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# The distribution function inequality for a finite sum of finite products of Toeplitz operators

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**Abstract**

A generalized area function associated with a finite sum of finite products of Toeplitz operators is introduced. A distribution function inequality is established for the generalized area function. By using the distribution function inequality, we characterize when a finite sum of finite products of Toeplitz operators on the Hardy space is a compact perturbation of a Toeplitz operator.


## 1. Introduction

Let $D$ be the open unit disk in the complex plane and $\partial D$ the unit circle. $d\sigma(w)$ denotes the normalized Lebesgue measure on the unit circle. Let $L^2$ denote the Lebesgue square integrable functions on the unit circle. For $1 \leqslant p < \infty$, and $f(z)$ an analytic function on $D$, we say $f \in H^p$ if

$$\sup_r \int_{\partial D} |f(re^{i\theta})|^p d\sigma(e^{i\theta}) = \|f\|_p^p < \infty.$$

$H^\infty$ denotes the set of bounded analytic functions on the unit disk.

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Let $P$ be the Hardy projection of $L^2$ onto $H^2$. For $A \in L^\infty$, the Toeplitz operator $T_A : H^2 \to H^2$ with symbol $A$ is defined by

$$T_A h = P(Ah).$$

The Hankel operator $H_A : H^2 \to L^2 \ominus H^2$ with symbol $A$ is defined by

$$H_A h = (I - P)(Ah).$$

For more details on Toeplitz operators, see [4,7,8,20,21].

The map $\xi: A \to T_A$, which is called the Toeplitz quantization, carries $L^\infty$ into the $C^*$-algebra of bounded operators on $H^2$. It is a contractive *-linear mapping [8]. However it is not multiplicative in general. On the other hand, Douglas [8] showed that $\xi$ is actually a cross-section for a *-homomorphism from the Toeplitz algebra, the $C^*$-algebra generated by all bounded Toeplitz operators on $H^2$, onto $L^\infty$. So modulo the commutator ideal of the Toeplitz algebra, $\xi$ is multiplicative.

Studying the Toeplitz algebra has shed light on the theory of Toeplitz operators [7,8,20]. In this paper we will study the (not closed) algebra of finite sums of finite products of Toeplitz operators, which is dense in the Toeplitz algebra. The main question to be considered in this paper is when a finite sum of finite products of Toeplitz operators is a compact perturbation of a Toeplitz operator. This problem is connected with the spectral theory of Toeplitz operators; see [4,7,8,20]. A theorem of Douglas [8] implies that $\sum_{l=1}^{L} \prod_{j=1}^{I_l} T_{A_{lj}}$ can be a compact perturbation of a Toeplitz operator only when it is a compact perturbation of $T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}}$.

In this paper we will introduce a generalized area function associated with a finite sum of finite products of Toeplitz operators and establish a distribution function inequality for the area function. By means of the key distribution function inequality we will prove that a finite sum $T$ of finite products of Toeplitz operators is a compact perturbation of a Toeplitz operator if and only if

$$\lim_{|z| \to 1} \| T - T^*_{\phi_z} T T_{\phi_z} \| = 0. \tag{1}$$

Here $\phi_z$ denotes the Möbius map,

$$\phi_z(w) = \frac{z - w}{1 - \bar{z} w}.$$

The above result is a variant of Theorem 4 in [14]. However, some crucial details are omitted from the proof in [14], especially, details in the proof of a key distribution function inequality.

One of our motivations is the result of Axler and the second author [2] that if an operator $S$ on the Bergman space equals a finite sum of finite products of Toeplitz operators, then $S$ is compact if and only if the Berezin transform of $S$ vanishes on the boundary of the unit disk. One may expect that the Berezin transform gives the analogous characterization for a finite sum of finite products of Toeplitz operators to be compact on the Hardy space. However, we will use examples from [12] to show that even if an operator $T$ on the Hardy space equals a finite sum of finite products of Toeplitz operators, the vanishing of the Berezin transform of $T$ does not have to imply that $T$ is compact.

Another motivation is the solution of the problem of characterizing when the product of two Toeplitz operator on the Hardy space $H^2$ is a compact perturbation of a Toeplitz operator, by Axler et al. [1] and Volberg [22]. Their beautiful result is that $T_f T_g$ is a compact perturbation of a Toeplitz operator if and only if $H^\infty[\bar{f}] \bigcap H^\infty[g] \subset H^\infty + C(\partial D)$; here $H^\infty[g]$ denotes the closed subalgebra of $L^\infty$ generated by $H^\infty$ and $g$.

Recently, the second author [23] showed that $T_f T_g$ is a compact perturbation of a Toeplitz operator if and only if

$$\lim_{|z|\to 1} \|H_{\bar{f}} k_z\|_2 \|H_g k_z\|_2 = 0;$$

here $k_z$ denotes the normalized reproducing kernel in $H^2$ for point evaluation at $z$. This is equivalent to

$$\lim_{|z|\to 1} \|[T_f T_g - T_{fg}] - T^*_{\phi_z}[T_f T_g - T_{fg}]T_{\phi_z}\| = 0.$$

The semicommutator $T_f T_g - T_{fg}$ can be written as a product of two bounded Hankel operators. To study a finite sum of finite products of Toeplitz operators we will decompose the finite sum as a finite sum of products of two (unbounded) Hankel operators in Section 3. Clearly, a much more involved cancellation may happen in the sum of products of two Hankel operators. We need to take care of the cancellation by introducing a generalized area integral associated with the sum in Section 4. Even in some special cases [13,15] some generalized area integral functions were introduced. Gorkin and the second author [13] have shown that the commutator $[T_f, T_g](= T_f T_g - T_g T_f)$ of two Toeplitz operators is compact on $H^2$ if and only if

$$\lim_{|z|\to 1} \|[T_f, T_g] - T^*_{\phi_z}[T_f, T_g]T_{\phi_z}\| = 0.$$

Condition (1) not only unifies the results on the compactness of commutators or semi-commutators of Toeplitz operators, but is also useful in understanding the Toeplitz algebra. In Section 7 we will give applications of our main result to the following two questions:

**Question 1.** For an inner function $b$, characterize the operators $X$ on $H^2$ such that $T_b^* X T_b - X$ is compact.

**Question 2.** For an inner function $b$, characterize the operators $X$ on $H^2$ such that the commutator $[T_b, X]$ is compact.

These questions are closely related to and inspired by the following Douglas problems:

**Douglas problem 1.** If $X$ is an operator on $H^2$ such that $T_b^* X T_b - X$ is compact for every inner function $b$, then is $X = T_\psi + K$ for some $\psi$ in $L^\infty$ and compact operator $K$? [7]

**Douglas problem 2.** If the commutator $[T_b, X]$ is compact for each $b$ in $H^\infty + C$, then is $X = T_\psi + K$ for some $\psi$ in $H^\infty + C$ and compact operator $K$? [9].

Douglas showed [9] that the solution of the first problem will give the solution of the second problem. Douglas [9] solved the first problem in the case that $X$ is in the Toeplitz algebra. Although the Douglas problem 1 remains open, Davidson [6] has solved the second problem. Clearly, the above questions localize the Douglas problems in some sense.

Another application of our main result is the solution of the problem of when a Hankel operator essentially commutes with a Toeplitz operator [16].

## 2. Examples and maximal ideal space

In this section we will recall examples from [12] to show that the Berezin transform does not characterize the compactness of a finite sum of finite products of Toeplitz operators on the Hardy space. Let $T$ be a bounded operator on $H^2$. The Berezin transform of $T$ is defined by

$$\hat{T}(z) = \langle T k_z, k_z \rangle$$

for $z$ in $D$. Perhaps the most important tool in the study of the Toeplitz algebra, the norm-closed algebra of operators generated by the Toeplitz operators, is the existence of a homomorphism, the so-called symbol mapping $\sigma$, from the Toeplitz algebra to $L^\infty$ such that $\sigma(T_f) = f$ for every $f \in L^\infty$. The key point here is that $\sigma$ is multiplicative. The symbol mapping was discovered and exploited by Douglas [7]. Barría and Halmos [3] showed the symbol mapping $\sigma$ is well defined for asymptotic Toeplitz operators. Recently Englis [10] showed that the nontangential limit of the Berezin transform of $T$ equals the symbol of $T$, for $T$ in the Toeplitz algebra.

To present the examples in [12], we need to introduce the maximal ideal space of $H^\infty$. Let $M(H^\infty)$ be the set of the multiplicative linear functionals on $H^\infty$. If $B$ is a Douglas algebra, i.e., a subalgebra of $L^\infty$ that contains $H^\infty$, then $M(B)$ can be identified with the set of nonzero linear functionals in $M(H^\infty)$ whose representing measures (on $M(L^\infty)$) are multiplicative on $B$. We identify a function $f$ in $B$ with its Gelfand transform on $M(B)$. In particular, $M(H^\infty + C) = M(H^\infty) - D$, and a function $f \in H^\infty$ may be thought of as a continuous function on $M(H^\infty)$.

**Examples.** Let $b$ be any interpolating Blaschke product with zeros $\{z_n\}$. Choose a sequence of positive integers $l_n \to \infty$ such that

$$\sum_{n=1}^{\infty} l_n(1-|z_n|) < \infty.$$

Let

$$b_1 = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n}\left(\frac{z_n - z}{1-\bar{z}_n z}\right)^{l_n}$$

denote the corresponding Blaschke product. It was proved in [12] that for each $m \in M(H^\infty + C)$,

$$m(b_1\bar{b}) = m(b_1)m(\bar{b}).$$

This is equivalent to

$$\lim_{|z|\to 1}[\widehat{b_1\bar{b}}(z) - b_1(z)\overline{b(z)}] = 0,$$

where $\widehat{b_1\bar{b}}(z)$ is the harmonic extension of $b_1\bar{b}$ at $z$ given by

$$\widehat{b_1\bar{b}}(z) = \int_{\partial D} b_1(w)\bar{b}(w)|k_z(w)|^2 d\sigma(w).$$

Let $T = T_{b_1\bar{b}} - T_{b_1}T_{\bar{b}}$. Clearly, $T$ is a finite sum of finite products of Toeplitz operators. An easy calculation gives that the Berezin transform of $T$ is

$$\begin{aligned}\hat{T}(z) &= \langle [T_{b_1\bar{b}} - T_{b_1}T_{\bar{b}}]k_z, k_z\rangle \\ &= \widehat{b_1\bar{b}}(z) - b_1(z)\overline{b(z)}.\end{aligned}$$

Thus

$$\lim_{|z|\to 1} \hat{T}(z) = 0.$$

Since $b_1\bar{b} = \frac{b_1}{b}$ is in $H^\infty$, we have

$$T_{\bar{b}_1 b}T = T_{\bar{b}_1 b}(T_{b_1\bar{b}} - T_{b_1}T_{\bar{b}}) = I - T_b T_{\bar{b}}$$

is an infinite dimensional projection, and hence $T$ is not compact.

Hoffman [17,18] has shown that for each $m\in M(H^\infty+C)$, $m$ has a unique extension to $L^\infty$, which is given by

$$m(f)=\int_{S_m} f\,d\mu_m$$

for $f\in L^\infty$. Here $S_m$ is the (closed) support of the representing measure $d\mu_m$. A subset $S$ of $M(L^\infty)$ is called a support set if it is the (closed) support of the representing measure for a functional in $M(H^\infty+C)$.

Let $H^2(m)$ be the closure of $H^\infty$ in $L^2(d\mu_m)$. Let $H_0^2(m)=\{f\in H^2(m): \int_S f\,d\mu_m=0\}$. Hoffman [17, p. 289] proved that $L^2(d\mu_m)=H^2(m)\oplus\overline{H_0^2(m)}$.

An inner function in $H^2(m)$ is a function $q\in H^\infty(m)$ with $|q|=1$ a.e. on $S_m$. An outer function in $H^2(m)$ is a function $o$ such that $H^\infty o$ is dense in $H^2(m)$. But Theorem 22 [19] says that every function $f$ in $H^2(m)$ with $f(m)\neq 0$ has the factorization $qo$ for an inner function $q$ and an outer function $o$.

The following lemma will be needed in Section 7.

**Lemma 1.** *If $m\in M(H^\infty+C)$ and $b$ is an inner function in $H^\infty$ not equal to a constant on the support set $S_m$, then $1-b$ is an outer function in $H^2(m)$.*

**Proof.** We assume that $b$ does not identically equal 1 on the support set $S_m$. Let $E=\{x\in S_m : b(x)\neq 1\}$, a subset of $S_m$ of positive measure. For $0<r<1$, the function $(1-rb)^{-1}$ is in $H^\infty$, and $(1-rb)^{-1}(1-b)\to\chi_E$ pointwise boundedly on $S_m$ as $r\to 1$. Hence $\chi_E$ is in the $H^2(m)$-closure of $(1-b)H^\infty$, and also in $H^\infty(m)$. Since $\mu_m$ is multiplicative on $H^\infty(m)$, we have

$$\mu_m(E)^2=\left(\int\chi_E\,d\mu_m\right)^2=\int\chi_E^2\,d\mu_m=\mu_m(E),$$

giving $\mu_m(E)=1$ (since $\mu_m(E)\neq 0$). Hence the constant function 1 is in the $H^2(m)$-closure of $(1-b)H^\infty$, showing that $b$ is outer in $H^2(m)$. $\square$

We thank D. Sarason for his suggesting the above proof.

## 3. Decomposition

Although our main concern is with bounded Toeplitz operators and Hankel operators, we will need to make use of densely defined unbounded Toeplitz operators and Hankel operators. Given two operators $S_1$ and $S_2$ densely defined on $H^2$, we say that $S_1=S_2$ if

$$S_1p=S_2p,$$

for each $p$ in the set $\mathcal{P}$ of analytic polynomials.

