic on D.

We fix, for the rest of this paper, a Douglas algebra B. Thus B is a closed subalgebra of $L^\infty(\partial D, |dz|)$ containing $H^\infty(\partial D)$. Let I_B denote the set of interpolating Blaschke products invertible in B. The Chang–Marshall theorem [6,14] tells us that B is the closed subalgebra of $L^\infty(\partial D, |dz|)$ generated by $H^\infty(\partial D)$ and the complex conjugates of the functions in I_B.

For $f \in H^\infty(\partial D)$, we often identify $\hat{f}$ with f, writing f instead of $\hat{f}$ for the analytic extension of f to D. Thus I_B could denote either a subset of $H^\infty(\partial D)$ or a subset of $H^\infty(D)$; the context will make clear which interpretation is appropriate.

Throughout this paper, $\mathcal{B}$ denotes the closed subalgebra of $L^\infty(D, dA)$ generated by $\{\hat{f}\colon f \in B\}$. Thus $\mathcal{B} \subset \mathcal{U}$. We will show that $\mathcal{B}$ is generated by $H^\infty(D)$ and the set $\{\bar{b}\colon b \in I_B\}$.

For the given Douglas algebra B, the *Toeplitz algebra* $\mathcal{T}_\mathcal{B}$ is the subalgebra of $\mathcal{L}(L_a^2)$ generated by $\{T_g\colon g \in \mathcal{B}\}$. Our goal in this paper is to study the Berezin transforms of the operators in $\mathcal{T}_\mathcal{B}$. Our main result describes the commutator ideal $\mathcal{C}_{\mathcal{T}_\mathcal{B}}$ (the smallest closed, two-sided ideal of $\mathcal{T}_\mathcal{B}$ containing all operators of the form $RS - SR$, where $R, S \in \mathcal{T}_\mathcal{B}$). Indeed, it will be shown that $S - T_{\tilde{S}}$ is in the commutator ideal $\mathcal{C}_{\mathcal{T}_\mathcal{B}}$ for every $S \in \mathcal{T}_\mathcal{B}$. Writing $S = T_{\tilde{S}} + (S - T_{\tilde{S}})$, this gives us a canonical way to express the (nondirect) sum

$$\mathcal{T}_\mathcal{B} = \{T_u\colon u \in B\} + \mathcal{C}_{\mathcal{T}_\mathcal{B}}.$$

This extends the McDonald–Sundberg theorem [15]. Indeed, we obtain a new proof of the McDonald–Sundberg theorem different from those in [15] and [18].

2. Berezin transform on algebras on the disc

In this section we will show that the Berezin transform maps the algebra $H^\infty(D)[\mathcal{V}]$ into itself. We will need explicit formulas for the reproducing kernel and the normalized reproducing kernel. As is well known,

$$K_z(w) = \frac{1}{(1-\bar{z}w)^2} \quad \text{and} \quad k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}$$

for $z, w \in D$.

Analytic automorphisms of the unit disk will play a key role here. For $z \in D$, let φ_z be the Möbius map on D defined by

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

The following formula for the Berezin transform of a product of Toeplitz operators comes from [2, Lemma 2.7]: if $u_1, \ldots, u_n \in L^\infty(D, dA)$, then

$$(T_{u_1} \ldots T_{u_n})\tilde{}(z) = \langle T_{u_1 \circ \varphi_z} \ldots T_{u_n \circ \varphi_z} 1, 1 \rangle$$

for every $z \in D$.

We will need to make extensive use of the *maximal ideal space* of $H^\infty(D)$, which we denote by $\mathcal{M}$. We define $\mathcal{M}$ to be the set of multiplicative linear maps from $H^\infty(D)$ onto the field of complex numbers. With the weak-star topology, $\mathcal{M}$ is a compact Hausdorff space. If z is a point in the unit disk D, then point evaluation at z is a multiplicative linear functional on $H^\infty(D)$. Thus we can think of z as an element of $\mathcal{M}$ and the unit disk D as a subset of $\mathcal{M}$. Carleson's corona theorem states that D is dense in $\mathcal{M}$.

Suppose $m \in \mathcal{M}$ and $z \mapsto \alpha_z$ is a mapping of D into some topological space E. Suppose also that $\beta \in E$. The notation

$$\lim_{z \to m} \alpha_z = \beta$$

means (as you should expect) that for each open set X in E containing β, there is an open set Y in $\mathcal{M}$ containing m such that $\alpha_z \in X$ for all $z \in Y \cap D$. Note that with this notation z is always assumed to lie in D. We must deal with these nets rather than sequences because the topology of $\mathcal{M}$ is not metrizable.

The Gelfand transform allows us to think of $H^\infty(D)$ as contained in $C(\mathcal{M})$, the algebra of continuous complex-valued functions on $\mathcal{M}$. By the Stone–Weierstrass theorem, the set of finite sums of functions of the form $f\bar{g}$, with $f, g \in H^\infty(D)$, is dense in $C(\mathcal{M})$, where $C(\mathcal{M})$ is endowed with the usual supremum norm. Because D is dense in $\mathcal{M}$, this supremum norm is the same as the usual supremum norm over D. Thus we can identify $C(\mathcal{M})$ with $\mathcal{U}$, the closure in $L^\infty(D, dA)$ of finite sums of functions of the form $f\bar{g}$, with $f, g \in H^\infty(D)$. In other words, given a function $u \in \mathcal{U}$, which we normally think of as a function on D, we can uniquely extend u

to a continuous complex-valued function on $\mathcal{M}$; this extension to $\mathcal{M}$ is also denoted by u. Thus for $u \in \mathcal{U}$ and $m \in \mathcal{M}$, the expression $u(m)$ makes sense—it is the complex number defined by

$$u(m) = \lim_{z \to m} u(z).$$

Conversely, we will sometimes use the identification of $\mathcal{U}$ with $C(\mathcal{M})$ to prove that a function is in $\mathcal{U}$. Specifically, if u is a continuous function on D and we can prove that u extends to a continuous function on $\mathcal{M}$, then we can conclude that $u \in \mathcal{U}$.

For $m \in \mathcal{M}$, let $\varphi_m : D \to \mathcal{M}$ denote the *Hoffman map*. This is defined by setting

$$\varphi_m(w) = \lim_{z \to m} \varphi_z(w)$$

for $w \in D$; here we are taking a limit in $\mathcal{M}$. The existence of this limit, as well as many other deep properties of φ_m, was proved by Hoffman [13]. An exposition of Hoffman's results can also be found in [7, Chapter X]. We shall use, without further comment, Hoffman's result that φ_m is a continuous mapping of D into $\mathcal{M}$. Note that $\varphi_m(0) = m$.

The set $\varphi_m(D)$ is called the *Gleason part* of m. If φ_m is constant on $\mathcal{M}$ (i.e., if $\varphi_m(z) = m$ for all $z \in D$), then m is called a *one-point Gleason part*. The subset of $\mathcal{M}$ consisting of all one-point Gleason parts is denoted by $\mathcal{M}_1$. The set of $m \in \mathcal{M}$ that are not one-point Gleason parts is denoted by $\mathcal{G}$. Thus $\mathcal{M}$ is the disjoint union of $\mathcal{M}_1$ and $\mathcal{G}$.

If $u \in \mathcal{U}$ and $m \in \mathcal{M}$, then $u \circ \varphi_m$ makes sense as a continuous function on D, because φ_m maps D into $\mathcal{M}$ and u can be thought of as a continuous function on $\mathcal{M}$, as discussed above.

Proposition 3.1 of [2] states that the Berezin transform maps $\mathcal{U}$ to $\mathcal{U}$. Thus if $u \in \mathcal{U}$ and $m \in \mathcal{M}$, then $\tilde{u}(m)$ makes sense.