As in [14], in this section we will show that a finite sum of finite products of Toeplitz operators can be written as a finite sum of products of two Toeplitz operators. The key here is a simple and useful idea used in [14]

$$T_{A_1}T_{A_2}T_{A_3} = T_{A_1[(A_2)_+ + c_1]}T_{A_3} + T_{A_1}T_{[(A_2)_- - c_1]A_3},$$

for three bounded functions $A_1$, $A_2$ and $A_3$, and a constant $c_1$. Here $A_+ = P(A)$ and $A_- = (I-P)(A)$. For four bounded functions $A_1$, $A_2$, $A_3$ and $A_4$; and three constants $c_1$, $c_2$ and $c_3$, we have

$$\begin{aligned} T_{A_1}T_{A_2}T_{A_3}T_{A_4} =& [T_{A_1[(A_2)_+ + c_1]}T_{A_3} + T_{A_1}T_{[(A_2)_- - c_1]A_3}]T_{A_4} \\ =& T_{A_1[(A_2)_+ + c_1][(A_3)_+ + c_2]}T_{A_4} + T_{A_1[(A_2)_+ + c_1]}T_{[(A_3)_- - c_2]A_4} \\ &+ T_{A_1\{[[(A_2)_- - c_1]A_3]_+ + c_3\}}T_{A_4} + T_{A_1}T_{\{[[(A_2)_- - c_1]A_3]_- - c_3\}A_4}. \end{aligned}$$

Clearly, for an integer $m \geqslant 2$, by induction, we see that a product of $m$ Toeplitz operators with bounded symbols can be written in a sum of $2^{m-2}$ terms that are products of two Toeplitz operators with (perhaps unbounded) symbols, and the decomposition is not unique. In order to deal with a finite sum of products of two Toeplitz operators with unbounded symbols we need to introduce systematic decompositions of the finite products. To do so, let $\boldsymbol{\lambda} = \{\lambda(l,k)\}$ be a sequence of complex numbers. For a sequence of functions $A_1, A_2, \ldots, A_n$ in $L^\infty$, we inductively define

$$\begin{array}{rclrcl} {}_\lambda A_1^0 &=& A_1, & {}_\lambda B_1^0 &=& A_2 \\ {}_\lambda A_{2k-1}^i &=& {}_\lambda A_k^{i-1}[({}_\lambda B_k^{i-1})_+ + \lambda(i-1,k)], & {}_\lambda B_{2k-1}^i &=& A_{i+2} \\ {}_\lambda B_{2k}^i &=& [({}_\lambda B_k^{i-1})_- - \lambda(i-1,k)]A_{i+2}, & {}_\lambda A_{2k}^i &=& {}_\lambda A_k^{i-1}, \end{array}$$

for $k \leqslant 2^{i-1}$.

**Lemma 2.** *Let $\boldsymbol{\lambda} = \{\lambda(l,k)\}$ be a sequence of complex numbers. If $A_1, A_2, \ldots, A_m$ are of functions in $L^\infty$, then ${}_\lambda A_j^i$ and ${}_\lambda B_j^i$ defined above are in $\cap_{\infty > p > 1} L^p$. Moreover,*

$$T_{A_1}T_{A_2}\cdots T_{A_m} = \sum_{j=1}^{2^{m-2}} T_{{}_\lambda A_j^{m-2}} T_{{}_\lambda B_j^{m-2}}$$

*and*

$$A_1A_2\cdots A_m = \sum_{j=1}^{2^{m-2}} {}_\lambda A_j^{m-2}\, {}_\lambda B_j^{m-2}.$$

**Proof.** We use induction to prove the theorem. When $n = 2$, from our definition we have

$$T_{A_1}T_{A_2} = T_{{}_\lambda A_1^0}T_{{}_\lambda B_1^0}$$

and

$$A_1A_2 = {}_\lambda A_1^0\,{}_\lambda B_1^0.$$

For $n = m$, we assume that

$$T_{A_1}T_{A_2}\cdots T_{A_m} = \sum_{j=1}^{2^{m-2}} T_{{}_\lambda A_j^{m-2}}T_{{}_\lambda B_j^{m-2}} \tag{2}$$

and

$$A_1A_2\cdots A_m = \sum_{j=1}^{2^{m-2}} {}_\lambda A_j^{m-2}\,{}_\lambda B_j^{m-2}. \tag{3}$$

Now

$$\begin{aligned}
\sum_{j=1}^{2^{m-1}} {}_\lambda A_j^{m-1}\,{}_\lambda B_j^{m-1} &= \sum_{j=1}^{2^{m-2}} [{}_\lambda A_{2j-1}^{m-1}\,{}_\lambda B_{2j-1}^{m-1} + {}_\lambda A_{2j}^{m-1}\,{}_\lambda B_{2j}^{m-1}] \\
&= \sum_{j=1}^{2^{m-2}} \Big\{ {}_\lambda A_j^{m-2}[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k)]A_{m+1} + {}_\lambda A_j^{m-2} \\
&\quad \times [({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)]\,{}_\lambda A_{m+1} \Big\} \\
&= \left\{ \sum_{j=1}^{2^{m-2}} {}_\lambda A_j^{m-2}[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k)] + {}_\lambda A_j^{m-2} \right. \\
&\quad \left. \times [({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)] \right\} {}_\lambda A_{m+1} \\
&= \left\{ \sum_{j=1}^{2^{m-2}} {}_\lambda A_j^{m-2}[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k) \right.
\end{aligned}$$

$$+ ({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)] \Big\} {}_\lambda A_{m+1}$$

$$= \left\{ \sum_{j=1}^{2^{m-2}} {}_\lambda A_j^{m-2} {}_\lambda B_j^{m-2} \right\} {}_\lambda A_{m+1}$$

$$= A_1 A_2 \cdots A_m A_{m+1}.$$

The last equality follows from (3).

Note that both $({}_\lambda B_k^{m-2})_+ + \lambda(m-2,k)$ and $\overline{[({}_\lambda B_k^{m-2})_- - \lambda(m-2,k)]}$ are in $H^2$. Thus

$$T_{{}_\lambda A_j^{m-2}} T_{[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k)]} = T_{{}_\lambda A_j^{m-2}[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k)]},$$

and

$$T_{[({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)]} T_{A_{m+1}} = T_{[({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)] A_{m+1}}.$$

So by (2) we obtain

$$\begin{aligned} T_{A_1} T_{A_2} \cdots T_{A_m} T_{A_{m+1}} &= \sum_{j=1}^{2^{m-2}} T_{{}_\lambda A_j^{m-2}} T_{{}_\lambda B_j^{m-2}} T_{A_{m+1}} \\ &= \sum_{j=1}^{2^{m-2}} [T_{{}_\lambda A_j^{m-2}} T_{[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k)]} T_{A_{m+1}} \\ &\quad + T_{{}_\lambda A_j^{m-2}} T_{[({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)]} T_{A_{m+1}}] \\ &= \sum_{j=1}^{2^{m-2}} [T_{{}_\lambda A_j^{m-2}[({}_\lambda B_j^{m-2})_+ + \lambda(m-2,k)]} T_{A_{m+1}} \\ &\quad + T_{{}_\lambda A_j^{m-2}} T_{[({}_\lambda B_j^{m-2})_- - \lambda(m-2,k)] A_{m+1}}]. \end{aligned}$$

Hence we conclude

$$T_{A_1} T_{A_2} \cdots T_{A_m} T_{A_{m+1}} = \sum_{j=1}^{2^{m-1}} T_{{}_\lambda A_j^{m-1}} T_{{}_\lambda B_j^{m-1}}.$$

Note that $\cap_{\infty > p > 1} L^p$ is an algebra, i.e., both $fg$ and $f+g$ are in $\cap_{\infty > p > 1} L^p$ if $f$ and $g$ are in $\cap_{\infty > p > 1} L^p$. In addition, $P_+$ and $P_-$ are bounded on $L^p$ for $1 < p < \infty$, and map $L^\infty$ into $BMO$. The John-Nirenberg theorem tells us that *BMO* is contained in the intersection $\cap_{\infty>p>1} L^p$. These imply that ${}_\lambda A^i_j$ and ${}_\lambda B^i_j$ are products of functions in $\cap_{\infty>p>1} L^p$. So they are also in $\cap_{\infty>p>1} L^p$. This completes the proof. □

The above lemma gives the following proposition. The decompositions of $A_i$ are different from those in [14].

**Proposition 3.** *Let* $\boldsymbol{\lambda} = \{\lambda(l,k)\}$ *be a sequence of complex numbers.*

$$T_{A_1}T_{A_2}\cdots T_{A_m} - T_{A_1A_2\cdots A_m} = \sum_{j=1}^{2^{m-2}} H^*_{\overline{{}_\lambda A^{m-2}_j}} H_{{}_\lambda B^{m-2}_j}.$$

**Proof.** By Lemma 2, we have

$$T_{A_1}T_{A_2}\cdots T_{A_m} = \sum_{j=1}^{2^{m-2}} T_{{}_\lambda A^{m-2}_j} T_{{}_\lambda B^{m-2}_j}$$

and

$$A_1A_2\cdots A_m = \sum_{j=1}^{2^{m-2}} {}_\lambda A^{m-2}_j \, {}_\lambda B^{m-2}_j.$$

Because

$$T_A T_B - T_{AB} = H^*_{\bar{A}} H_B,$$

we get

$$\begin{aligned} T_{A_1}T_{A_2}\cdots T_{A_m} - T_{A_1A_2\cdots A_m} &= \sum_{j=1}^{2^{m-2}} \left[ T_{{}_\lambda A^{m-2}_j} T_{{}_\lambda B^{m-2}_j} - T_{{}_\lambda A^{m-2}_j \, {}_\lambda B^{m-2}_j} \right] \\ &= \sum_{j=1}^{2^{m-2}} H^*_{\overline{{}_\lambda A^{m-2}_j}} H_{{}_\lambda B^{m-2}_j}. \end{aligned}$$

This completes the proof. □

Although the representation of a finite product of Toeplitz operators as a sum of products of two Toeplitz operators is not unique, it has the advantage of letting us to choose $\lambda(j,k)$. In order to establish our distribution function inequality we need to choose those constants $\lambda(j,k)$ appropriately at each point $z\in D$. The following lemma tells us that we can do so.

Let $A_1,\ldots,A_m$ be in $L^\infty$. Given a point $z\in D$, inductively define a sequence $\{\lambda(l,k)\}$ of complex numbers

$$\lambda(i-1,k) = ({}_\lambda B_k^{i-1})_-(z).$$

From the definition of $({}_\lambda B_k^{i-1})_-(z)$, it depends on only $\lambda(j,k)$ for $j<i-1$.

**Lemma 4.** *Let $A_1,\ldots,A_m$ be in $L^\infty$. Suppose that*

$$\sup_i \|A_i\|_\infty \leqslant M$$

*for some constant M. For a fixed z in D, let $\lambda(i-1,k) = ({}_\lambda B_k^{i-1})_-(z)$. Then for $1<p<\infty$ there are constants $M_{pi}$, such that*

$$\max_j \max\{\|{}_\lambda A_j^{i-2}\circ\phi_z\|_p, \|{}_\lambda B_j^{i-2}\circ\phi_z\|_p\} \leqslant M_{pi}.$$

*Moreover $M_{pi}$ depends on M and p, but does not depend on z.*

**Proof.** We will prove this lemma by induction. When $i=2$, we have

$${}_\lambda A_1^0 = A_1,\ {}_\lambda B_1^0 = A_2.$$

For each $1<p<\infty$,

$$\|{}_\lambda A_1^0\circ\phi_z\|_p = \|A_1\circ\phi_z\|_p \leqslant \|A_1\|_\infty \leqslant M$$

and

$$\|{}_\lambda B_1^0\circ\phi_z\|_p = \|A_2\circ\phi_z\|_p \leqslant \|A_2\|_\infty \leqslant M.$$

When $i=n$, for each $1<p<\infty$, assume

$$\max_j \max\{\|{}_\lambda A_j^{n-2}\circ\phi_z\|_p, \|{}_\lambda B_j^{n-2}\circ\phi_z\|_p\} \leqslant M_{pn}.$$

Let $N_p$ be the positive constant such that

$$\|P_+ f\|_p \leqslant N_p \|f\|_p,$$
$$\|P_- f\|_p \leqslant N_p \|f\|_p$$

for $f \in L^p$. When $i = n+1$,

$$\begin{gathered} {}_\lambda A^{n-1}_{2k-1} \circ \phi_z = {}_\lambda A^{n-2}_k \circ \phi_z [({}_\lambda B^{n-2}_k)_+ \circ \phi_z + \lambda(n-2,k)], \\ {}_\lambda B^{n-1}_{2k-1} \circ \phi_z = A_{n+1} \circ \phi_z, \\ {}_\lambda B^{n-1}_{2k} \circ \phi_z = [({}_\lambda B^{n-2}_k)_- \circ \phi_z - \lambda(n-2,k)] A_{n+1} \circ \phi_z, \\ {}_\lambda A^{n-1}_{2k} \circ \phi_z = {}_\lambda A^{n-2}_k \circ \phi_z. \end{gathered}$$

Clearly,

$$\max_k \max\{\|{}_\lambda A^{n-1}_{2k} \circ \phi_z\|_p, \|{}_\lambda B^{n-1}_{2k-1} \circ \phi_z\|_p\} \leqslant \max\{M_{pn}, M\}.$$

Note that for each function $f \in L^2$,

$$f_+ \circ \phi_z = (f \circ \phi_z)_+ - f_-(z), \qquad f_- \circ \phi_z = (f \circ \phi_z)_- + f_-(z).$$

Thus

$$({}_\lambda B^{n-2}_k)_+ \circ \phi_z = ({}_\lambda B^{n-2}_k \circ \phi_z)_+ - ({}_\lambda B^{n-2}_k)_-(z),$$

and

$$({}_\lambda B^{n-2}_k)_- \circ \phi_z = ({}_\lambda B^{n-2}_k \circ \phi_z)_- + ({}_\lambda B^{n-2}_k)_-(z).$$

By our choice, we have

$$\lambda(n-2,k) = ({}_\lambda B^{n-2}_k)_-(z).$$

So

$$({}_\lambda B^{n-2}_k)_+ \circ \phi_z + \lambda(n-2,k) = ({}_\lambda B^{n-2}_k \circ \phi_z)_+,$$

and

$$({}_\lambda B^{n-2}_k)_- \circ \phi_z - \lambda(n-2,k) = ({}_\lambda B^{n-2}_k \circ \phi_z)_-.$$

Hence we conclude

$$\begin{aligned}\|{}_\lambda A^{n-1}_{2k-1}\circ\phi_z\|_p &= \|{}_\lambda A^{n-2}_k\circ\phi_z[({}_\lambda B^{n-2}_k)_+\circ\phi_z+\lambda(n-2,k)]\|_p\\ &= \|{}_\lambda A^{n-2}_k\circ\phi_z({}_\lambda B^{n-2}_k\circ\phi_z)_+\|_p\\ &\leqslant \|{}_\lambda A^{n-2}_k\circ\phi_z\|_{2p}\|({}_\lambda B^{n-2}_k\circ\phi_z)_+\|_{2p}\\ &\leqslant N_p M^2_{(2p)n},\end{aligned}$$

and

$$\|{}_\lambda B^{n-1}_{2k}\circ\phi_z\|_p = \|[({}_\lambda B^{n-2}_k)_-\circ\phi_z-\lambda(n-2,k)]A_{n+1}\circ\phi_z\|_p$$

$$= \|({}_\lambda B^{n-2}_k\circ\phi_z)_- A_{n+1}\circ\phi_z\|_p \leqslant \|({}_\lambda B^{n-2}_k\circ\phi_z)_-\|_p\|A_{n+1}\circ\phi_z\|_\infty \leqslant N_p M_{pn} M.$$

The last inequality follows because the Hardy projection is bounded on $L^p$ for $1<p<\infty$. Letting $M_{p(n+1)}=\max\{N_p M^2_{(2p)n}, N_p M_{pn}M, M_{pn}, M\}$, we complete the proof. □

Summarily, Proposition 3 suggests the first part of the following theorem and Lemma 4 gives the second part of the following theorem.

**Theorem 5.** *Let $M$ be a positive constant. Suppose that $T$ is a finite sum of finite products of Toeplitz operators, i.e., for $A_{lj}$ in $L^\infty$ with $\max_{l,j}\|A_{lj}\|_\infty\leqslant M$,*

$$T=\sum_{l=1}^{L}\prod_{j=1}^{I_l}T_{A_{lj}}.$$

(1) *For any sequence $\lambda=\{\lambda(l,j)\}$ of complex numbers, then*

$$T-T_{\sum_{l=1}^{L}\prod_{j=1}^{I_l}A_{lj}}=\sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}H^*_{\overline{{}_\lambda A^{I_l-2}_{lj}}}H_{{}_\lambda B^{I_l-2}_{lj}}.$$

(2) *For each $z\in D$ we can find a sequence $\lambda_z=\{\lambda(l,j)(z)\}$ of complex numbers so that for $1<p<\infty$*

$$\max_{l,j}\max\{\|{}_{\lambda_z}A^{I_l-2}_{lj}\circ\phi_z\|_p,\|{}_{\lambda_z}B^{I_l-2}_{lj}\circ\phi_z\|_p\}\leqslant M_p,$$

*for some constant $M_p$ depending only on $M$ and $p$.*

## 4. A generalized area integral function

For a point $w$ of $\partial D$, let $\Gamma(w)$ denote the angle with vertex $w$ and opening $\pi/2$ which is bisected by the radius to $w$. The set of points $z$ in $\Gamma(w)$ satisfying $|z-w| < \varepsilon$ will be denoted by $\Gamma_\varepsilon(w)$. For $h$ in $L^1(\partial D)$, define the truncated Lusin area integral of $h$ to be

$$A_\varepsilon(h)(w) = \left[\int_{\Gamma_\varepsilon(w)} |grad\, h(z)|^2 dA(z)\right]^{1/2},$$

where $(grad\, h)(z)$ denotes the gradient of the harmonic extension $h$ at $z = x + iy$:

$$grad\, h(z) = \left(\frac{\partial h}{\partial x}(z), \frac{\partial h}{\partial y}(z)\right),$$

$dA(z)$ denotes the Lebesgue measure on the unit disk $D$ and $h(z)$ denotes the harmonic extension of $h$ at $z \in D$, via the Poisson integral

$$h(z) = \int_{\partial D} h(w) \frac{(1-|z|^2)}{|1-w\bar{z}|^2} d\sigma(w).$$

Observe that if $h$ is holomorphic, $A_\varepsilon(h)(w)$ equals the area of the image of $\Gamma_\varepsilon(w)$ under the mapping $z \to h(z)$, with points counted according to their multiplicity.

Suppose that $T$ is a finite sum of finite products of Toeplitz operators, i.e., for some functions $A_{lj}$ in $L^\infty$,

$$T = \sum_{l=1}^{L} \prod_{j=1}^{I_l} T_{A_{lj}}.$$

By Theorem 5, for any sequence $\lambda$ of complex numbers, we have the representation

$$T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{[\lambda A_{lj}^{I_l-2}]}} H_{\lambda B_{lj}^{I_l-2}}.$$

Let $\lambda_0$ be the sequence $\{\lambda(l,j)\}$ with $\lambda(l,j)=0$ for $l,j$. Let $u$ and $v$ be in the class $\mathcal{P}$ of analytic polynomials on the unit disk. Define a generalized area integral by

$$_T B_\varepsilon(u,v)(w) = \int_{\Gamma_\varepsilon(w)} \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z)) \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z)) \right| dA(z).$$

Here $((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z)) \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z))$ denotes the inner product of the two complex vectors $((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z))$ and $((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z))$.

The main result in this section is that $_T B_\varepsilon(u,v)(w)$ does not depend on $\lambda_0$. That is, for any sequence $\lambda$ of complex numbers,

$$_T B_\varepsilon(u,v)(w) = \int_{\Gamma_\varepsilon(w)} \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda B_{lj}^{I_l-2}} u)(z)) \bullet ((grad\, H_{\overline{\lambda A_{lj}^{I_l-2}}} v)(z)) \right| dA(z).$$

Note that both $H_{\lambda B_{lj}^{I_l-2}} u$ and $H_{\overline{\lambda A_{lj}^{I_l-2}}} v$ are in $\overline{H^2}$. Thus

$$\sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda B_{lj}^{I_l-2}} u)(z)) \bullet ((grad\, H_{\overline{\lambda A_{lj}^{I_l-2}}} v)(z)) = 2\frac{\partial^2}{\partial z \partial \bar{z}} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((H_{\lambda B_{lj}^{I_l-2}} u)(z)) \overline{((H_{\overline{\lambda A_{lj}^{I_l-2}}} v)(z))} \right]. \tag{4}$$

So

$$_T B_\varepsilon(u,v)(w) = 2\int_{\Gamma_\varepsilon(w)} \left| \frac{\partial^2}{\partial z \partial \bar{z}} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} \left( \left( H_{\lambda_0 B_{lj}^{I_l-2}} u \right)(z) \right) \times \overline{((H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z))} \right] \right| dA(z). \tag{5}$$

We need to introduce some notation. For $x$ and $y$ two vectors in $L^2$. $x \otimes y$ is the operator of rank one defined by

$$(x \otimes y)(f) = \langle f, y \rangle x.$$

Observe that the norm of the operator $x \otimes y$ equals

$$\|x\|_2\|y\|_2.$$

We thank D. Sarason for suggesting the following lemma that gives a way to estimate the norm of the operators with finite rank. Let *trace* be the trace on the trace class of operators on a Hilbert space.

**Lemma 6.** *Let $x_1, \ldots, x_N, y_1, \ldots, y_N$ be vectors in a Hilbert space, let $S = \sum_{i=1}^N x_i \otimes y_i$. Then there is an $N \times N$ unitary matrix $U$ such that*

$$S = \sum_{i=1}^N \tilde{x}_i \otimes \tilde{y}_i \tag{6}$$

*and*

$$trace\, SS^* = \sum_{i=1}^N \|\tilde{x}_i\|^2 \|\tilde{y}_i\|^2, \tag{7}$$

*where*

$$\begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_N \end{pmatrix} = U \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_N \end{pmatrix} = U \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}.$$

For the proof of the above lemma, a computation shows (6) holds for any $N \times N$ unitary matrix $U$. To get (7) one just takes $U$ to diagonalize the Grammian matrix of the vectors $y_1, \ldots, y_N$. The details are left to the reader.

Note that if $f_1, \ldots, f_N$ are in $L^p$, $U$ is an $N \times N$ unitary matrix, and

$$\begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} = U \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix},$$

then

$$\|h_j\|_p \leqslant N \max_j \|f_i\|_p$$

for $j = 1, \ldots, N$. Let $x_i = H_{f_i} k_z$ and $y_i = H_{g_i} k_z$. Applying the above lemma, we obtain the following lemma.

**Lemma 7.** *Let* $S = \sum_{i=1}^{N} H_{f_i}k_z \otimes H_{g_i}k_z$. *Then there is a unitary* $N \times N$ *matrix* $U_z = (a_{ij}(z))_{N\times N}$ *such that*

$$trace\, SS^* = \sum_{i=1}^{N} \|H_{\tilde{f}_i}k_z\|_2^2 \|H_{\tilde{g}_i}k_z\|_2^2,$$

*where* $(\tilde{f}_i)^T = U_z(f_i)^T$ *and* $(\tilde{g}_i)^T = U_z(g_i)^T$. *Moreover,*

$$S = \sum_{i=1}^{N} H_{\tilde{f}_i}k_z \otimes H_{\tilde{g}_i}k_z,$$

*and if for some* $p \in (1, \infty)$, *there is a positive constant* $M_p$ *such that*

$$\max_i \max\{\|f_i \circ \phi_z\|_p, \|g_i \circ \phi_z\|_p\} \leqslant M_p,$$

*then*

$$\max_i \max\{\|\tilde{f}_i \circ \phi_z\|_p, \|\tilde{g}_i \circ \phi_z\|_p\} \leqslant NM_p.$$

Define an antiunitary operator $V$ on $L^2$ by

$$(Vh)(w) = \overline{wh(w)}.$$

The operator enjoys many nice properties such as $V^{-1}(I-P)V = P$ and $V = V^{-1}$. These properties easily leads to the relation

$$V^{-1}H_f V = H_f^*.$$

To show that ${}_TB_\varepsilon(u, v)(w)$ does not depend on $\lambda_0$, we need the following lemma.

**Lemma 8.** *Let* $\phi$ *and* $\psi$ *be polynomials in z. Suppose that f and g are in* $\cap_{p>1} L^p$. *Then*

$$(1-|z|^2)H_g\phi(z)\overline{H_f\psi(z)} = |z|^2\langle [VH_f k_z \otimes VH_g k_z]\phi, \psi\rangle.$$

**Proof.** For each $z \in D$, $f \to f(z)$ is a bounded linear functional on $[H^2]^\perp$, and $\{\bar{w}^n\}$ is an orthonormal basis for $[H^2]^\perp$. Thus the reproducing kernel at $z$ is given by

$$\sum_{n=1}^{\infty} \bar{w}^n z^n = z\bar{w}K_{\bar{z}}(\bar{w}) = zVK_z.$$

So

$$H_g\phi(z) = \bar{z}\langle H_g\phi, VK_z\rangle$$

and

$$H_f\psi(z) = \bar{z}\langle H_f\psi, VK_z\rangle.$$

This gives

$$\begin{aligned} &H_g\phi(z)\overline{H_f\psi(z)} \\ &\quad = |z|^2\langle H_g\phi, VK_z\rangle\overline{\langle H_f\psi, VK_z\rangle} \\ &\quad = |z|^2\langle \phi, H_g^* Vk_z\rangle\overline{\langle \psi, H_f^* Vk_z\rangle} \\ &\quad = |z|^2\langle [H_f^* Vk_z] \otimes [H_g^* Vk_z]\phi, \psi\rangle \\ &\quad = |z|^2\langle [VH_f k_z \otimes VH_g k_z]\phi, \psi\rangle, \end{aligned}$$

to complete the proof. □

The proof of Lemma 1 [23] leads to the following lemma.