Lemma 2.1. *Suppose that $u \in \mathcal{U}$ and $m \in \mathcal{M}$. If $u \circ \varphi_m \in H^\infty(D)$, then $\tilde{u}(m) = u(m)$.*

Proof. From [2, Proposition 3.1], we have

$$\tilde{u}(m) = \int_D u \circ \varphi_m \, dA.$$

If $u \circ \varphi_m \in H^\infty(D)$, then the mean value property of analytic functions implies that the integral above equals $(u \circ \varphi_m)(0)$, which equals $u(m)$, which gives the desired result. □

A subset E of $\mathcal{M}$ is called *antisymmetric* for $H^\infty(D)[\mathcal{V}]$ if every function in $H^\infty(D)[\mathcal{V}]$ that is real-valued on E is constant on E. Every antisymmetric set is contained in a maximal antisymmetric set. For $g \in C(\mathcal{M})$, the E. Bishop Antisymmetric Decomposition Theorem says that $g \in H^\infty(D)[\mathcal{V}]$ if and only if $g|_E \in H^\infty(D)[\mathcal{V}]|_E$ for each maximal antisymmetric set E for $H^\infty(D)[\mathcal{V}]$. C. Bishop has obtained a distance formula from $f \in L^\infty(D)$ to the algebra $H^\infty(D)[\mathcal{V}]$; see [3].

The following lemma is a slight extension of [19, Corollary 1].

Lemma 2.2. *Let $\mathcal{V}$ be a family of bounded harmonic functions on D. Let E be an antisymmetric set for $H^\infty(D)[\mathcal{V}]$ containing more than one point. Then $u \circ \varphi_m \in H^\infty(D)$ for every $m \in E$ and every $u \in H^\infty(D)[\mathcal{V}]$.*

Proof. Fix $m \in E$. First suppose $v \in \mathcal{V}$. Because $H^\infty(D)[v] \subset H^\infty(D)[\mathcal{V}]$, we see that E is an antisymmetric set for $H^\infty(D)[v]$. Thus E is contained in a maximal antisymmetric set for $H^\infty(D)[v]$. Corollary 1 of [19] now implies that $v \circ \varphi_m \in H^\infty(D)$.

Because $v \circ \varphi_m \in H^\infty(D)$ for every generator $v \in \mathcal{V}$ of $H^\infty(D)[\mathcal{V}]$, a similar conclusion holds for every $u \in H^\infty(D)[\mathcal{V}]$. $\square$

The next result shows that the Berezin transform maps $H^\infty(D)[\mathcal{V}]$ into itself. In the special case where $\mathcal{V}$ is the set of all bounded harmonic functions on D, this result was proved as [2, Proposition 3.1].

Theorem 2.3. *Let $\mathcal{V}$ be a family of bounded harmonic functions on D. If $u \in H^\infty(D)[\mathcal{V}]$, then $\tilde{u} \in H^\infty(D)[\mathcal{V}]$.*

Proof. Let $u \in H^\infty(D)[\mathcal{V}]$. Let E be a maximal antisymmetric set for $H^\infty(D)[\mathcal{V}]$. By the E. Bishop Antisymmetric Decomposition Theorem, to conclude that $\tilde{u}$ is in $H^\infty(D)[\mathcal{V}]$ we need only show that $\tilde{u}|_E \in H^\infty(D)[\mathcal{V}]|_E$.

If E contains only one point, then clearly $\tilde{u}|_E \in H^\infty(D)[\mathcal{V}]|_E$. Thus we can assume that E contains more than one point. Lemma 2.2 now implies that $u \circ \varphi_m \in H^\infty(D)$ for every $m \in E$. Lemma 2.1 then implies that $\tilde{u}|_E = u|_E$. Thus $\tilde{u}|_E \in H^\infty(D)[\mathcal{V}]|_E$, completing the proof. $\square$

Because $\mathcal{B} = H^\infty(D)[\mathcal{V}]$ for $\mathcal{V} = \{\hat{f}\colon\ f \in B\}$, the theorem above implies that the Berezin transform maps $\mathcal{B}$ into $\mathcal{B}$.

For our given Douglas algebra B, let $\mathcal{M}(B)$ denote the maximal ideal space of B. The map from $\mathcal{M}(B)$ to $\mathcal{M}$ defined by $m \mapsto m|_{H^\infty(\partial D)}$ is injective (because $H^\infty(\partial D)$ is a logmodular algebra; see [5]). Thus we can think of $\mathcal{M}(B)$ as a subset of $\mathcal{M}$.

Define $\mathcal{M}_B$ by

$$\mathcal{M}_B = \bigl\{m \in \mathcal{M}\colon\ \hat{f} \circ \varphi_m \in H^\infty(D) \text{ for every } f \in B\bigr\}.$$

With this notation, Lemma 2.2 implies that if E is an antisymmetric set for $\mathcal{B}$ containing more than one point, then $E \subset \mathcal{M}_B$.

The next theorem states that $\mathcal{M}_B$ is the union of the maximal ideal space of B and the one-point Gleason parts.

Theorem 2.4. $\mathcal{M}_B = \mathcal{M}(B) \cup \mathcal{M}_1$.

Proof. If $m \in \mathcal{M}_1$, then φ_m is a constant function. Thus $\mathcal{M}_B \supset \mathcal{M}_1$.

To prove that $\mathcal{M}_B \supset \mathcal{M}(B)$, recall that the Chang–Marshall theorem tells us that

$$\mathcal{M}(B) = \bigcap_{b \in I_B} \bigl\{m \in \mathcal{M}\colon\ \bigl|b(m)\bigr| = 1\bigr\}.$$

Fix $m \in \mathcal{M}(B)$ and $b \in I_B$. Then $|m(b)| = 1$ and hence the analytic function $\hat{b} \circ \varphi_m$ attains its maximum absolute value on D at 0, which implies that $\hat{b} \circ \varphi_m$ is constant on D, which implies that $\varphi_m(D) \subset \mathcal{M}(B)$. Thus if $g \in H^\infty(\partial D)$, we have

$$(g\bar{b})\hat{}\circ \varphi_m = (\hat{g} \circ \varphi_m)(\hat{\bar{b}} \circ \varphi_m).$$

The last term above on the right is a constant function, which implies that $(g\bar{b})\hat{} \circ \varphi_m \in H^\infty(D)$. Functions of the form $g\bar{b}$, where g ranges over $H^\infty(\partial D)$ and b ranges over I_B, generate B, and so we conclude that $m \in \mathcal{M}_B$. Thus $\mathcal{M}_B \supset \mathcal{M}(B)$, as desired.

We have now shown that $\mathcal{M}_B \supset \mathcal{M}(B) \cup \mathcal{M}_1$. To prove the inclusion in the other direction, suppose that $m \in \mathcal{M}$ but $m \notin \mathcal{M}(B) \cup \mathcal{M}_1$. Because $m \notin \mathcal{M}(B)$, there exist $b \in I_B$ and $\epsilon > 0$ such that $|b(m)| < 1 - \epsilon$. Because $m \notin \mathcal{M}_1$, there is an interpolating sequence $\{z_n\} \subset D$ such that m is in the closure of $\{z_n\}$ in $\mathcal{M}$ and

$$\left|b(z_n)\right| < 1 - \epsilon$$

for each n. Let u be the interpolating Blaschke product with zeros $\{z_n\}$. By a theorem in [17], $\bar{u} \in H^\infty(\partial D)[\bar{b}]$, and hence $\bar{u} \in B$. However, $u \circ \varphi_m$ is not constant on D (because interpolating Blaschke products, composed with any Hoffman map, have isolated zeros on D) and thus $\bar{u} \circ \varphi_m \notin H^\infty(D)$. Thus $m \notin \mathcal{M}_B$. This proves that $\mathcal{M}_B \subset \mathcal{M}(B) \cup \mathcal{M}_1$, completing the proof. $\square$

The next lemma, which will be used in the proof of Theorem 2.6, was essentially proved as [19, Theorem 2]. The context there is that E is a maximal antisymmetric set for $H^\infty[f]$, where f is a bounded harmonic function on D. However, an examination of the proof of Theorem 2 of [19] shows that what is actually proved there is the following lemma. The restatement given here holds because a connected subset of $\mathcal{M}_1$ can contain at most one point (because $\mathcal{M}_1$ is totally disconnected, which is proved as [16, Theorem 3.4]).