**Lemma 9.** *Suppose that f and g are in* $\cap_{p>1} L^p$. *Then the operator* $H_f^* H_g - T_{\phi_z}^* H_f^* H_g T_{\phi_z}$ *equals*

$$[VH_f k_z] \otimes [VH_g k_z]$$

**Theorem 10.** *For any sequence* $\lambda = \{\lambda(l, j)\}$ *of complex numbers,*

$$\begin{aligned} {}_T B_\varepsilon(u, v)(w) = \int_{\Gamma_\varepsilon(w)} \Bigg| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} \left( \left( grad\, H_{{}_\lambda B_{lj}^{I_l-2}} u \right)(z) \right) \\ \bullet \left( \left( grad\, H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v \right)(z) \right) \Bigg| dA(z). \end{aligned}$$

**Proof.** Let

$$T = \sum_{l=1}^{L} \prod_{j=1}^{I_l} T_{A_{lj}}.$$

By Theorem 5, for any sequence $\lambda = \{\lambda(l, j)\}$ of complex numbers, we have the following representation:

$$T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{[\lambda A_{lj}^{I_l - 2}]}} H_{\lambda B_{lj}^{I_l - 2}}.$$

Note that for each $\eta \in D$,

$$T^*_{\phi_\eta} T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} T_{\phi_\eta} = T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}}.$$

Thus for each $\eta \in D$,

$$T - T^*_{\phi_\eta} T T_{\phi_\eta} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{[\lambda A_{lj}^{I_l - 2}]}} H_{\lambda B_{lj}^{I_l - 2}} - T^*_{\phi_\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{[\lambda A_{lj}^{I_l - 2}]}} H_{\lambda B_{lj}^{I_l - 2}} \right] T_{\phi_\eta}. \tag{8}$$

By Lemma 9, we get

$$\sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{[\lambda A_{lj}^{I_l - 2}]}} H_{\lambda B_{lj}^{I_l - 2}} - T^*_{\phi_\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{[\lambda A_{lj}^{I_l - 2}]}} H_{\lambda B_{lj}^{I_l - 2}} \right] T_{\phi_\eta} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} [V H_{\overline{[\lambda A_{lj}^{I_l - 2}]}} k_\eta] \otimes [V H_{\lambda B_{lj}^{I_l - 2}} k_\eta]. \tag{9}$$

By Lemma 8, we have

$$(1 - |\eta|^2) \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} \left( H_{\lambda B_{lj}^{I_l - 2}} u \right) (\eta) \overline{(H_{\overline{\lambda A_{lj}^{I_l - 2}}} v)(\eta)} = |\eta|^2 \left\langle \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} (V H_{\lambda B_{lj}^{I_l - 2}} k_\eta) \otimes (V H_{\overline{\lambda A_{lj}^{I_l - 2}}} k_\eta) \right] u, v \right\rangle. \tag{10}$$

Combining (9) with (10) gives

$$(1 - |\eta|^2) \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} (H_{\lambda B_{lj}^{I_l - 2}} u)(\eta) \overline{(H_{\overline{\lambda A_{lj}^{I_l - 2}}} v)(\eta)}$$

$$= |\eta|^2 \left\langle \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{[{}_\lambda A_{lj}^{I_l-2}]}} H_{{}_\lambda B_{lj}^{I_l-2}} - T^*_{\phi_\eta} \right.\right.$$
$$\left.\left. \times \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{[{}_\lambda A_{lj}^{I_l-2}]}} H_{{}_\lambda B_{lj}^{I_l-2}} \right] T_{\phi_\eta} \right] u, v \right\rangle$$
$$= |\eta|^2 \langle [T - T^*_{\phi_\eta} T T_{\phi_\eta}] u, v \rangle.$$

The last equality follows from (8). Clearly, the last term does not involve $\lambda$. Hence we conclude that

$$\frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{{}_\lambda B_{lj}^{I_l-2}} u)(\eta) \overline{(H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v)(\eta)} \right]$$

does not depend on the choice of $\lambda$. That is,

$$\frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{{}_\lambda B_{lj}^{I_l-2}} u)(\eta) \overline{(H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v)(\eta)} \right]$$
$$= \frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{{}_{\lambda_0} B_{lj}^{I_l-2}} u)(\eta) \overline{(H_{\overline{{}_{\lambda_0} A_{lj}^{I_l-2}}} v)(\eta)} \right].$$

Hence (5) gives that

$$_T B_\varepsilon(u, v)(w)$$
$$= 2 \int_{\Gamma_\varepsilon(w)} \left| \frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{{}_{\lambda_0} B_{lj}^{I_l-2}} u)(\eta) \overline{(H_{\overline{{}_{\lambda_0} A_{lj}^{I_l-2}}} v)(\eta)} \right] \right| dA(\eta)$$
$$= 2 \int_{\Gamma_\varepsilon(w)} \left| \frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{{}_\lambda B_{lj}^{I_l-2}} u)(\eta) \overline{(H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v)(\eta)} \right] \right| dA(\eta)$$
$$= \int_{\Gamma_\varepsilon(w)} \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\ H_{{}_\lambda B_{lj}^{I_l-2}} u)(\eta)) \bullet ((grad\ H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v)(\eta)) \right| dA(\eta).$$

The last equality follows from (4). This completes the proof. □

## 5. A distribution function inequality

In this section we will establish a distribution function inequality for the generalized area integral introduced in Section 4. The distribution function inequality involves the Lusin area integral and the Hardy–Littlewood maximal function. The idea to use distribution function inequalities in the theory of Toeplitz operators and Hankel operators first appeared in [1]. Chang [5] also used a distribution function inequality to study the commutator of the Szegö projection and multiplication operators.

Write $|I|$ for the length of an arc $I$. The Hardy–Littlewood maximal function of $h$ is

$$Mh(e^{i\theta}) = \sup_{e^{i\theta}\in I} \frac{1}{|I|}\int_I |h(e^{i\theta})| d\sigma(e^{i\theta})$$

for $h$ integrable on the unit circle $\partial D$. The Hardy–Littlewood maximal theorem ([11, Theorem 4.3]) states that for $1 < p \leqslant \infty$,

$$\|Mh\|_p \leqslant N_p \|h\|_p$$

for $h \in L^p$ where $N_p$ is a constant depending only on $p$. For $r > 1$, let

$$\Lambda_r h(e^{i\theta}) = [M|h|^r(e^{i\theta})]^{1/r}.$$

Then

$$\|\Lambda_r h\|_p \leqslant N_{\frac{p}{r}}^{\frac{1}{r}} \|h\|_p,$$

for $p > r$.

For $z \in D$, we let $I_z$ denote the closed subarc of $\partial D$ with center $\frac{z}{|z|}$ and length $\delta(z) = 1 - |z|$. The Lebesgue measure of a subset $E$ of $\partial D$ will be denoted by $|E|$.

Recall the area integral function $A_\varepsilon(h)(w)$ for a function $h$ in $L^1$:

$$A_\varepsilon(h)(w) = \left[\int_{\Gamma_\varepsilon(w)} |grad\, h(z)|^2 dA(z)\right]^{1/2},$$

where $h(z)$ denotes the harmonic extension of $h$ at $z \in D$.

The following distribution function inequality was established in [23].

### 5.1. The distribution function inequality

*Let $f$ and $g$ be in $L^2$, and $\phi$ and $\psi$ in the Hardy space $H^2$. Fix $s > 2$. Then there are numbers $p,\ r \in (1, 2)$ with $\frac{1}{s} + \frac{1}{r} = \frac{1}{p}$, such that for $|z| > 1/2$ and $a > 0$*

*sufficiently large,*

$$\Big|\Big\{ w\in I_z : A_{2\delta(z)}(H_f\phi)(w)A_{2\delta(z)}(H_g\psi)(w) < a^2\|f_-\circ\phi_z - f_-(z)\|_s\|g_-\circ\phi_z - g_-(z)\|_s \times \inf_{w\in I_z} \Lambda_r(\phi)(w)\inf_{w\in I_z}\Lambda_r(\psi)(w)\Big\}\Big| \geqslant C_a|I_z|.$$

*Moreover, the constant* $C_a = 1 - C'a^{-p}$ *and* $C'$ *is a constant depending only on* $s$.

For each $f$ in $L^2$, write $f = f_+ + f_-$. Given $z$ in $D$, an easy calculation gives

$$H_f k_z = [f_- - f_-(z)]k_z.$$

Thus by a change of variable, we have

$$\|H_f k_z\|_2 = \|[f_- - f_-(z)]k_z\|_2 = \|f_-\circ\phi_z - f_-(z)\|_2.$$

If $f$ is in $\cap_{p>1} L^p$, by the Cauchy–Schwarz inequality, for $s > 2$, we have

$$\|f\|_s^s = \int_{\partial D} |f(w)|^s d\sigma(w) \leqslant \left[\int_{\partial D} |f(w)|^2 d\sigma(w)\right]^{1/2}\left[\int_{\partial D} |f(w)|^{2s-2} d\sigma(w)\right]^{1/2},$$

to get

$$\|f_-\circ\phi_z - f_-(z)\|_s \leqslant \|f_-\circ\phi_z - f_-(z)\|_2^{1/s}\|f_-\circ\phi_z - f_-(z)\|_{2s-2}^{(s-1)/s}.$$

The above distribution function inequality implies the following form, which will be needed later on.

*Let* $f$ *and* $g$ *be in* $L^2$, *and* $\phi$ *and* $\psi$ *in the Hardy space* $H^2$. *Suppose that for some* $s > 2$ *there is a constant* $M_{2s-2}$ *such that*

$$\sup_{z\in D} \max\{\|f_-\circ\phi_z - f_-(z)\|_{2s-2}, \|g_-\circ\phi_z - g_-(z)\|_{2s-2}\} \leqslant M_{2s-2}.$$

*Then there are numbers* $p,\ r\in(1,2)$ *with* $\frac{1}{s}+\frac{1}{r}=\frac{1}{p}$, *such that for* $|z|>1/2$ *and* $a>0$ *sufficiently large,*

$$\Bigg|\Bigg\{w\in I_z : A_{2\delta(z)}(H_f\phi)(w)A_{2\delta(z)}(H_g\psi)(w) < a^2 M_{2s-2}^{\frac{2s-2}{s}}[\|H_fk_z\|_2\|H_gk_z\|_2]^{1/s}\inf_{w\in I_z}\Lambda_r(\phi)(w)\inf_{w\in I_z}\Lambda_r(\psi)(w)\Bigg\}\Bigg| \geqslant C_a|I_z|. \tag{11}$$

*Moreover, the constant* $C_a=1-C'a^{-p}$ *and* $C'$ *is a constant depending only on* $s$.

The following distribution function inequality is the main result in this section and is the key to the proof of Theorem 12.

**Theorem 11.** *Let* $M$ *be a positive constant. Suppose that* $T$ *is a finite sum of finite products of Toeplitz operators, i.e., for* $A_{lj}$ *in* $L^\infty$ *with* $\max_{l,j}\|A_{lj}\|_\infty\leqslant M$,

$$T=\sum_{l=1}^{L}\prod_{j=1}^{I_l}T_{A_{lj}}.$$

*Let* $u$ *and* $v$ *be in* $\mathcal{P}$. *Let* $z$ *be a point in* $D$ *such that* $|z|>1/2$. *Then for any* $s>2$, *for* $a>0$ *sufficiently large and* $\delta(z)=1-|z|$,

$$\Bigg|\Bigg\{w\in I_z : {}_TB_{2\delta(z)}(u,v)(w) < a^2\|T-T^*_{\phi_z}TT_{\phi_z}\|^{1/s}M_{2s-2}^{2(s-1)/s}\left[\inf_{w\in I_z}\Lambda_r(u)(w)\right]\times\left[\inf_{w\in I_z}\Lambda_r(v)(w)\right]\Bigg\}\Bigg|\geqslant C_a|I_z|,$$

*where* $C_a$ *depends only on* $s$ *and* $a$, $\lim_{a\to\infty}C_a=1$, *and* $\frac{1}{s}+\frac{1}{r}=\frac{1}{p}$ *for some* $p$ *and* $r$ *in* $(1,2)$ *and* $M_{2s-2}$ *is the constant in Theorem* 5 *depending only on* $M$ *and* $s$.

**Proof.** Assume that

$$T=\sum_{l=1}^{L}\prod_{j=1}^{I_l}T_{A_{lj}}.$$

Let $L(T)$ denote the integer $\sum_{l=1}^{L}I_l$. Fix $s>2$. Choose two numbers $1<p<r<2$ such that $\frac{1}{s}+\frac{1}{r}=\frac{1}{p}$. Fix a point $z\in D$. Let $S_z=T-T^*_{\phi_z}TT_{\phi_z}$.

Since for some positive constant $M$,

$$\max_{l,j} \|A_{lj}\|_\infty \leqslant M,$$

for such $z$, by Theorem 5, we can choose $\lambda = \{\lambda(l,j)(z)\}$ so that

$$T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{{}_\lambda A_{lj}^{I_l-2}}} H_{{}_\lambda B_{lj}^{I_l-2}},$$

and

$$\max_{l,j} \max\{\|{}_\lambda A_{lj}^{I_l-2} \circ \phi_z\|_{2s-2}, \|{}_\lambda B_{lj}^{I_l-2} \circ \phi_z\|_{2s-2}\} \leqslant M_{2s-2},$$

where $M_{2s-2}$ is the constant in Theorem 5, depending only on $2s-2$ and $M$.

Let $E$ be the subset of $I_z$ such that

$${}_T B_{2\delta(z)}(u,v)(w) \leqslant a^2 M_{2s-2}^{2(s-1)/s} \|S_z\|^{1/s} \left[\inf_{w\in I_z} \Lambda_r(u)(w)\right] \left[\inf_{w\in I_z} [\Lambda_r(v)(w)\right].$$

To complete the proof, we need only to prove that

$$|E| \geqslant C_a |I_z| \tag{12}$$

for some positive constant $C_a$ depending only on $a$, $s$ and $L(T)$ and satisfying

$$\lim_{a\to\infty} C_a = 1. \tag{13}$$

Because

$$T^*_{\phi_z} T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} T_{\phi_z} = T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}},$$

we have

$$S_z = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{{}_\lambda A_{lj}^{I_l-2}}} H_{{}_\lambda B_{lj}^{I_l-2}} - T^*_{\phi_z} \left[\sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{{}_\lambda A_{lj}^{I_l-2}}} H_{{}_\lambda B_{lj}^{I_l-2}}\right] T_{\phi_z}.$$

Lemma 9 gives

$$S_z = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} [VH_{\overline{\lambda A_{lj}^{I_l-2}}} k_z] \otimes [VH_{\lambda B_{lj}^{I_l-2}} k_z].$$

By Lemma 7, there are functions $\{\lambda f_i\}_{i=1}^{J}$ in the space spanned by $\{\overline{\lambda A_{lj}^{I_l-2}}\}_{l=1,j=1}^{L,2^{I_l-2}}$ and $\{\lambda g_i\}_{i=1}^{J}$ in the space spanned by $\{\lambda B_{lj}^{I_l-2}\}_{l=1,j=1}^{L,2^{I_l-2}}$ such that

$$S_z = \sum_{i=1}^{J} [VH_{\lambda f_i} k_z] \otimes [VH_{\lambda g_i} k_z],$$

and

$$trace(S_z S_z^*) = \sum_{i=1}^{J} \|H_{\lambda f_i} k_z\|_2^2 \|H_{\lambda g_i} k_z\|_2^2. \tag{14}$$

Lemma 7 also gives that $J = \sum_{l=1}^{L} 2^{I_l-2}$. Thus

$$J \leqslant 2^{L(T)}. \tag{15}$$

By Lemma 8, we obtain

$$\begin{aligned}
&(1-|\eta|^2) \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{\lambda B_{lj}^{I_l-2}} u)(\eta) \overline{(H_{\overline{\lambda A_{lj}^{I_l-2}}} v)(\eta)} \\
&= |\eta|^2 \left\langle \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (VH_{\lambda B_{lj}^{I_l-2}} k_\eta) \otimes (VH_{\overline{\lambda A_{lj}^{I_l-2}}} k_\eta) \right] u, v \right\rangle = |\eta|^2 \langle S_\eta u, v \rangle \\
&= |\eta|^2 \left\langle \left[ \sum_{i=1}^{J} (VH_{\lambda f_i} k_\eta) \otimes (VH_{\lambda g_i} k_\eta) \right] u, v \right\rangle \\
&= (1-|\eta|^2) \sum_{i=1}^{J} [H_{\lambda g_i} u(\eta)][\overline{H_{\lambda f_i} v(\eta)}].
\end{aligned} \tag{16}$$

Thus Theorem 10 gives

$$
\begin{aligned}
{}_T B_{2\delta(z)}(u,v)(w) &= \int_{\Gamma_{2\delta(z)}(w)} \left| \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (grad\, H_{{}_\lambda B_{lj}^{I_l-2}} u)(\eta) \right.\right. \\
&\qquad \left.\left. \bullet (grad (H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v)(\eta) \right] \right| dA(\eta) \\
&= 2 \int_{\Gamma_{2\delta(z)}(w)} \left| \frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} (H_{{}_\lambda B_{lj}^{I_l-2}} u)(\eta) \right.\right. \\
&\qquad \left.\left. \times \overline{(H_{\overline{{}_\lambda A_{lj}^{I_l-2}}} v)(\eta)} \right] \right| dA(\eta) \\
&\qquad\qquad \text{(by (4))} \\
&= 2 \int_{\Gamma_{2\delta(z)}(w)} \left| \frac{\partial^2}{\partial\eta\partial\bar\eta} \left[ \sum_{i=1}^{J} (H_{\lambda g_i} u)(\eta) \overline{(H_{\lambda f_i} v)(\eta)} \right] \right| dA(\eta) \\
&\qquad\qquad \text{(by (16))} \\
&= \int_{\Gamma_{2\delta(z)}(w)} \left| \sum_{i=1}^{J} (grad (H_{\lambda g_i} u)(\eta)) \bullet (grad (H_{\lambda f_i} v)(\eta)) \right| dA(\eta).
\end{aligned}
$$

The last equality also follows from (4).

Let $E_i$ be the subset of $I_z$ such that

$$
\begin{aligned}
& A_{2\delta(z)}(H_{\lambda f_i} v)(w) A_{2\delta(z)}(H_{\lambda g_i} u)(w) \\
&\quad \leqslant a^2 \frac{(JM_{2s-2})^{(2s-2)/s}}{J^{1+(2s-2)/s}} [\|H_{\lambda f_i} k_z\|_2 \|H_{\lambda g_i} k_z\|_2]^{1/s} \\
&\quad \times \left[ \inf_{w\in I_z} \Lambda_r(u)(w) \right] \left[ \inf_{w\in I_z} \Lambda_r(v)(w) \right].
\end{aligned}
$$

Note that Lemma 7 gives, for $s > 2$,

$$
\max_i \max\{\|{}_\lambda f_i \circ \phi_z\|_{2s-2}, \|{}_\lambda g_i \circ \phi_z\|_{2s-2}\} \leqslant JM_{2s-2}.
$$

The distribution function inequality (11) gives

$$
|E_i| \geqslant (1 - a^{-p} J^{\frac{p}{2} + \frac{p(s-1)}{s}} C')|I_z|. \tag{17}
$$

The Cauchy–Schwarz inequality gives

$$
\begin{aligned}
{}_T B_{2\delta(z)}(u,v)(w) &= \int_{\Gamma_{2\delta(z)}(w)} \left| \sum_{i=1}^{J} ((grad\ H_{\lambda g_i} u)(\eta)) \bullet ((grad\ H_{\lambda f_i} v)(\eta)) \right| dA(\eta) \\
&\leqslant \sum_{i=1}^{J} \int_{\Gamma_{2\delta(z)}(w)} |((grad\ H_{\lambda g_i} u)(\eta)) \bullet ((grad\ H_{\lambda f_i} v)(\eta))| dA(\eta) \\
&\leqslant \sum_{i=1}^{J} \left[ \int_{\Gamma_{2\delta(z)}(w)} |(grad\ H_{\lambda g_i} u)(\eta)|^2 dA(\eta) \right]^{1/2} \\
&\quad \times \left[ \int_{\Gamma_{2\delta(z)}(w)} |(grad\ H_{\lambda f_i} v)(\eta)|^2 dA(\eta) \right]^{1/2} \\
&\leqslant \sum_{i=1}^{J} A_{2\delta(z)}(H_{\lambda f_i} v)(w) A_{2\delta(z)}(H_{\lambda g_i} u)(w).
\end{aligned}
$$

Thus for $w$ in the intersection $\cap_{i=1}^{J} E_i$, we have

$$
\begin{aligned}
{}_T B_{2\delta(z)}(u,v)(w) &\leqslant \sum_{i=1}^{J} A_{2\delta(z)}(H_{\lambda f_i} v)(w) A_{2\delta(z)}(H_{\lambda g_i} u)(w) \\
&\leqslant \sum_{i=1}^{J} \frac{a^2 M_{2s-2}^{(2s-2)/s}}{J} [\| H_{\lambda f_i} k_z \|_2^2 \| H_{\lambda g_i} k_z \|_2^2]^{1/(2s)} \\
&\quad \times \left[ \inf_{w \in I_z} \Lambda_r(u)(w) \right] \left[ \inf_{w \in I_z} \Lambda_r(v)(w) \right] \\
&= \frac{a^2 M_{2s-2}^{(2s-2)/s}}{J} \left\{ \sum_{i=1}^{J} [\| H_{\lambda f_i} k_z \|_2^2 \| H_{\lambda g_i} k_z \|_2^2]^{1/(2s)} \right\} \\
&\quad \times \left[ \inf_{w \in I_z} \Lambda_r(u)(w) \right] \left[ \inf_{w \in I_z} \Lambda_r(v)(w) \right] \\
&\leqslant \frac{a^2 M_{2s-2}^{(2s-2)/s}}{J^{1/(2s)}} \left\{ \sum_{i=1}^{J} [\| H_{\lambda f_i} k_z \|_2^2 \| H_{\lambda g_i} k_z \|_2^2] \right\}^{1/(2s)} \\
&\quad \times \left[ \inf_{w \in I_z} \Lambda_r(u)(w) \right] \left[ \inf_{w \in I_z} \Lambda_r(v)(w) \right] \\
&\qquad \text{(by the Hölder inequality)}
\end{aligned}
$$

$$\leqslant \frac{a^2 M_{2s-2}^{(2s-2)/s}}{J^{1/(2s)}}[trace(S_z S_z^*)]^{1/(2s)}$$

$$\times \left[\inf_{w \in I_z} \Lambda_r(u)(w)\right]\left[\inf_{w \in I_z} \Lambda_r(v)(w)\right]$$

(by (14))

$$\leqslant a^2 M_{2s-2}^{(2s-2)/s} \|S_z\|^{1/s}\left[\inf_{w \in I_z} \Lambda_r(u)(w)\right]\left[\inf_{w \in I_z} \Lambda_r(v)(w)\right]. \tag{18}$$

The last inequality follows from that $S_z S_z^*$ is a finite rank operator of rank at most $J$ and

$$trace(S_z S_z^*) \leqslant J\|S_z S_z^*\| = J\|S_z\|^2.$$

So (18) gives

$$\cap_{i=1}^J E_i \subset E.$$

Since $E_1 \cup E_2 \subset I_z$,

$$|E_1 \cap E_2| = |E_1| + |E_2| - |E_1 \cup E_2| \geqslant |E_1| + |E_2| - |I_z|,$$

By induction, we get

$$|\cap_{i=1}^J E_i| \geqslant \left[\sum_{i=1}^J |E_i|\right] - (J-1)|I_z|.$$

Thus (17) gives

$$|\cap_{i=1}^J E_i| \geqslant (1 - a^{-p} J^{1+\frac{p}{2}+\frac{p(s-1)}{s}} C')|I_z|.$$

So

$$|E| \geqslant (1 - a^{-p} J^{1+\frac{p}{2}+\frac{p(s-1)}{s}} C')|I_z|.$$

By (15) we have

$$|E| \geqslant (1 - a^{-p} 2^{L(T)(1+\frac{p}{2}+\frac{p(s-1)}{s})} C')|I_z|.$$

Letting $C_a = (1 - a^{-p} 2^{L(T)(1+\frac{p}{2}+\frac{p(s-1)}{s})} C')$, we obtain (12) and (13) to complete the proof. $\square$

**Remark.** The above proof shows that the constant $C_a$ depends also on the "length" $L(T)$ of $T$. We thank the referee for pointing out the fact. Also the constant $M$ in Theorem 11 may be chosen as $\max_{l,j} \|A_{lj}\|_\infty$ that is finite. So Theorem 11 holds only for a finite sum $T$ of finite products of Toeplitz operators. Certainly, we would like that Theorem 11 holds for $T$ in the Toeplitz algebra. But it remains open.

## 6. Finite sums of finite products of Toeplitz operators

In this section, using the key distribution function inequality in the previous section, we will prove the main result in this paper about a finite sum of finite products of Toeplitz operators.

**Theorem 12.** *A finite sum $T$ of finite products of Toeplitz operators is a compact perturbation of a Toeplitz operator if and only if*

$$\lim_{|z|\to 1} \|T - T^*_{\phi_z} T T_{\phi_z}\| = 0. \tag{19}$$

**Proof.** Suppose $T = T_A + K$ where $K$ is a compact operator on $H^2$ and $A$ is a function in $L^\infty$. Note that

$$T_A = T^*_{\phi_z} T_A T_{\phi_z}.$$

An easy calculation gives

$$T - T^*_{\phi_z} T T_{\phi_z} = K - T^*_{\phi_z} K T_{\phi_z}.$$

By Lemma 2 [23],

$$\lim_{|z|\to 1} \|K - T^*_{\phi_z} K T_{\phi_z}\| = 0.$$

Thus

$$\lim_{|z|\to 1} \|T - T^*_{\phi_z} T T_{\phi_z}\| = 0.$$

Conversely, suppose that $T$ is a finite sum of finite products of Toeplitz operators and

$$\lim_{|z|\to 1} \|T - T^*_{\phi_z} T T_{\phi_z}\| = 0. \tag{20}$$

We need to prove that $T$ is a compact perturbation of a Toeplitz operators. We may assume that

$$T = \sum_{l=1}^{L} \prod_{j=1}^{I_l} T_{A_{lj}},$$

where $A_{lj}$ are in $L^\infty$ and satisfy

$$\|A_{lj}\|_\infty \leqslant M,$$

if $M = \max_{l,j} \|A_{lj}\|_\infty$.

Let $\lambda_0$ be the sequence $\{\lambda(l,j)\}$ of complex numbers satisfying $\lambda(l,j) = 0$, for $l, j$. By Theorem 5, we have the following representation:

$$T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{{}_{\lambda_0}A_{lj}^{I_l-2}}} H_{{}_{\lambda_0}B_{lj}^{I_l-2}}.$$

Now let $u$ and $v$ be two functions in $\mathcal{P}$. In order to estimate the distance of the operator $T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}}$ to the set of compact operators we consider the inner product,

$$\begin{aligned}
&\langle [T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}}]u, v\rangle \\
&= \left\langle \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} H^*_{\overline{{}_{\lambda_0}A_{lj}^{I_l-2}}} H_{{}_{\lambda_0}B_{lj}^{I_l-2}} u, v \right\rangle \\
&= \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} \langle H_{{}_{\lambda_0}B_{lj}^{I_l-2}} u, H_{\overline{{}_{\lambda_0}A_{lj}^{I_l-2}}} v\rangle.
\end{aligned}$$

Since $H_{{}_{\lambda_0}B_{lj}^{I_l-2}} u$ is orthogonal to $H^2$, we see that

$$H_{{}_{\lambda_0}B_{lj}^{I_l-2}} u(0) = 0.$$

By the Littlewood–Paley formula ([11, Lemma 3.1]), we have

$$\begin{aligned}
&\langle [T - T_{\sum_{l=1}^{L}\prod_{j=1}^{I_l} A_{lj}}]u, v\rangle \\
&= \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}} \int_D ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)) \\
&\quad \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z)) \log \frac{1}{|z|} dA(z) \\
&= \int_D \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)) \\
&\quad \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z)) \log \frac{1}{|z|} dA(z). \qquad (21)
\end{aligned}$$

For each $1/2 < R < 1$, denote

$$\begin{aligned}
\mathcal{W}_R = &\int_{|z|>R} \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}})u(z)) \\
&\bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z)) \log \frac{1}{|z|}\, dA(z)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Z}_R = &\int_{|z|\leqslant R} \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)) \\
&\bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z)) \log \frac{1}{|z|}\, dA(z).
\end{aligned}$$

Thus (21) gives

$$\langle [T - T_{\sum_{l=1}^{L}\prod_{j=1}^{I_l} A_{lj}}]u, v\rangle = \mathcal{W}_R + \mathcal{Z}_R. \qquad (22)$$

First we show that there is a compact operator $K_R$ such that

$$\mathcal{Z}_R = \langle K_R u, v\rangle. \qquad (23)$$

Note that

$$\sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}((grad\, H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)) \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z))$$
$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}(H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)\overline{(H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z)}\right].$$

From the proof of Theorem 10, we know that

$$\sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}(H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)\overline{(H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z)}$$
$$= \frac{|z|^2}{(1-|z|^2)}\langle [T - T^*_{\phi_z}TT_{\phi_z}]u, v\rangle.$$

Thus

$$\sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}((grad\, H_{\lambda_0 B_{lj}^{I_l-2}}u)(z)) \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}}v)(z))$$
$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{|z|^2}{(1-|z|^2)}\langle [T - T^*_{\phi_z}TT_{\phi_z}]u, v\rangle\right].$$

So

$$\mathcal{Z}_R = \int_{\{|z|\leqslant R\}} 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{|z|^2}{(1-|z|^2)}\langle [T - T^*_{\phi_z}TT_{\phi_z}]u, v\rangle\right]\log\frac{1}{|z|}\,dA(z)$$
$$= \left\langle \int_{\{|z|\leqslant R\}} 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{|z|^2}{(1-|z|^2)}[T - T^*_{\phi_z}TT_{\phi_z}]\right]\log\frac{1}{|z|}\,dA(z)u, v\right\rangle.$$

Let

$$K_R = \int_{\{|z|\leqslant R\}} 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{|z|^2}{(1-|z|^2)}[T - T^*_{\phi_z}TT_{\phi_z}]\right]\log\frac{1}{|z|}\,dA(z).$$

Because $T$ is a finite sum of finite products of Toeplitz operators and the integral is taken over the compact subset $\{|z|\leqslant R\}$ of the unit disk $D$, $K_R$ is an integral operator with kernel in $L^2(D\times D, dAdA)$. Thus it is a compact operator on $H^2$. This gives (23).