Lemma 2.5. *Suppose $E \subset \mathcal{M}$ is closed, connected, and contains more than one point. Then $E \cap \mathcal{G}$ is dense in E.*

Gorkin and Izuchi [10] showed that there exists an inner function b (not in the little Bloch space) such that $H^\infty(D)[\bar{b}]$ is not generated by $H^\infty(D)$ and the complex conjugates of the interpolating Blaschke products in $H^\infty(D)[\bar{b}]$. The following theorem, which can be thought of as a Chang–Marshall theorem for the disk, shows that this phenomenon cannot occur with algebras generated by harmonic extensions of a Douglas algebra (and thus the Gorkin–Izuchi algebra $H^\infty(D)[\bar{b}]$ is not equal to $\mathcal{B}$ for any choice of B).

Theorem 2.6. *The algebra $\mathcal{B}$ is generated by $H^\infty(D)$ and the complex conjugates of the analytic extensions of the interpolating Blaschke products that are invertible in B. In other words, $\mathcal{B} = H^\infty(D)[\bar{I}_B]$.*

Proof. Obviously $\mathcal{B} \supset H^\infty(D)[\bar{I}_B]$.

To prove the inclusion in the other direction, fix $f \in B$. We need to show that $\hat{f} \in H^\infty(D)[\bar{I}_B]$. Let E be a maximal antisymmetric set for $H^\infty(D)[\bar{I}_B]$. By the E. Bishop Antisymmetric Decomposition Theorem, it suffices to show that $\hat{f}|_E \in H^\infty(D)[\bar{I}_B]|_E$. This holds trivially if E contains only a single point, so we will assume that E contains more than one point.

Suppose that there exists $m \in E$ such that $m \notin \mathcal{M}_B$. By Theorem 2.4, $m \notin \mathcal{M}(B) \cup \mathcal{M}_1$. As in the proof of Theorem 2.4, this implies that there exists $u \in I_B$ such that $\bar{u} \circ \varphi_m \notin H^\infty(D)$, which contradicts Lemma 2.2. This contradiction means that we can conclude that $E \subset \mathcal{M}_B$.

Now by Theorem 2.4, $E \cap \mathcal{G}$ is contained in $\mathcal{M}(B)$. By the Chang–Marshall theorem, for each $\epsilon > 0$, there are functions $g_j \in H^\infty(\partial D)$ and $b_j \in I_B$ such that

$$\left\| f - \sum_{j=1}^{n} g_j \bar{b}_j \right\|_{L^\infty(\partial D)} < \epsilon.$$

Thus for each $m \in \mathcal{M}(B)$,

$$\epsilon \geqslant \left| \hat{f}(m) - m\left(\sum_{j=1}^{n} g_j \bar{b}_j \right) \right| = \left| \hat{f}(m) - \sum_{j}^{n} g_j(m) \overline{b_j(m)} \right|.$$

This implies that $\hat{f}$ is approximated by functions in $H^\infty(D)[\bar{I}_B]$ on $E \cap \mathcal{G}$. By Lemma 2.5, $E \cap \mathcal{G}$ is dense in E (where the connectedness of E is shown in the discussion preceding [19, Corollary 1]). Thus $\hat{f}|_E \in H^\infty(D)[\bar{I}_B]|_E$, completing the proof. □

A special case of the theorem above occurs when $B = L^\infty(\partial D, |dz|)$. In this case, the theorem above implies that $\mathcal{U}$ is equal to the closed algebra generated by $H^\infty(D)$ and the complex conjugates of all the interpolating Blaschke products. This special case also follows from a deep theorem of Garnett and Nicolau [8] stating that the span of the interpolating Blaschke products is dense in $H^\infty(\partial D)$. However, the proof given here seems to be a simpler approach in this special case.

The next two corollaries will be used in proving Theorem 3.9.

Corollary 2.7. *If $u_1, u_2, \ldots, u_n \in \mathcal{B}$, then $T_{u_1} T_{u_2} \ldots T_{u_n} - T_{u_1 u_2 \ldots u_n} \in \mathcal{C}_{\mathcal{T}_\mathcal{B}}$.*

Proof. First suppose that $n = 2$. Consider initially the case where $u_2 = h\bar{b}$, where $h, b \in H^\infty(D)$ and $\bar{b} \in B$. Then

$$\begin{aligned} T_{u_1} T_{u_2} - T_{u_1 u_2} &= T_{u_1} T_{h\bar{b}} - T_{u_1 h\bar{b}} \\ &= T_{u_1} T_{\bar{b}} T_h - T_{\bar{b}} T_{u_1} T_h \\ &= (T_{u_1} T_{\bar{b}} - T_{\bar{b}} T_{u_1}) T_h, \end{aligned}$$

which shows that $T_{u_1} T_{u_2} - T_{u_1 u_2} \in \mathcal{C}_{\mathcal{T}_\mathcal{B}}$, as desired.

The proof of the corollary when $n = 2$ is completed by noting that an arbitrary $u_2 \in \mathcal{B}$ can be approximated by a finite sum of functions of the form $h\bar{b}$, where $h, b \in H^\infty(D)$ and $\bar{b} \in B$ (by Theorem 2.6, which shows that we can take b to be a finite product of interpolating Blaschke products that are invertible in B).

Now we use induction on n. For $n > 2$ we have

$$\begin{aligned} T_{u_1} T_{u_2} \ldots T_{u_n} - T_{u_1 u_2 \ldots u_n} &= (T_{u_1} T_{u_2} - T_{u_1 u_2}) T_{u_3} \ldots T_{u_n} \\ &\quad + (T_{u_1 u_2} T_{u_3} \ldots T_{u_n} - T_{u_1 u_2 u_3 \ldots u_n}). \end{aligned}$$

The first term on the right-hand side of the equation above is in $\mathcal{C}_{\mathcal{T}_\mathcal{B}}$ by Corollary 2.7; the second term on the right-hand side of the equation above comes from the $n - 1$ functions $u_1 u_2, u_3, \ldots, u_n$ in $\mathcal{B}$ and thus (by an induction hypothesis) it also is in $\mathcal{C}_{\mathcal{T}_\mathcal{B}}$. □

The following corollary will be strengthened in the next section—see the remark before Corollary 3.11.

Corollary 2.8. *Suppose $S \in \mathcal{T}_{\mathcal{B}}$ and $\epsilon > 0$. Then there exist $u \in \mathcal{B}$ and $R \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ such that*

$$\|S - T_u - R\| < \epsilon.$$

Proof. We can assume that $S = T_{u_1} T_{u_2} \dots T_{u_n}$, where each $u_j \in \mathcal{B}$ (because the set of finite sums of operators of this form is dense in $\mathcal{T}_{\mathcal{B}}$). Let $u = u_1 u_2 \dots u_n$ and let $R = T_{u_1} T_{u_2} \dots T_{u_n} - T_u$. Thus $S - T_u - R = 0$. The proof is completed by noting that Corollary 2.7 implies that $R \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$. □

3. Berezin transform on Toeplitz algebras

The next theorem shows that the Berezin transform maps $\mathcal{T}_{\mathcal{B}}$ into $\mathcal{B}$. In the special case where $B = L^\infty(\partial D, |dz|)$, this result was proved as [2, Theorem 2.11].

Theorem 3.1. *If $S \in \mathcal{T}_{\mathcal{B}}$, then $\tilde{S} \in \mathcal{B}$. Moreover, if $u_1, \dots, u_n \in \mathcal{B}$, then*

$$(T_{u_1} \dots T_{u_n})\tilde{\ }(m) = u_1(m) \dots u_n(m),$$

for each $m \in \mathcal{M}_B$.