For any $\tau > 0$, recall that $\Gamma_\tau(w)$ is the cone at $w$ truncated at height $\tau$ and the generalized area integral is given by

$$ {}_T B_\tau(u,v)(w) = \int_{\Gamma_\tau(w)} \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z)) \right. $$
$$ \left. \bullet((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z)) \right| dA(z). $$

Note that ${}_T B_\tau(u,v)(w)$ is increasing with $\tau$. We define the “Stopping time” $\tau(w)$ by

$$ \tau(w) = \sup \left\{ \tau > 0 : {}_T B_\tau(u,v)(w) \right. $$
$$ \left. \leqslant M_{2s-2}^{2(s-1)/s} a^2 \sup_{|z|>R} \|T - T_{\phi_z}^* T T_{\phi_z}\|^{1/s} [\Lambda_r(u)(w)][\Lambda_r(v)(w)] \right\}. $$

Here $M_{2s-2}$ is the constant in Theorem 11 and $a$ is sufficiently large so that $C_a \geqslant \frac{1}{2}$, for the constant $C_a$ in Theorem 11. For $z \in D$, let $\delta(z) = 1 - |z|$. The distribution function inequality (Theorem 11) gives that for each $z \in D$,

$$ |\{w \in I_z : \tau(w) \geqslant 2\delta(z)\}| \geqslant C_a |I_z|. $$

Let $E_z = \{w \in I_z : \tau(w) \geqslant 2|I_z|\}$. Let $\chi_w(z)$ be the characteristic function of the truncated cone $\Gamma_{\tau(w)}(w)$. Now, for $w \in E_z$, write $z = te^{i\theta}$ and note that $\tau(w) \geqslant \frac{3}{2}(1 - |z|)$. We have

$$ |t\, e^{i\,\theta} - w| \leqslant |t\, e^{i\,\theta} - e^{i\,\theta}| + |e^{i\,\theta} - w| \leqslant (1 - |z|) + \frac{(1 - |z|)}{2} \leqslant \tau(w). $$

Therefore, for $w \in E_z$, we have that $z \in \Gamma_{\tau(w)}(w)$ and that $\chi_w(z) = 1$ on $E_z$. So,

$$ \int_{\partial D} \chi_w(z) d\sigma(w) \geqslant |E_z| \geqslant C_a |I_z| = C_a(1 - |z|). \qquad (24) $$

Fubini's theorem gives

$$C_a \int_{|z|>R} \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z)) \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z)) \right| (1-|z|) dA(z)$$

$$\leqslant \int_{|z|>R} \int_{\partial D} \chi_w(z) \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z)) \right.$$

$$\left. \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z)) \right| d\sigma(w) dA(z)$$

(by (24))

$$= \int_{\partial D} \int_{\Gamma_{\tau(w)}(w)} \left| \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l-2}} ((grad\, H_{\lambda_0 B_{lj}^{I_l-2}} u)(z)) \right.$$

$$\left. \bullet ((grad\, H_{\overline{\lambda_0 A_{lj}^{I_l-2}}} v)(z)) \right| dA(z) d\sigma(w)$$

$$= \int_{\partial D} {}_T B_{\tau(w)}(u, v)(w) d\sigma(w)$$

$$\leqslant \int_{\partial D} M_{2s-2}^{2(s-1)/s} a^2 \sup_{|z|>R} \|T - T_{\phi_z}^* T T_{\phi_z}\|^{1/s} [\Lambda_r(u)(w)][\Lambda_r(v)(w)] d\sigma(w)$$

$$\leqslant M_{2s-2}^{2(s-1)/s} a^2 \sup_{|z|>R} \|T - T_{\phi_z}^* T T_{\phi_z}\|^{1/s} [\|\Lambda_r(u)\|_2][\|\Lambda_r(v)\|_2]$$

$$\leqslant N_{\frac{2}{r}}^{\frac{2}{r}} M_{2s-2}^{2(s-1)/s} a^2 \sup_{|z|>R} \|T - T_{\phi_z}^* T T_{\phi_z}\|^{1/s} \|u\|_2 \|v\|_2.$$

The last inequality follows from that

$$||\Lambda_r u||_2 \leqslant N_{\frac{2}{r}}^{\frac{1}{r}} ||u||_2$$

since $\frac{2}{r} > 1$. Note that

$$\log \frac{1}{|z|} \leqslant 1 - |z|$$

for $1/2 \leqslant |z| < 1$. Thus we obtain

$$|\mathcal{W}_R| \leqslant M_{2s-1}^{2(s-1)/s} C_a^{-1} N_{\frac{2}{r}}^{\frac{2}{r}} a^2 \sup_{|z|>R} \|T - T_{\phi_z}^* T T_{\phi_z}\|^{1/s} \|u\|_2 \|v\|_2,$$

so (22) and (23) give

$$\|T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} - K_R\| \leqslant M_{2s-1}^{2(s-1)/s} C_a^{-1} N_{\frac{2}{r}}^{\frac{2}{r}} a^2 \sup_{|z|>R} \|T - T_{\phi_z}^* T T_{\phi_z}\|^{1/s},$$

because $\mathcal{P}$ is dense in $H^2$. Therefore (20) implies

$$\lim_{R\to 1} \|T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} - K_R\| = 0.$$

We conclude that $T - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}}$ is compact. This completes the proof. $\square$

## 7. Two applications

In this section we will completely answer Questions 1 and 2 if $X$ is a finite sum of finite products of Toeplitz operators. First let the operator $S_A$ with symbol $A \in L^2$ be densely defined on $[H^2]^\perp$, by

$$S_A h = P_-(Ah).$$

For two functions $F$ and $G$, an easy calculation gives

$$H_{\bar{G}\bar{F}}^* = T_G H_{\bar{F}}^* + H_{\bar{G}}^* S_F, \tag{25}$$

and

$$S_A H_G = H_{AG} \tag{26}$$

if $A$ is in $H^2$.

For a function $f$ on the unit disk $D$ and $m \in M(H^\infty + C)$, we say

$$\lim_{z\to m} f(z) = 0$$

if for every net $\{z_\alpha\} \subset D$ converging to $m$,

$$\lim_{z_\alpha \to m} f(z_\alpha) = 0.$$

Let $\mathcal{T}$ be the Toeplitz algebra, generated by Toeplitz operators with symbols in $L^\infty$. Theorem 4 in [7] implies that there exists a symbol map from $\mathcal{T}$ to $L^\infty$, and for an operator in $\mathcal{T}$, its symbol is zero if and only if the operator is in the commutator ideal of $\mathcal{T}$.

The following theorem answers Question 1 for a finite sum of finite products of Toeplitz operators.

**Theorem 13.** *Suppose that $X$ is a finite sum of finite products of Toeplitz operators on $H^2$ and $b$ is an inner function. Then $T_b^* X T_b - X$ is compact if and only if for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,*

$$\lim_{z \to m} \|X - T_{\phi_z}^* X T_{\phi_z}\| = 0.$$

Theorem 13 implies the following theorem, which gives the answer to Question 2 for a finite sum of finite products of Toeplitz operators.

**Theorem 14.** *Suppose that $X$ is a finite sum of finite products of Toeplitz operators on $H^2$ and $b$ is an inner function. Then $T_b X - X T_b$ is compact if and only if there are $F \in L^\infty$ and an operator $X_1$ in the commutator ideal of $\mathcal{T}$ such that $X = T_F + X_1$ and for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,*

$$\lim_{z \to m} \|X_1 - T_{\phi_z}^* X_1 T_{\phi_z}\| = 0,$$

*and*

$$\lim_{z \to m} \|H_F k_z\|_2 = 0.$$

**Proof.** Assume that

$$X = \sum_{l=1}^{L} \prod_{j=1}^{I_l} T_{A_{lj}}.$$

Let

$$M = \max_{l,j} \|A_{lj}\|_\infty.$$

Then $M < \infty$. Theorem 5 implies that for each $z \in D$, there is a sequence $\lambda_z$ of complex numbers such that

$$X - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}} = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}} H_{\lambda_z B_{lj}^{I_l-2}}, \tag{27}$$

and

$$\max_{l,j} \max\{\|_{\lambda_z} A_{lj}^{I_l-2} \circ \phi_z\|_4, \|_{\lambda_z} B_{lj}^{I_l-2} \circ \phi_z\|_4\} \leqslant M_4.$$

for some positive constant $M_4$.

Let $F = \sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}$ and $X_1 = X - T_F$. By Theorem 4 in [8], the symbol of $X_1$ is zero.

Suppose that $T_b X - X T_b$ is compact. We need to show that for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,

$$\lim_{z \to m} \|X_1 - T^*_{\phi_z} X_1 T_{\phi_z}\| = 0, \tag{28}$$

and

$$\lim_{z \to m} \|H_F k_z\|_2 = 0. \tag{29}$$

Since $T_{\bar{b}} T_b = I$ and $T_{\bar{b}} T_F T_b = T_F$, we obtain that

$$T_{\bar{b}} X T_b - X = T_{\bar{b}}[X T_b - T_b X]$$

is compact and hence

$$T_{\bar{b}} X_1 T_b - X_1 = T_{\bar{b}}[X - T_F]T_b - [X - T_F] = T_{\bar{b}} X T_b - X$$

is also compact.

By Theorem 13, for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,

$$\lim_{z \to m} \|X_1 - T^*_{\phi_z} X_1 T_{\phi_z}\| = 0. \tag{30}$$

We obtain (28).

*To prove* (29), *first we show that Condition* (30) *implies that* $T_b X_1 - X_1 T_b$ *is compact. This result will be also used at the end of this proof.*

Let $Z = T_b X_1 - X_1 T_b$. Since $T_b X_1 - X_1 T_b$ is a finite sum of finite products of Toeplitz operators, to prove that $Z$ is compact, by Theorem 12 we need only to show that

$$\lim_{|z|\to 1} \|Z - T^*_{\phi_z} Z T_{\phi_z}\| = 0.$$

By the Corona theorem, this is equivalent to the requirement that for each $m \in M(H^\infty + C)$,

$$\lim_{z\to m} \|Z - T^*_{\phi_z} Z T_{\phi_z}\| = 0. \tag{31}$$

Since

$$\begin{aligned}\|Z - T^*_{\phi_z} Z T_{\phi_z}\| &= \|X_1 - T^*_{\phi_z} X_1 T_{\phi_z} + T^*_b [X_1 - T^*_{\phi_z} X_1 T_{\phi_z}] T_b\| \\ &\leqslant \|X_1 - T^*_{\phi_z} X_1 T_{\phi_z}\| + \|T^*_b\| \|X_1 - T^*_{\phi_z} X_1 T_{\phi_z}\| \|T_b\| \\ &\leqslant 2\|X_1 - T^*_{\phi_z} X_1 T_{\phi_z}\|,\end{aligned}$$

for each $m \in M(H^\infty + C)$ satisfying $|b(m)| < 1$, by (30), we have

$$\lim_{z\to m} \|Z - T^*_{\phi_z} Z T_{\phi_z}\| = 0.$$

So we need only to prove (31) for $m \in M(H^\infty + C)$ satisfying $|b(m)| = 1$. In this case, $b$ is constant on the support set of $m$. Thus

$$\lim_{z\to m} \int |b - b(z)|^4 |k_z|^2 d\sigma = \mathbf{0}.$$

Making a change of variable gives

$$\lim_{z\to m} \|b \circ \phi_z - b(z)\|_4 = 0.$$

By (27), we have

$$X_1 = \sum_{l=1}^{L} \sum_{j=1}^{2^{I_l - 2}} H^*_{\overline{\lambda_z A^{I_l-2}_{lj}}} H_{\lambda_z B^{I_l-2}_{lj}}. \tag{32}$$

Let $G$ be either ${}_{\lambda_z}A_{lj}^{I_l-2}$ or ${}_{\lambda_z}B_{lj}^{I_l-2}$. Then,

$$\begin{aligned}\|T^*_{b-b(z)}VH_Gk_z\|_2 &= \|U_zT_{b\circ\phi_z-b(z)}VH_{G\circ\phi_z}1\|_2\\ &= \|T_{b\circ\phi_z-b(z)}VH_{G\circ\phi_z}1\|_2 = \|P[(b\circ\phi_z-b(z))VH_{G\circ\phi_z}1]\|_2\\ &\leqslant \|(b\circ\phi_z-b(z))VH_{G\circ\phi_z}1]\|_2 \leqslant \|b\circ\phi_z-b(z)\|_4\|VH_{G\circ\phi_z}1\|_4\\ &\leqslant \|b\circ\phi_z-b(z)\|_4(1+N_4)\|G\circ\phi_z\|_4\\ &\leqslant (1+N_4)M_4\|b\circ\phi_z-b(z)\|_4.\end{aligned}$$

Here $N_4$ is the norm of the Hardy projection $P$ on $L^4$, and $U_z$ is a unitary operator defined on $L^2$ by

$$U_zh = h\circ\phi_zk_z.$$

Similarly, we also have

$$\|T_{b-b(z)}VH_Gk_z\|_2 \leqslant (1+N_4)M_4\|b\circ\phi_z-b(z)\|_4.$$

Those give

$$\lim_{z\to m}\max\{\|T^*_{b-b(z)}VH_Gk_z\|_2, \|T_{b-b(z)}VH_Gk_z\|_2\} = 0. \tag{33}$$

For each $z\in D$, (32) gives

$$\begin{aligned}Z &= \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}\left[T_bH^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}} - H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}}T_b\right]\\ &= \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}\left[T_{b-b(z)}H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}} - H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}}T_{b-b(z)}\right].\end{aligned}$$

Thus

$$\begin{aligned}&Z - T^*_{\phi_z}ZT_{\phi_z}\\ &= \sum_{l=1}^{L}\sum_{j=1}^{2^{I_l-2}}\Bigg\{[T_{b-b(z)}H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}} - T^*_{\phi_z}T_{b-b(z)}H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}}T_{\phi_z}]\\ &\quad - [H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}}T_{b-b(z)} - T^*_{\phi_z}H^*_{\overline{{}_{\lambda_z}A_{lj}^{I_l-2}}}H_{{}_{\lambda_z}B_{lj}^{I_l-2}}T_{b-b(z)}T_{\phi_z}]\Bigg\}.\end{aligned}$$

To prove (31) it suffices to show that for each $l$, $j$,

$$\lim_{z\to m}\|T_{b-b(z)}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}-T^*_{\phi_z}T_{b-b(z)}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{\phi_z}\|=0, \tag{34}$$

and

$$\lim_{z\to m}\|H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{b-b(z)}-T^*_{\phi_z}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{b-b(z)}T_{\phi_z}\|=0. \tag{35}$$

Since $T_{b-b(z)}T_{\phi_z}=T_{\phi_z}T_{b-b(z)}$, by Lemma 9, we have

$$\begin{aligned}
&H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{b-b(z)}-T^*_{\phi_z}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{b-b(z)}T_{\phi_z}\\
&=\left\{[VH^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}k_z]\otimes[VH_{\lambda_z B_{lj}^{I_l-2}}k_z]\right\}T_{b-b(z)}\\
&=[VH^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}k_z]\otimes[T^*_{b-b(z)}VH_{\lambda_z B_{lj}^{I_l-2}}k_z].
\end{aligned}$$

Thus (33) implies (35). Using two well-known identities (see, e.g., (1.2) and Lemma 5 in [16]),

$$T^*_{\phi_z}T_{b-b(z)}-T_{b-b(z)}T^*_{\phi_z}=H^*_{\overline{b-b(z)}}H_{\overline{\phi_z}}$$

and

$$H_{\overline{\phi_z}}=-Vk_z\otimes k_z,$$

by Lemma 9 again, we have

$$\begin{aligned}
&T_{b-b(z)}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}-T^*_{\phi_z}T_{b-b(z)}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{\phi_z}\\
&=[T_{b-b(z)}VH_{\overline{\lambda_z A_{lj}^{I_l-2}}}k_z]\otimes[VH_{\lambda_z B_{lj}^{I_l-2}}k_z]+H^*_{\overline{b-b(z)}}H_{\overline{\phi_z}}H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{\phi_z}\\
&=[T_{b-b(z)}VH_{\overline{\lambda_z A_{lj}^{I_l-2}}}k_z]\otimes[VH_{\lambda_z B_{lj}^{I_l-2}}k_z]+[VH_{\overline{b-b(z)}}k_z]\\
&\quad\otimes[(H^*_{\overline{\lambda_z A_{lj}^{I_l-2}}}H_{\lambda_z B_{lj}^{I_l-2}}T_{\phi_z})^*k_z].
\end{aligned}$$

Thus (33) implies (34). Therefore, we conclude

$$\lim_{z\to m}\|Z-T^*_{\phi_z}ZT_{\phi_z}\|=0.$$

Hence by Theorem 12, $Z = T_b X_1 - X_1 T_b$ is compact.

Noting that

$$T_b T_F - T_F T_b = T_b[X - X_1] - [X - X_1]T_b = T_b X - X T_b - Z$$

we have that $T_b T_F - T_F T_b$ is compact. Since

$$T_b T_F - T_F T_b = T_b T_F - T_{Fb} = H_{\bar{b}}^* H_F,$$

by the main result in [23] we obtain

$$\lim_{|z|\to 1} \|\bar{b} \circ \phi_z - \bar{b}(z)\|_2 \|F_- \circ \phi_z - F_-(z)\|_2 = 0.$$

Because

$$\lim_{z\to m} \|\bar{b} \circ \phi_z - \bar{b}(z)\|_2 = \lim_{z\to m} (1 - |b(z)|^2)^{1/2} = (1 - |b(m)|^2)^{1/2} > 0,$$

for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$, the above limit gives

$$\lim_{z\to m} \|F_- \circ \phi_z - F_-(z)\|_2 = 0.$$

Thus we get

$$\lim_{z\to m} \|H_F k_z\|_2 = \lim_{z\to m} \|F_- \circ \phi_z - F_-(z)\|_2 = 0,$$

we get (29), as desired.

Conversely, suppose that there are $F \in L^\infty$ and an operator $X_1$ in the commutator ideal of $\mathcal{T}$ such that $X = T_F + X_1$ and for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,

$$\lim_{z\to m} \|X_1 - T_{\phi_z}^* X_1 T_{\phi_z}\| = 0, \tag{36}$$

and

$$\lim_{z\to m} \|H_F k_z\|_2 = 0. \tag{37}$$

We need to show that $T_b X - X T_b$ is compact. Since

$$T_b X - X T_b = T_b X_1 - X_1 T_b + T_b T_F - T_F T_b,$$

it suffices to show that both $T_bX_1 - X_1T_b$ and $T_bT_F - T_FT_b$ are compact. In the proof of (29) we have shown that Condition (36)((30)) implies that $Z = T_bX_1 - X_1T_b$ is compact. Also for each $m \in M(H^\infty + C)$ satisfying $|b(m)| = 1$,

$$\lim_{z\to m} \|H_{\bar{b}}k_z\|_2 = \lim_{z\to m}(1 - |b(z)|^2)^{1/2} = 0,$$

and hence (37) gives that for each $m \in M(H^\infty + C)$,

$$\lim_{z\to m} \|H_{\bar{b}}k_z\|_2\|H_F k_z\|_2 = 0.$$

Thus by the main result in [23] again, $T_bT_F - T_FT_b$ is compact. This completes the proof. □

To prove Theorem 13 we need the following lemmas:

**Lemma 15.** *Let $\{g_j\}$ be functions in $L^2$. Suppose that for a fixed $z \in D$, $\{VH_{g_j}k_z\}_{j=1}^N$ are linearly independent. Let*

$$A_z = (\langle [VH_{g_i}k_z], [VH_{g_j}k_z]\rangle)_{N\times N},$$

*and*

$$B_z = (\langle [VH_{g_i}k_z], [VH_{g_jb}k_z]\rangle)_{N\times N}.$$

*If $c$ is an eigenvalue of the matrix $A_z^{-1}B_z$, then $|c| \leqslant 1$.*

**Proof.** Letting $(x_1, \ldots, x_N)^T$ be the eigenvector for the eigenvalue $c$ of $A_z^{-1}B_z$, we have

$$cA_z \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = B_z \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Taking inner product of $(x_1, \ldots, x_N)^T$ with both sides of the above vector equations we obtain

$$\begin{aligned} c\|VH_{\sum_{j=1}^N x_jg_j}k_z\|^2 &= \langle VS_bH_{\sum_{j=1}^N x_jg_j}k_z, VH_{\sum_{j=1}^N x_jg_j}k_z\rangle \\ &= \langle T_{\bar{b}}VH_{\sum_{j=1}^N x_jg_j}k_z, VH_{\sum_{j=1}^N x_jg_j}k_z\rangle. \end{aligned}$$

The Cauchy–Schwarz inequality gives

$$|c|\|VH_{\sum_{j=1}^N x_j g_j} k_z\|^2 \leqslant \|T_{\bar{b}}\| \|VH_{\sum_{j=1}^N x_j g_j} k_z\|^2.$$

Thus $|c| \leqslant 1$ because $\|T_{\bar{b}}\| \leqslant 1$ and $\|VH_{\sum_{j=1}^N x_j g_j} k_z\|^2 \neq 0$.

**Lemma 16.** *Suppose that $A$ is a $N \times N$ matrix with eigenvalues $|c_i| \leqslant 1$ and for some positive constant $M_4$,*

$$\sup_{z \in D, j} \|f_j \circ \phi_z\|_p \leqslant M_4.$$

*If for $m \in M(H^\infty + C)$ with $|b(m)| < 1$,*

$$\lim_{z \to m} \left\| \begin{pmatrix} VH_{f_1} k_z \\ \vdots \\ VH_{f_N} k_z \end{pmatrix} - A \begin{pmatrix} VH_{f_1 b} k_z \\ \vdots \\ VH_{f_N b} k_z \end{pmatrix} \right\|_2 = 0,$$

*then*

$$\lim_{z \to m} \left\| \begin{pmatrix} VH_{f_1} k_z \\ \vdots \\ VH_{f_N} k_z \end{pmatrix} \right\|_2 = 0.$$

**Proof.** By the Jordan theory there is a unitary matrix $U$ such that

$$U^* A U = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 & 0 \\ \varepsilon_{21} & c_2 & 0 & \cdots & 0 & 0 \\ \varepsilon_{31} & \varepsilon_{32} & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} & \vdots & \vdots & \varepsilon_{NN-1} & c_N \end{pmatrix}.$$

Let

$$\begin{pmatrix} VH_{\tilde{f}_1} k_z \\ \vdots \\ VH_{\tilde{f}_N} k_z \end{pmatrix} = U^* \begin{pmatrix} VH_{f_1} k_z \\ \vdots \\ VH_{f_N} k_z \end{pmatrix}.$$

We get

$$U^* \begin{pmatrix} VH_{f_1}k_z \\ \vdots \\ VH_{f_N}k_z \end{pmatrix} - U^*AUU^* \begin{pmatrix} VH_{f_1b}k_z \\ \vdots \\ VH_{f_Nb}k_z \end{pmatrix}$$
$$= \begin{pmatrix} VH_{\tilde{f}_1}k_z \\ \vdots \\ VH_{\tilde{f}_N}k_z \end{pmatrix} - \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 & 0 \\ \varepsilon_{21} & c_2 & 0 & \cdots & 0 & 0 \\ \varepsilon_{31} & \varepsilon_{32} & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{N1} & \varepsilon_{N2} & \vdots & \vdots & \varepsilon_{NN-1} & c_N \end{pmatrix} \begin{pmatrix} VH_{\tilde{f}_1b}k_z \\ \vdots \\ VH_{\tilde{f}_Nb}k_z \end{pmatrix}. \tag{38}$$

The first equality in the above vector equation gives

$$\lim_{z\to m} \|VH_{\tilde{f}_1(1-\bar{c}_1b)}k_z\|_2 = 0.$$

Making a change of variable yields

$$\lim_{z\to m} \|(1-P)[\tilde{f}_1 \circ \phi_z(1-\bar{c}_1 b\circ\phi_z)]\|_2 = 0.$$

Since $|c_1|\leqslant 1$ and $b$ is not constant on the support set of $m$, by Lemma 1, $(1-\bar{c}_1b)$ is an outer function on the support set of $m$. For any $\varepsilon > 0$, there is a function $p\in H^\infty$ such that

$$\int_{S_m} |p(1-\bar{c}_1b)-1|^2\, d\mu_m < \varepsilon.$$

For such $\varepsilon$, there is also a neighborhood $W$ of $m$ such that for $z\in W\cap D$,

$$\left| \int_{S_m} |p(1-\bar{c}_1b)-1|^2\, d\mu_m - \int_{S_m} |p(1-\bar{c}_1b)-1|^2 |k_z|^2 d\sigma \right| < \varepsilon.$$

Making a change of variable we obtain

$$\int |p\circ\phi_z(1-\bar{c}_1b\circ\phi_z)-1|^2 d\sigma < 2\varepsilon.$$

For $t = \frac{4}{3}$, the Hölder inequality gives

$$\|(1-P)(\tilde{f}_1\circ\phi_z[p\circ\phi_z(1-\bar{c}_1b\circ\phi_z)-1])\|_t$$

$$\leqslant C_t \|(\tilde{f}_1 \circ \phi_z[p \circ \phi_z(1 - \bar{c}_1 b \circ \phi_z) - 1])\|_t$$
$$\leqslant C_t \|\tilde{f}_1 \circ \phi_z\|_{(2t)/(2-t)} \|p \circ \phi_z(1 - \bar{c}_1 b \circ \phi_z) - 1\|_2$$
$$= C_t \|\tilde{f}_1 \circ \phi_z\|_4 \|p \circ \phi_z(1 - \bar{c}_1 b \circ \phi_z) - 1\|_2$$
$$\leqslant C_t M_4 \varepsilon^{1/2}.$$

Thus

$$\|(1-P)\tilde{f}_1 \circ \phi_z\|_t \leqslant C_t M_4 \varepsilon^{1/2} + \|(1-P)[\tilde{f}_1 \circ \phi_z(p \circ \phi_z)(1 - \bar{c}_1 b \circ \phi_z)]\|_t$$
$$\leqslant C_t M_4 \varepsilon^{1/2} + \|p\|_\infty \|(1-P)(\tilde{f}_1 \circ \phi_z(1 - \bar{c}_t b \circ \phi_z))\|_2.$$

The last inequality follows from

$$(1-P)[\tilde{f}_1 \circ \phi_z(p \circ \phi_z)(1 - \bar{c}_1 b \circ \phi_z)]$$
$$= H_{\tilde{f}_1 \circ \phi_z (p \circ \phi_z)}(1 - \bar{c}_1 b \circ \phi_z)$$
$$= S_{p \circ \phi_z} H_{\tilde{f}_1}(1 - \bar{c}_1 b \circ \phi_z). \qquad \text{(by (26))}$$

So

$$\lim_{z \to m} \|(1-P)\tilde{f}_1 \circ \phi_z\|_t \leqslant C_t M_4 \varepsilon^{1/2}.$$

Hence we get

$$\lim_{z \to m} \|(1-P)\tilde{f}_1 \circ \phi_z\|_t = 0.$$

This implies

$$\lim_{z \to m} \|V H_{\tilde{f}_1} k_z\|_2 = \lim_{z \to m} \|(1-P)\tilde{f}_1 \circ \phi_z\|_2 = 0$$

because

$$\|(1-P)\tilde{f}_1 \circ \phi_z\|_2 \leqslant \|(1-P)\tilde{f}_1 \circ \phi_z\|_t^{1/2} \|(1-P)\tilde{f}_1 \circ \phi_z\|_{4)}$$
$$\leqslant M_4 \|(1-P)\tilde{f}_1 \circ \phi_z\|_t^{1/2}.$$

The second equality in (38) yields

$$\lim_{z \to m} \|V H_{\tilde{f}_2(1-\bar{c}_2 b)} k_z + \varepsilon_{21} V H_{\tilde{f}_1} k_z\|_2 = 0.$$

Hence

$$\lim_{z\to m} \|VH_{\tilde{f}_2(1-\bar{c}_2 b)} k_z\|_2 = 0.$$

Repeating the above argument gives

$$\lim_{z\to m} \|VH_{\tilde{f}_2} k_z\|_2 = 0.$$

By induction we conclude that

$$\lim_{z\to m} \|VH_{\tilde{f}_j} k_z\|_2 = 0,$$

for all $j$. Therefore

$$\lim_{z\to m} \left\| \begin{pmatrix} VH_{f_1}k_z \\ \vdots \\ VH_{f_N}k_z \end{pmatrix} \right\|_2 = \lim_{z\to m} \left\| U \begin{pmatrix} VH_{\tilde{f}_1}k_z \\ \vdots \\ VH_{\tilde{f}_N}k_z \end{pmatrix} \right\|_2 = 0.$$

Now we are ready to prove Theorem 13.