Proof. First we will prove the second part of the theorem. Suppose $u_1, \dots, u_n \in \mathcal{B}$. By [2, Theorem 2.11], $(T_{u_1} \dots T_{u_n})\tilde{\ }$ extends to a continuous function on $\mathcal{M}$, and that the extension is given by

$$(T_{u_1} \dots T_{u_n})\tilde{\ }(m) = \langle T_{u_1 \circ \varphi_m} \dots T_{u_n \circ \varphi_m} 1, 1\rangle \tag{3.2}$$

for every $m \in \mathcal{M}$.

Suppose $m \in \mathcal{M}_B$, which means that $\hat{f} \circ \varphi_m \in H^\infty(D)$ for each $f \in B$. This implies that $u \circ \varphi_m \in H^\infty(D)$ for each $u \in \mathcal{B}$. Thus each Toeplitz operator $T_{u_j \circ \varphi_m}$ is simply multiplication by $u_j \circ \varphi_m$. Hence

$$\begin{aligned}
(T_{u_1} \dots T_{u_n})\tilde{\ }(m) &= \langle T_{u_1 \circ \varphi_m} \dots T_{u_n \circ \varphi_m} 1, 1\rangle \\
&= \langle u_1 \circ \varphi_m \dots u_n \circ \varphi_m, 1\rangle \\
&= u_1\big(\varphi_m(0)\big) \dots u_n\big(\varphi_m(0)\big) \\
&= u_1(m) \dots u_n(m),
\end{aligned}$$

where the first equality comes from Eq. (3.2) and the third equality follows from the mean value property of analytic functions. The last equation above completes the proof of the second part of the theorem.

Now we will show that

$$(T_{u_1} \dots T_{u_n})\tilde{\ } \in \mathcal{B}.$$

By the E. Bishop Antisymmetric Decomposition Theorem, it suffices to show that $(T_{u_1} \dots T_{u_n})\tilde{}|_E \in \mathcal{B}|_E$ for each maximal antisymmetric set E of $\mathcal{B}$.

Let E be a maximal antisymmetric set for $\mathcal{B}$. If E contains only one point, then our desired conclusion trivially holds. Hence we can assume that E contains more than one point, which by Lemma 2.2 implies that $E \subset \mathcal{M}_B$. Thus the second part of the theorem, which we have already proved, shows that

$$(T_{u_1} \dots T_{u_n})\tilde{}|_E = (u_1 \dots u_n)|_E \in \mathcal{B}|_E,$$

completing the proof that $(T_{u_1} \dots T_{u_n})\tilde{} \in \mathcal{B}$.

The linearity and continuity of the Berezin transform now imply that the Berezin transform maps $\mathcal{T}_{\mathcal{B}}$ into $\mathcal{B}$. □

The special case of the next corollary when $B = L^\infty(\partial D, |dz|)$ was proved as [2, Corollary 2.15] (in that special case, $\mathcal{M}_{\mathcal{B}} = \mathcal{M}_1$).

Corollary 3.3. *If $R, S \in \mathcal{T}_{\mathcal{B}}$, then*

$$\widetilde{(RS)}(m) = \tilde{R}(m)\tilde{S}(m)$$

for every $m \in \mathcal{M}_B$.

Proof. If R, S are each products of Toeplitz operators with symbols in $\mathcal{B}$, then the desired result follows from the theorem above. The proof is completed by recalling that sums of such operators are dense in $\mathcal{T}_{\mathcal{B}}$. □

Let b be the Blaschke product with zeros $\{z_n\} \subset D$. Define

$$\delta(b) = \inf_n \prod_{m \neq n} \frac{|z_n - z_m|}{|1 - \bar{z}_n z_m|}.$$

Recall that b is an interpolating Blaschke product if and only if $\delta(b) > 0$.

The following result is the first of five lemmas that we need before getting to Theorem 3.9 and its proof.

Lemma 3.4. *Suppose $m \in \mathcal{M} \setminus \mathcal{M}_B$. Then there exists $b \in I_B$ such that $m(b) = 0$ and $\delta(b) > \frac{1}{2}$.*

Proof. Because $m \notin \mathcal{M}_B$, the Chang–Marshall theorem implies that $g \circ \varphi_m$ is not constant for some $g \in I_B$. Let $\lambda = m(g)$. Because $\lambda = (g \circ \varphi_m)(0)$, we see that $|\lambda| < 1$. Note that $m(\varphi_\lambda \circ g) = 0$, but $\varphi_\lambda \circ g$ does not vanish identically on the Gleason part $\varphi_m(D)$. By [13, Theorem 3.3], $m(g_1) = 0$ for some interpolating Blaschke factor g_1 of $\varphi_\lambda \circ g$. By [13, Theorem 3.2], there is a subfactor b of g_1 with $m(b) = 0$ and $\delta(b) > \frac{1}{2}$. Because for each $g \in I_B$, each subfactor of Blaschke product g is contained in I_B, we see that b is also in I_B, as desired. □

The pseudohyperbolic distance $\rho(w, z)$ between two points $w, z \in D$ is defined by

$$\rho(w, z) = \frac{|z - w|}{|1 - \bar{z}w|}.$$

The pseudohyperbolic disk with pseudohyperbolic center $z \in D$ and pseudohyperbolic radius $r \in (0, 1)$ is defined by

$$D(z, r) = \{w \in D\colon\ \rho(w, z) < r\}.$$

The following lemma is just a restatement of [13, Lemmas 4.1 and 4.2].

Lemma 3.5. *Let b be an interpolating Blaschke product with zeros $\{z_n\}$ and $\delta(b) > \frac{1}{2}$. Then there are constants $0 < \delta_1 < \delta_2 < 1$ such that*

$$\{z \in D\colon\ |b(z)| < \delta_1\} \subset \bigcup_n D(z_n, \delta_2) \subset \{z \in D\colon\ |b(z)| < 2\delta_2\},$$

and for each $|\lambda| < 3\delta_2$, the level set $E_\lambda = \{z \in D\colon\ b(z) = \lambda\}$ is an interpolating sequence. Furthermore, δ_1 and δ_2 can be chosen to be less than any specific positive constant.

The next lemma provides a convenient open cover of the set where a continuous function on $\mathcal{M}$ that is small on $\mathcal{M}_B$ is bounded away from 0.

Lemma 3.6. *Let $u \in C(\mathcal{M})$ and $\epsilon > 0$. If $|u(m)| < \epsilon$ for every $m \in \mathcal{M}_B$, then there is a finite number of interpolating Blaschke products $b_1, \ldots, b_N \in I_B$ such that $\delta(b_n) > \frac{1}{2}$ for each n and*

$$\{m \in \mathcal{M}\colon\ |u(m)| \geqslant \epsilon\} \subset \bigcup_{n=1}^{N} \{m \in \mathcal{M}\colon\ |b_n(m)| < \delta_1(b_n)\},$$

where $\delta_1(b_n)$ is the constant in Lemma 3.5.

Proof. Suppose $m \in \mathcal{M}$ is such that $|u(m)| \geqslant \epsilon$. Our hypothesis then implies that $m \notin \mathcal{M}_B$. Lemma 3.4 now implies that there exists $b_m \in I_B$ such that $b_m(m) = 0$ and $\delta(b_m) > \frac{1}{2}$. Let $\mathcal{O}_m$ denote the open set defined by $\mathcal{O}_m = \{\tau \in \mathcal{M}\colon\ |b_m(\tau)| < \delta_1(b_m)\}$, and note that $m \in \mathcal{O}_m$.

Thus

$$\{m \in \mathcal{M}\colon\ |u(m)| \geqslant \epsilon\} \subset \bigcup_{\{m \in \mathcal{M}\colon\ |u(m)| \geqslant \epsilon\}} \mathcal{O}_m.$$

Because the set $\{m \in \mathcal{M}\colon\ |u(m)| \geqslant \epsilon\}$ is compact, there is a finite number of $m_1, \ldots, m_N$ such that $|u(m_n)| \geqslant \epsilon$ for each n and

$$\{m \in \mathcal{M}\colon\ |u(m)| \geqslant \epsilon\} \subset \bigcup_{n=1}^{N} \mathcal{O}_{m_n}.$$

Letting $b_n = b_{m_n}$, we obtain the desired result. $\square$

Recall that a positive measure μ on the unit disk is called a Carleson measure if there is a $C > 0$ such that $\mu(\{w \in D\colon\ |z - w| < t\}) \leqslant Ct$ for every $z \in \partial D$ and $t > 0$.

Lemma 3.7. *Suppose b be an interpolating Blaschke product with zeros $\{z_n\}$ and with $\delta(b) > \frac{1}{2}$. Let φ be a bounded smooth function on D such that $\sup_{z\in D} |\nabla\varphi(z)|(1-|z|^2) < \infty$, $|\nabla\varphi| dA$ is a Carleson measure, and φ is supported in $\bigcup_n D(z_n, 3\delta_2)$, where δ_2 is the constant in Lemma 3.5. Then $u\varphi \in H^\infty(D)[\bar{b}]$ for every $u \in C(\mathcal{M})$.*

Proof. The C. Bishop geometric characterization theorem [4, Theorem 1.1] implies that φ is in $C(\mathcal{M})$.

Let $u \in C(\mathcal{M})$. In order to show that $u\varphi \in H^\infty(D)[\bar{b}]$, by the E. Bishop Antisymmetric Decomposition Theorem, it suffices to show that $u\varphi|_E$ is in $H^\infty(D)[\bar{b}]|_E$ for each maximal antisymmetric set E of $H^\infty(D)[\bar{b}]$. To do so, let E be a maximal antisymmetric set of $H^\infty(D)[\bar{b}]$. Noting that b is in $H^\infty(D)$, we have that E is contained in some level set $E_\lambda = \{m \in \mathcal{M}\colon b(m) = \lambda\}$. If $|\lambda| < 3\delta_2$, by Lemma 3.5, E_λ is an interpolating set for $H^\infty(D)$. Thus there is a function $\psi \in H^\infty(D)$ such that $u\varphi|_{E_\lambda} = \psi|_{E_\lambda}$ and $u\varphi|_E = \psi|_E$. If $|\lambda| \geqslant 3\delta_2$, $\varphi|_{E_\lambda} = 0$ and hence $u\varphi|_E = 0$. This gives that $u\varphi|_E$ is in $H^\infty(D)[\bar{b}]|_E$, completing the proof. □

For a measurable set $E \subset D$, let $|E|$ denote the normalized area of E. In other words, $|E| = \int_E 1\, dA$.

Lemma 3.8. *There exists a constant $c > 1$ such that*

$$\int\limits_{\bigcup_{n=1}^{\infty} D(z_n, r)} |f|^2\, dA \leqslant cr^2 \|f\|_2^2$$

whenever $f \in L_a^2$, $r \in (0, \frac{1}{2})$, and $\{z_n\}$ is an interpolating sequence with

$$\inf_n \prod_{m\neq n} \frac{|z_n - z_m|}{|1 - \overline{z_n} z_m|} > \frac{1}{2}.$$

Proof. Suppose $\{z_n\}$ is an interpolating sequence satisfying the inequality above, and suppose $r \in (0, \frac{1}{2})$. Let Q denote a typical Carleson square

$$Q = \left\{se^{i\theta}\colon\ 1 - h < s < 1 \text{ and } \theta_0 < \theta < \theta_0 + h\right\}$$

and let $T(Q)$ denote the half of Q away from ∂D:

$$T(Q) = \left\{se^{i\theta}\colon\ 1 - h < s < 1 - \frac{h}{2} \text{ and } \theta_0 < \theta < \theta_0 + h\right\}.$$

First we show that the number of points in $T(Q) \cap \{z_n\}$ is less than 100. To do this, suppose that $z_j, z_k \in T(Q)$. Then

$$\begin{aligned}
|1 - \overline{z_k} z_j| &\leqslant 1 - |z_k|^2 + |z_k||z_k - z_j| \\
&\leqslant \big(1 + |z_k|\big)h + \frac{3}{2}h \\
&< \frac{7}{2}h.
\end{aligned}$$

Thus this gives

$$1-|z_j|^2<\frac{49h^2(1-|z_j|^2)}{4|1-\overline{z_k}z_j|^2}=\frac{49h^2}{4(1-|z_k|^2)}\frac{(1-|z_k|^2)(1-|z_j|^2)}{|1-\overline{z_k}z_j|^2}.$$

So

$$\begin{aligned}
\sum_{z_j\in T(Q)}\left(1-|z_j|^2\right)&<\frac{49h^2}{4(1-|z_k|^2)}\sum_{z_j\in T(Q)}\frac{(1-|z_k|^2)(1-|z_j|^2)}{|1-\overline{z_k}z_j|^2}\\
&=\frac{49h^2}{4(1-|z_k|^2)}\sum_{z_j\in T(Q)}\left(1-\left|\frac{z_k-z_j}{1-\overline{z_k}z_j}\right|^2\right)\\
&<\frac{49h}{3}\sum_{z_j\in T(Q)}\left(1-\left|\frac{z_k-z_j}{1-\overline{z_k}z_j}\right|^2\right)\\
&<\frac{49h}{3}\left[1+2\sum_{z_j\in T(Q),\,j\neq k}\log\frac{1}{|\frac{z_k-z_j}{1-\overline{z_k}z_j}|}\right]\\
&=\frac{49h}{3}\left[1+2\log\left(\prod_{j\neq k}\left|\frac{z_k-z_j}{1-\overline{z_k}z_j}\right|\right)^{-1}\right]\\
&<\frac{49h}{3}(1+\log 4)<39h.
\end{aligned}$$

Here the third line follows from the inequality $1-|z_k|^2\geqslant\frac{3h}{4}$, and the fourth line follows from the inequality $1-|z|^2<2\log\frac{1}{|z|}$ for $0<|z|<1$. Because $1-|z_j|^2\geqslant\frac{3h}{4}$ for each z_j in $T(Q)$, letting N be the number of points in $T(Q)\cap\{z_n\}$ we have $\frac{3h}{4}N<39h$. Therefore N is less than 100.

On the other hand,

$$\left|D(z,r)\right|\leqslant\frac{r^2(1-|z|^2)}{1-r^2}$$

for any z in $T(Q)$. Thus

$$\left|D(z,r)\right|\leqslant 32r^2\left|T(Q)\right|,$$

and therefore

$$\left|\left[\bigcup_{n=1}^{\infty}D(z_n,r)\right]\cap T(Q)\right|\leqslant 10^5r^2\left|T(Q)\right|.$$

But each Carleson square Q is $\bigcup_{k=1}^{\infty}T(Q_k)$ for a family of Carleson squares Q_k with disjoint interior. Hence

$$\begin{aligned}\left|\left[\bigcup_{n=1}^{\infty} D(z_n,r)\right]\cap Q\right| &= \sum_{k=1}^{\infty}\left|\left[\bigcup_{n=1}^{\infty} D(z_n,r)\right]\cap T(Q_k)\right| \\ &\leqslant \sum_{k=1}^{\infty} 10^5 r^2 \bigl|T(Q_k)\bigr| = 10^5 r^2 \bigl|T(Q)\bigr|.\end{aligned}$$