**Proof of Theorem 13.** Assume that

$$X = \sum_{l=1}^{L} \prod_{j=1}^{I_l} T_{A_{lj}}.$$

Let

$$M = \max_{l,j} \|A_{lj}\|_\infty.$$

Theorem 5 implies that for each $z \in D$, there is a sequence $\lambda_z$ of complex numbers such that $X - T_{\sum_{l=1}^{L} \prod_{j=1}^{I_l} A_{lj}}$ is a finite sum of products of two Hankel operators:

$$\sum_{k=1}^{N} H^*_{\lambda_z f_k} H_{\lambda_z g_k},$$

and

$$\max_k \max\{\|_{\lambda_z} f_k\|_4, \|_{\lambda_z} g_k\|_4\} \leqslant M_4.$$

Let $Y = T_{\bar{b}}XT_b - X$. Then $Y$ is also a finite sum of finite products of Toeplitz operators and

$$Y = \sum_{k=1}^{N} H^*_{b_{\lambda_z} f_k} H_{b_{\lambda_z} g_k} - \sum_{k=1}^{N} H^*_{\lambda_z f_k} H_{\lambda_z g_k}.$$

Suppose that for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,

$$\lim_{z\to m} \|X - T^*_{\phi_z} X T_{\phi_z}\| = 0.$$

In order to prove that $Y$ is compact, by Theorem 12 we need only to show

$$\lim_{|z|\to 1} \|Y - T^*_{\phi_z} Y T_{\phi_z}\| = 0.$$

This is equivalent to requirement that for each $m \in M(H^\infty + C)$,

$$\lim_{z\to m} \|Y - T^*_{\phi_z} Y T_{\phi_z}\| = 0. \tag{39}$$

Because

$$\begin{aligned}\|Y - T^*_{\phi_z} Y T_{\phi_z}\| &= \|T_{\bar{b}}[X - T^*_{\phi_z} Y T_{\phi_z}]T_b - [X - T^*_{\phi_z} X T_{\phi_z}]\| \\ &\leqslant \|T_{\bar{b}}\| \|X - T^*_{\phi_z} Y T_{\phi_z}\| \|T_b\| + \|X - T^*_{\phi_z} X T_{\phi_z}\| \\ &\leqslant 2\|X - T^*_{\phi_z} X T_{\phi_z}\|,\end{aligned}$$

for $m$ satisfying $|b(m)| < 1$, we get

$$\lim_{z\to m} \|Y - T^*_{\phi_z} Y T_{\phi_z}\| = 0.$$

For $m$ satisfying that $|b(m)| = 1$, $b$ is constant on the support set of $m$. Thus

$$\lim_{z\to m} \|b \circ \phi_z - b(m)\|_4^4 = \lim_{z\to m} \int |b(w) - b(m)|^4 |k_z(w)|^2 d\sigma(w) = 0.$$

Let $f$ be either ${}_{\lambda_z} f_k$ or ${}_{\lambda_z} g_k$. Then

$$\sup_{z\in D} \|f \circ \phi_z\|_4 \leqslant M_4.$$

Thus we have

$$\begin{aligned}\|H_{fb}k_z - b(m)H_f k_z\|_2 &= \|H_f T_{b-b(m)}k_z\|_2 \\ &= \|(1-P)[f\circ\phi_z(b\circ\phi_z - b(m))]\|_2 \\ &\leqslant \|f\circ\phi_z(b\circ\phi_z - b(m))\|_2 \\ &\leqslant \|f\circ\phi_z\|_4\|(b\circ\phi_z - b(m))\|_4 \\ &\leqslant M_4\|(b\circ\phi_z - b(m))\|_4.\end{aligned}$$

This implies

$$\lim_{z\to m}\|H_{fb}k_z - b(m)H_f k_z\|_2 = 0. \tag{40}$$

By Lemma 9, we have

$$Y - T^*_{\phi_z} Y T_{\phi_z} = \left(\sum_{k=1}^{K}\{[VH_{\lambda_z f_k b}k_z]\otimes[VH_{\lambda_z g_k b}k_z] - [VH_{\lambda_z f_k}k_z]\otimes[VH_{\lambda_z g_k}k_z]\}\right).$$

Thus (40) gives

$$\lim_{z\to m}\left\|\sum_{k=1}^{N}[VH_{\lambda_z f_k b}k_z]\otimes[VH_{\lambda_z g_k b}k_z] - \sum_{k=1}^{N}[VH_{\lambda_z f_k b(m)}k_z]\otimes[VH_{\lambda_z g_k b(m)}k_z]\right\| = 0.$$

On the other hand, we have

$$\begin{aligned}&\sum_{k=1}^{N}[VH_{\lambda_z f_k b(m)}k_z]\otimes[VH_{\lambda_z g_k b(m)}k_z] \\ &\quad = |b(m)|^2\sum_{k=1}^{N}[VH_{\lambda_z f_k}k_z]\otimes[VH_{\lambda_z g_k}k_z] \\ &\quad = \sum_{k=1}^{N}[VH_{\lambda_z f_k}k_z]\otimes[VH_{\lambda_z g_k}k_z].\end{aligned}$$

This leads to

$$\lim_{z\to m}\left\|\sum_{k=1}^{N}[VH_{\lambda f_k b}k_z]\otimes[VH_{\lambda_z g_k b}k_z] - \sum_{k=1}^{N}[VH_{\lambda_z f_k}k_z]\otimes[VH_{\lambda_z g_k}k_z]\right\| = 0.$$

Therefore we obtain

$$\begin{aligned}
&\lim_{z\to m} \|Y - T^*_{\phi_z} Y T_{\phi_z}\| \\
&= \lim_{z\to m} \left\| \sum_{k=1}^{N} [VH_{\lambda_z f_k b} k_z] \otimes [VH_{\lambda_z g_k b} k_z] - \sum_{k=1}^{N} [VH_{\lambda_z f_k} k_z] \otimes [VH_{\lambda_z g_k} k_z] \right\| \\
&= 0.
\end{aligned}$$

This completes the proof of (39).

Conversely suppose that $Y$ is compact. By Theorem 12, we have

$$\lim_{|z|\to 1} \|Y - T^*_{\phi_z} Y T_{\phi_z}\| = 0.$$

Thus

$$\lim_{|z|\to 1} \left\| \sum_{k=1}^{N} [VH_{\lambda_z f_k b} k_z] \otimes [VH_{\lambda_z g_k b} k_z] - \sum_{k=1}^{N} [VH_{\lambda_z f_k} k_z] \otimes [VH_{\lambda_z g_k} k_z] \right\| = 0.$$

Note that

$$X - T^*_{\phi_z} X T_{\phi_z} = \sum_{k=1}^{N} [VH_{\lambda_z f_k} k_z] \otimes [VH_{\lambda_z g_k} k_z].$$

It suffices to show that for each $m \in M(H^\infty + C)$ with $|b(m)| < 1$,

$$\lim_{|z|\to 1} \left\| \sum_{k=1}^{N} [VH_{\lambda_z f_k} k_z] \otimes [VH_{\lambda_z g_k} k_z] \right\| = 0.$$

Let $S_z = \sum_{j=1}^{N} [VH_{\lambda_z f_j} k_z] \otimes [VH_{\lambda_z g_j} k_z]$. By Lemma 7, we may assume that $\{VH_{\lambda_z g_j} k_z\}_{j=1}^{N}$ are orthogonal and

$$trace(S_z S_z^*) = \sum_{j=1}^{N} \|VH_{\lambda_z f_j} k_z\|_2^2 \|VH_{\lambda_z g_j} k_z\|_2^2.$$

Since

$$\|S_z\|^2 \leqslant trace(S_z S_z^*) \leqslant N \|S_z\|^2,$$

it is sufficient to show that

$$\lim_{z\to m} trace(S_z S_z^*) = 0.$$

Now we may assume that

$$\lim_{z\to m} \|VH_{\lambda_z g_j} k_z\|_2^2 = c_j \neq 0$$

for $j \leqslant N_1 \leqslant N$ and

$$\lim_{z\to m} \|VH_{\lambda_z g_j} k_z\|_2^2 = 0,$$

for $j > N_1$. Note that $H_{fb}k_z = H_f T_b k_z = S_b H_f k_z$. Thus

$$\|H_{fb}k_z\|_2 \leqslant \|S_b\| \|H_f k_z\|_2,$$

so

$$\lim_{z\to m} \|VH_{\lambda_z g_j b} k_z\|_2 = 0,$$

for $j > N_1$. This gives

$$\lim_{z\to m} \left\| \sum_{j=1}^{N_1} \{[VH_{\lambda_z f_j b} k_z] \otimes [VH_{\lambda_z g_j b} k_z] - [VH_{\lambda_z f_j} k_z] \otimes [VH_{\lambda_z g_j} k_z]\} \right\| = 0.$$

Let

$$R_z = \sum_{j=1}^{N_1} [VH_{\lambda_z f_j b} k_z] \otimes [VH_{\lambda_z g_j b} k_z] - [VH_{\lambda_z f_j} k_z] \otimes [VH_{\lambda_z g_j} k_z].$$

Let

$$A_z = (\langle [VH_{\lambda_z g_i} k_z], [VH_{\lambda_z g_j} k_z] \rangle)_{N_1 \times N_1},$$

and

$$B_z = (\langle [VH_{\lambda_z g_i} k_z], [VH_{\lambda_z g_j b} k_z] \rangle)_{N_1 \times N_1}.$$

Then

$$\begin{pmatrix} VH_{\lambda_z f_1}k_z \\ \vdots \\ VH_{\lambda_z f_{N_1}}k_z \end{pmatrix} = A_z^{-1}B_z \begin{pmatrix} VH_{\lambda_z f_1 b}k_z \\ \vdots \\ VH_{\lambda_z f_{N_1} b}k_z \end{pmatrix} + A_z^{-1} \begin{pmatrix} R_z VH_{\lambda_z g_1}k_z \\ \vdots \\ R_z VH_{\lambda_z g_{N_1}}k_z \end{pmatrix}.$$

By Lemma 15, the absolute values of the eigenvalues of the matrix $A_z^{-1}B_z$ are less than or equal to 1. Moreover

$$\lim_{z\to m} \left\| \begin{pmatrix} VH_{\lambda_z f_1}k_z \\ \vdots \\ VH_{\lambda_z f_{N_1}}k_z \end{pmatrix} - A_z^{-1}B_z \begin{pmatrix} VH_{\lambda_z f_1 b}k_z \\ \vdots \\ VH_{\lambda_z f_{N_1} b}k_z \end{pmatrix} \right\|_2 = 0.$$

By Lemma 16 we conclude that

$$\lim_{z\to m} \left\| \begin{pmatrix} VH_{\lambda_z f_1}k_z \\ \vdots \\ VH_{\lambda_z f_{N_1}}k_z \end{pmatrix} \right\|_2 = 0.$$

This implies

$$\lim_{z\to m} trace(S_z S_z^*) = 0,$$

to complete the proof. □

## 8. Block Toeplitz operators

Let $L^2(C^n)$ be the space of $C^n$-valued Lebesgue square integrable functions on the unit circle. The Hardy space $H^2(C^n)$ is the Hilbert space consisting of $C^n$-valued analytic functions on $D$ that are also in $L^2(C^n)$. Let $L^\infty_{n\times n}$ denote the space of $M_{n\times n}$-valued essentially bounded Lebesgue measurable functions on the unit circle and $H^\infty_{n\times n}$ denote the space of $M_{n\times n}$-valued essentially bounded analytic functions in the disk.

Let $P$ be the projection of $L^2(C^n)$ onto $H^2(C^n)$. For $F \in L^\infty_{n\times n}$, the block Toeplitz operator $T_F : H^2(C^n) \to H^2(C^n)$ with symbol $F$ is defined by

$$T_F h = P(Fh).$$

The main result in Section 6 extends to block Toeplitz operators. That is, a finite sum $T$ of finite products of block Toeplitz operators is a compact perturbation of a block Toeplitz operator if and only if

$$\lim_{|z|\to 1} \|T - T^*_{\Phi_z} T T_{\Phi_z}\| = 0.$$

Here $\Phi_z$ denotes the function $diag\{\phi_z, \ldots, \phi_z\} \in H^\infty_{n\times n}$. This result also extends the main results in [15] on block Toeplitz operators.

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