By the theorem (unnumbered) in [12], we conclude that there exists a constant c (independent of everything) such that

$$\int_{\bigcup_{n=1}^{\infty} D(z_n,r)} |f|^2\,dA \leqslant cr^2\|f\|_2^2,$$

for all $f \in L_a^2$, as desired. □

Suppose $R_n \subset P_n \subset D(z_n,r)$. If for fixed constants $0<\delta<1$ and $C>0$ we have $|(P_n \setminus R_n)\circ \varphi_{z_n}| \leqslant C\delta^2$ for all n, then making the change of variable $z=\varphi_{z_n}(w)$ in the area formula

$$|P_n \setminus R_n| = \int_{P_n \setminus R_n} dA(z)$$

gives

$$\begin{aligned}|P_n \setminus R_n| &= \int_{(P_n \setminus R_n)\circ\varphi_{z_n}} \bigl|\varphi_{z_n}'(w)\bigr|^2\,dA(w) \\ &\leqslant \frac{(1-|z_n|^2)^2}{(1-r)^4}\int_{(P_n \setminus R_n)\circ\varphi_{z_n}} dA(w) \\ &= \frac{(1-|z_n|^2)^2}{(1-r)^4}\bigl|(P_n \setminus R_n)\circ\varphi_{z_n}\bigr| \\ &\leqslant \frac{C\delta^2(1-|z_n|^2)^2}{(1-r)^4}.\end{aligned}$$

For a Carleson square Q, if z_n is in $T(Q)$, then

$$\bigl|T(Q)\bigr| > \frac{1}{32}\bigl(1-|z_n|^2\bigr)^2.$$

Hence

$$|P_n \setminus R_n| \leqslant 32C\delta^2\bigl|T(Q)\bigr|.$$

From the proof above of Lemma 3.8 we see that

$$\int_{\bigcup_{n=1}^{\infty} P_n \setminus R_n} |f|^2 \, dA \leqslant cC\delta^2 \|f\|_2^2.$$

We will use the remark above in the proof of the following theorem.

Theorem 3.9. *Suppose $S \in \mathcal{T}_{\mathcal{B}}$. Then S is in the commutator ideal $\mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ if and only if $\tilde{S}|_{\mathcal{M}_B} = 0$.*

Proof. The commutator ideal $\mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ is the closure of the set of finite sums of operators of the form $S_1(S_2S_3 - S_3S_2)S_4$, where $S_1, S_2, S_3, S_4 \in \mathcal{T}_{\mathcal{B}}$. By Corollary 3.3, each such operator has a Berezin transform that vanishes on $\mathcal{M}_B$. Thus if $S \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$, then $\tilde{S}|_{\mathcal{M}_B} = 0$, proving one direction of the theorem.

To prove the other direction, suppose $\tilde{S}|_{\mathcal{M}_B} = 0$. Let $\epsilon > 0$. It suffices to show that $\operatorname{dist}(S, \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}) < 6\epsilon$.

By Corollary 2.8 there exist $u \in \mathcal{B}$ and $R \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ such that

$$\|S - T_u - R\| < \epsilon.$$

Thus $|\tilde{S} - \tilde{u} - \tilde{R}| < \epsilon$ on $\mathcal{M}_B$. However $\tilde{S}|_{\mathcal{M}_B} = 0$ by hypothesis, and $\tilde{R}|_{\mathcal{M}_B} = 0$ by the other direction of this theorem. Hence $|\tilde{u}| < \epsilon$ on $\mathcal{M}_B$. But $\tilde{u} = u$ on $\mathcal{M}_B$ (by Lemma 2.1 and the definition of $\mathcal{M}_B$). So we have $|u| < \epsilon$ on $\mathcal{M}_B$.

We are going to show that

$$\operatorname{dist}(T_u, \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}) \leqslant 5\epsilon,$$

which will then imply that $\operatorname{dist}(S, \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}) < 6\epsilon$, as desired.

By Lemma 3.6, there are a finite number of interpolating Blaschke products $b_1, \ldots, b_N \in I_B$ such that

$$\bigl\{m \in \mathcal{M}\colon \bigl|u(m)\bigr| \geqslant 3\epsilon\bigr\} \subset \bigcup_{n=1}^{N} \bigl\{m \in \mathcal{M}\colon \bigl|b_n(m)\bigr| < \delta_1(b_n)\bigr\}.$$

By Lemma 3.5, there are interpolating sequences $\{z_{n,k}\}_{n=1}^{\infty}$ for all $1 \leqslant k \leqslant N$ such that for the interpolating Blaschke products b_k with zeros $\{z_{n,k}\}_{n=1}^{\infty}$, $\delta(b_k) > \frac{1}{2}$ and

$$\bigl\{z \in D\colon \bigl|u(z)\bigr| \geqslant 3\epsilon\bigr\} \subset \bigcup_{k=1}^{N} \bigcup_{n=1}^{\infty} D\bigl(z_{n,k}, \delta_2(b_k)\bigr).$$

For any two functions f and g in L_a^2,

$$\left| \int_D (T_u f)\bar{g} \, dA - \int_{\bigcup_{k=1}^{N} \bigcup_{n=1}^{\infty} D(z_{n,k}, \delta_2(b_k))} u f \bar{g} \, dA \right| \leqslant 3\epsilon \|f\|_2 \|g\|_2.$$

Now write

$$\bigcup_{k=1}^{N}\bigcup_{n=1}^{\infty} D\big(z_{n,k},\delta_2(b_k)\big)=\bigcup_{k=1}^{N}\bigcup_{n=1}^{\infty} R_{n,k}$$

with $R_{n,k}\subset D(z_{n,k},\delta_2(b_k))$ and $|R_{n,k}\cap R_{m,l}|=0$ for $n\neq m$ or $k\neq l$.

Letting $0<\delta<\min\{\delta_2(b_k)\colon\ k=1,\dots,N\}$, we are going to construct functions $\psi_{n,k}$ such that $\psi_{n,k}=1$ on $R_{n,k}$, $\psi_{n,k}=0$ on $D\setminus R'_{n,k}$, $0\leqslant\psi_{n,k}\leqslant 1$, and

$$\big|\nabla\psi_{n,k}(z)\big|\leqslant\frac{C}{1-|z|^2},$$

where

$$R'_{n,k}=\varphi_{z_{n,k}}\big(\big\{z\in D\colon\ \operatorname{dist}\big(z,\varphi_{z_{n,k}}(R_{n,k})\big)<\delta\big\}\big)$$

and φ_z denotes the Möbius transformation taking z to 0. To start this construction, let $U_{n,k}=\varphi_{z_{n,k}}(R_{n,k})$. Then $U_{n,k}$ is a subset of $D(0,\delta_2(b_k))$. We can find a function $f_{n,k}$ to satisfy $f_{n,k}=1$ on $U_{n,k}$ and $f_{n,k}(z)=0$ if $\operatorname{dist}(z,U_{n,k})\geqslant\delta$, $0\leqslant f_{n,k}\leqslant 1$, and $|\nabla f_{n,k}(z)|\leqslant C_\delta$.

Define functions $\psi_{n,k}=f_{n,k}\circ\varphi_{z_{n,k}}$. The chain rule gives

$$\nabla\psi_{n,k}(z)=\left(\frac{\partial f_{n,k}}{\partial z}\circ\varphi_{z_{n,k}}\varphi'_{z_{n,k}}(z),\ \frac{\partial f_{n,k}}{\partial\bar z}\circ\varphi_{z_{n,k}}\overline{\varphi'_{z_{n,k}}(z)}\right).$$

For each $z\in D(z_{n,k},\delta_2(b_k)(1+\delta))$, we have $(1-|z|^2)|\varphi'_{z_{n,k}}(z)|=1-|\varphi_{z_{n,k}}(z)|^2$; thus

$$\big|\varphi'_{z_{n,k}}(z)\big|\leqslant\frac{1}{1-|z|^2}.$$

This gives

$$\big|\nabla\psi_{n,k}(z)\big|\leqslant\frac{C}{1-|z|^2}.$$

Thus the functions $\psi_{n,k}$ satisfy the properties desired as above.

Let $\psi_k=\sum_{n=1}^{\infty}\psi_{n,k}$. Since those $\psi_{n,k}$ have disjoint supports,

$$|\nabla\psi_k|\leqslant\frac{C_\delta}{1-|z|^2}$$

and

$$\begin{aligned}|\nabla\psi_k|\,dA&\leqslant\frac{C_\delta}{1-|z|^2}\chi_{\bigcup_{n=1}^{\infty}D(z_{n,k},\delta_2(b_k)(1+\delta))}\,dA\\&\leqslant\frac{C_\delta}{1-|z|^2}\chi_{\bigcup_{n=1}^{\infty}D(z_{n,k},3\delta_2(b_k))}\,dA.\end{aligned}$$

By [9, Proposition 1], $|\nabla\psi_k|\,dA$ is a Carleson measure. By Lemma 3.7, those functions ψ_k are in $\mathcal{B}$. For any integers l, we have

$$\left|\int\limits_{\bigcup_{n=1}^{\infty} R_{n,k}} uf\bar{g}\,dA - \int\limits_{D} \psi_k^l uf\bar{g}\,dA\right| \leqslant \|u\|_\infty \Bigg(\int\limits_{\bigcup_{n=1}^{\infty} R'_{n,k}\setminus R_{n,k}} |f|^2\,dA\Bigg)^{1/2} \|g\|_2.$$

If we choose δ sufficiently close to 0 such that

$$\left|\left\{z\colon \operatorname{dist}(z, U_{n,k}) < \delta\right\} \setminus U_{n,k}\right| \leqslant \delta_2(b_k)^2\delta^{1/2},$$

then by the remark after the proof of Lemma 3.8, we have

$$\int\limits_{\bigcup_{n=1}^{\infty}[R'_{n,k}\setminus R_{n,k}]} |f|^2\,dA \leqslant c\delta_2(b_k)^2\delta^{1/2}\|f\|_2^2;$$

here c is the constant in Lemma 3.8. Thus

$$\left|\int\limits_{\bigcup_{n=1}^{\infty} R_{n,k}} uf\bar{g}\,dA - \int\limits_{D} \psi_k^l uf\bar{g}\,dA\right| \leqslant c\|u\|_\infty\delta_2(b_k)\delta^{1/4}\|f\|_2\|g\|_2.$$

Now we observe that $T_{\psi_k^l u} - T_u T_{\psi_k}^l$ is in the commutator ideal $\mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ (by Corollary 2.7). Let $A_{k,l} = T_{\psi_k^l u} - T_u T_{\psi_k}^l$. The estimate above gives

$$\begin{aligned}
&\left|\int\limits_{\bigcup_{n=1}^{\infty} R_{n,k}} uf\bar{g}\,dA - \int\limits_{D} (A_{k,l}f)\bar{g}\,dA\right| \\
&\quad\leqslant c\|u\|_\infty\delta_2(b_k)\delta^{1/4}\|f\|_2\|g\|_2 + \left|\int\limits_{D} \left(T_u T_{\psi_k}^l f\right)\bar{g}\,dA\right| \\
&\quad\leqslant c\|u\|_\infty\left[\delta_2(b_k)\delta^{1/4} + \left\|T_{\psi_k}^l\right\|\right]\|f\|_2\|g\|_2.
\end{aligned}$$

Applying the inequality above for all k yields

$$\begin{aligned}
&\left|\int\limits_{D} (T_u f)\bar{g}\,dA - \int\limits_{D} \left(\sum_{k=1}^{N} A_{k,l} f\right)\bar{g}\,dA\right| \\
&\quad\leqslant \left[3\epsilon + c\|u\|_\infty\left(\delta^{1/4}\sum_{k=1}^{N}\delta_2(b_k) + \sum_{k=1}^{N}\|T_{\psi_k}\|^l\right)\right]\|f\|_2\|g\|_2.
\end{aligned}$$

This implies

$$\left\|T_u - \sum_{k=1}^{N} A_{k,l}\right\| \leqslant \left[3\epsilon + c\|u\|_\infty\left(\delta^{1/4}\sum_{k=1}^{N}\delta_2(b_k) + \sum_{k=1}^{N}\|T_{\psi_k}\|^l\right)\right]$$

and since $\sum_{k=1}^{N} A_{k,l} \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$ for each l, this shows that

$$\operatorname{dist}(T_u, \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}) \leqslant 5\epsilon$$

if δ and l are chosen such that

$$c\|u\|_\infty \delta^{1/4} \sum_{k=1}^{N} \delta_2(b_k) \leqslant \epsilon$$

and

$$c\|u\|_\infty \sum_{k=1}^{N} \|T_{\psi_k}\|^l \leqslant \epsilon.$$

Noting that ψ_k is supported on $\bigcup_{n=1}^{\infty} D(z_{n,k}, 3\delta_2(b_k))$, by Lemma 3.8 we have

$$\|T_{\psi_k}\| \leqslant 3c^{1/2}\delta_2(b_k) < 1,$$

for suitably chosen interpolating sequence $\{z_{n,k}\}_{n=1}^{\infty}$ for $k = 1, \ldots, N$. This completes the proof. □

We now present a series of corollaries that follow from the last theorem.

Corollary 3.10. *If $u \in C(\mathcal{M})$ vanishes on $\mathcal{M}_B$, then $T_u \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$.*

Proof. This follows from Theorem 3.9 and Lemma 2.1. □

Given an operator $S \in \mathcal{T}_{\mathcal{B}}$, the following corollary tells us that there is a canonical choice of u (namely the Berezin transform of S) such that S can be written in the form $S = T_u + R$ with $u \in \mathcal{B}$ and $R \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$.

Corollary 3.11. *If $S \in \mathcal{T}_{\mathcal{B}}$, then $S - T_{\tilde{S}} \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$.*

Proof. Suppose $S \in \mathcal{T}_{\mathcal{B}}$. Then by Theorem 3.1, $\tilde{S} \in \mathcal{B}$. If $m \in \mathcal{M}_B$, then using Theorem 3.1 (with $n = 1$) we get

$$(S - T_{\tilde{S}})\tilde{}\,(m) = \tilde{S}(m) - \tilde{S}(m) = 0.$$

In other words, $(S - T_{\tilde{S}})\tilde{}\,|_{\mathcal{M}_B} = 0$. Thus Theorem 3.9 implies that $S - T_{\tilde{S}} \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$, completing the proof. □

The next result provides a useful short exact sequence.

Corollary 3.12. *A function $u \in \mathcal{B}$ is identically* 0 *on $\mathcal{M}_B$ if and only if $T_u \in \mathcal{C}_{\mathcal{T}_{\mathcal{B}}}$. Thus*

$$0 \to \{u \in \mathcal{B}\colon\ u|_{\mathcal{M}_B} = 0\} \to \mathcal{B} \to \mathcal{T}_{\mathcal{B}}/\mathcal{C}_{\mathcal{T}_{\mathcal{B}}} \to 0$$

is an exact sequence, where the first map above is the trivial map taking 0 *to* 0*, the second map is the inclusion, the third map takes* $u \in \mathcal{B}$ *to* $T_u + \mathcal{C}_{\mathcal{T}_\mathcal{B}}$*, and the fourth map takes everything to* 0*.*

Proof. Suppose $u \in \mathcal{B}$. By Theorem 3.9, $T_u \in \mathcal{C}_{\mathcal{T}_\mathcal{B}}$ if and only if $\tilde{u}|_{\mathcal{M}_B} = 0$. But Theorem 3.1 shows that $\tilde{u}|_{\mathcal{M}_B} = u|_{\mathcal{M}_B}$. Thus u is identically 0 on $\mathcal{M}_B$ if and only if $T_u \in \mathcal{C}_{\mathcal{T}_\mathcal{B}}$, completing the proof of the first part of the corollary.

The second part of the corollary now follows easily from the first part of the corollary. □

The next corollary shows that inclusions work in the expected fashion.

Corollary 3.13. *Let* $\mathcal{A}$ *be a subalgebra of* $C(\mathcal{M})$ *generated by harmonic functions. If* $\mathcal{T}_\mathcal{A}$ *is a subalgebra of* $\mathcal{T}_\mathcal{B}$*, then* $\mathcal{A}$ *is a subalgebra of* $\mathcal{B}$*.*

Proof. Let u be a harmonic function in $\mathcal{A}$. It suffices to show that $u \in \mathcal{B}$. Since $\mathcal{T}_\mathcal{A}$ is a subalgebra of $\mathcal{T}_\mathcal{B}$, Corollary 3.12 shows that there is a function v in $\mathcal{B}$ such that

$$T_u - T_v \in \mathcal{C}_{\mathcal{T}_\mathcal{B}}.$$

Thus T_{u-v} is in $\mathcal{C}_{\mathcal{T}_\mathcal{B}}$. So

$$(\tilde{u} - \tilde{v})|_{\mathcal{M}_B} = 0.$$

Note that $\tilde{v}|_{\mathcal{M}_B} = v|_{\mathcal{M}_B}$ and $\tilde{u} = u$. Hence

$$(u - v)|_{\mathcal{M}_B} = 0.$$

In order to show that $u \in \mathcal{B}$, by the E. Bishop Antisymmetric Decomposition Theorem, it suffices to show that $u|_E$ is in $\mathcal{B}|_E$ for each maximal antisymmetric set E of $\mathcal{B}$. To show this, let E be a maximal antisymmetric set of $\mathcal{B}$. Suppose that there exists $m \in E$ such that $m \notin \mathcal{M}_B$. By Theorem 2.4, $m \notin \mathcal{M}(B) \cup \mathcal{M}_1$. As in the proof of Theorem 2.4, this implies that there exists $u \in I_B$ such that $\bar{u} \circ \varphi_m \notin H^\infty(D)$, which contradicts Lemma 2.2. This contradiction means that we can conclude that $E \subset \mathcal{M}_B$. Thus $u|_E = v|_E \in \mathcal{B}|_E$, completing the proof. □

The next corollary shows that $\mathcal{T}_\mathcal{B}$ and $\mathcal{B}$ are determined by B.

Corollary 3.14. *Suppose* B_1 *and* B_2 *are Douglas algebras, with corresponding algebras on the disk* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *generated by* $\{\hat{f}\colon f \in B_1\}$ *and* $\{\hat{f}\colon f \in B_2\}$*, respectively. Then the following are equivalent*:

(a) $B_1 = B_2$;
(b) $\mathcal{B}_1 = \mathcal{B}_2$;
(c) $\mathcal{T}_{\mathcal{B}_1} = \mathcal{T}_{\mathcal{B}_2}$.

Proof. Clearly (a) implies (b) and (b) implies (c). Corollary 3.13 shows that (c) implies (b). Thus to complete the proof we only need to show that (b) implies (a).

Suppose that (b) holds; in other words, suppose that $\mathcal{B}_1 = \mathcal{B}_2$. The definition of $\mathcal{M}_{B_j}$ implies that for $j = 1, 2$ we have

$$\mathcal{M}_{B_j} = \{m \in \mathcal{M}\colon\ u \circ \varphi_m \in H^\infty(D) \text{ for every } u \in \mathcal{B}_j\}.$$

Thus $\mathcal{M}_{B_1} = \mathcal{M}_{B_2}$. Theorem 2.4 now implies that

$$\mathcal{M}(B_1) \cap \mathcal{G} = \mathcal{M}(B_2) \cap \mathcal{G}.$$

To complete the proof we need only to show that

$$\mathcal{M}(B_1) \cap \mathcal{M}_1 = \mathcal{M}(B_2) \cap \mathcal{M}_1.$$

This result, along with the result from the previous paragraph, would show that $\mathcal{M}(B_1) = \mathcal{M}(B_2)$, which by the Chang–Marshall theorem would imply our desired result that $B_1 = B_2$.

We will show that

$$\mathcal{M}(B_1) \cap \mathcal{M}_1 \subset \mathcal{M}(B_2) \cap \mathcal{M}_1,$$

which by the symmetry between B_1 and B_2 will give the desired result. To prove the inclusion above, let $m \in \mathcal{M}(B_1) \cap \mathcal{M}_1$. If $m \in M(L^\infty(\partial D, |dz|))$, then we also have $m \in \mathcal{M}(B_2)$ and we have nothing further to prove. Thus we can assume that $m \notin M(L^\infty(\partial D, |dz|))$. By [11, Corollary 3.2], this implies that m lies in the closure of the set V defined by

$$V = \{x \in \mathcal{M}\colon\ \operatorname{supp} x \subset \operatorname{supp} m\} \cap \mathcal{G},$$

where $\operatorname{supp} x$ denotes the smallest closed subset of $M(L^\infty(\partial D, |dz|))$ on which a representing measure for x is supported. Clearly V is contained in $\mathcal{M}(B_1) \cap \mathcal{G}$, which we have already shown equals $\mathcal{M}(B_2) \cap \mathcal{G}$. Because $\mathcal{M}(B_2)$ is closed, this implies that $m \in \mathcal{M}(B_2)$, completing the proof. $\square$

Acknowledgment

The authors thank the referee for several suggestions that helped improve this paper.

References

[1] Sheldon Axler, Bergman spaces and their operators, in: John B. Conway, Bernard B. Morrel (Eds.), Surveys of Some Recent Results in Operator Theory, vol. 1, Pitman Res. Notes Math., Longman, Harlow, 1988, pp. 1–50.
[2] Sheldon Axler, Dechao Zheng, The Berezin transform on the Toeplitz algebra, Studia Math. 127 (1998) 113–136.
[3] Christopher Bishop, A distance formula for algebras on the disk, Pacific J. Math. 174 (1996) 1–27.
[4] Christopher Bishop, Some characterizations of $C(\mathcal{M})$, Proc. Amer. Math. Soc. 124 (1996) 2695–2701.
[5] Andrew Browder, Introduction to Function Algebras, Benjamin, New York, 1969.
[6] Sun-Yung A. Chang, A characterization of Douglas subalgebras, Acta Math. 137 (1976) 81–89.
[7] John B. Garnett, Bounded Analytic Functions, Academic Press, 1981.
[8] John Garnett, Artur Nicolau, Interpolating Blaschke products generate H^∞, Pacific J. Math. 173 (1996) 501–510.
[9] Pratibha Ghatage, Dechao Zheng, Analytic functions of bounded mean oscillation and the Bloch space, Integral Equation Operator Theory 17 (1993) 501–515.
[10] Pamela Gorkin, Keiji Izuchi, Some counterexamples in subalgebras of $L^\infty(D)$, Indiana Univ. Math. J. 40 (1991) 1301–1313.

[11] Pamela Gorkin, Raymond Mortini, Interpolating Blaschke products and factorization in Douglas algebras, Michigan Math. J. 38 (1991) 147–160.
[12] William Hastings, A Carleson measure theorem for Bergman spaces, Proc. Amer. Math. Soc. 52 (1975) 237–241.
[13] Kenneth Hoffman, Bounded analytic functions and Gleason parts, Ann. of Math. 86 (1967) 74–111.
[14] Donald Marshall, Subalgebras of L^∞ containing H^∞, Acta Math. 137 (1976) 91–98.
[15] Gerard McDonald, Carl Sundberg, Toeplitz operators on the disc, Indiana Univ. Math. J. 28 (1979) 595–611.
[16] Daniel Suárez, Trivial Gleason parts and the topological stable rank of H^∞, Amer. J. Math. 118 (1996) 879–904.
[17] Carl Sundberg, A constructive proof of the Chang–Marshall theorem, J. Funct. Anal. 46 (1982) 239–245.
[18] Carl Sundberg, Exact sequences for generalized Toeplitz operators, Proc. Amer. Math. Soc. 101 (1987) 634–636.
[19] Rahman Younis, Dechao Zheng, Algebras on the unit disk and Toeplitz operators on the Bergman space, Integral Equations Operator Theory 37 (2000) 106–123.