Standard Deviation and Schatten Class Hankel Operators on the Segal-Bargmann Space

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ABSTRACT. We consider Hankel operators on the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$. Our main result is a necessary and sufficient condition for the simultaneous membership of H_f and $H_{\bar{f}}$ in the Schatten class C_p, $1 \le p < \infty$. We will explain that, since this condition is valid in the case $1 \le p \le 2$ as well as in the case $2 \le p < \infty$, this result reflects the structural difference between the Segal-Bargmann space and other reproducing-kernel spaces such as the Bergman space $L^2_a(B_n, dv)$.

1. INTRODUCTION

Let $d\mu$ be the Gaussian measure on $\mathbb{C}^n$ centered at 0 and normalized so that $\mu(\mathbb{C}^n) = 1$. In terms of the standard volume measure dV on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we can write $d\mu$ as

$$d\mu(z) = \pi^{-n} e^{-|z|^2}\, dV(z).$$

Recall that the Segal-Bargmann space (also called Fock space) $H^2(\mathbb{C}^n, d\mu)$ is defined to be the subspace $\{f \in L^2(\mathbb{C}^n, d\mu) \mid f \text{ is analytic on } \mathbb{C}^n\}$ of $L^2(\mathbb{C}^n, d\mu)$. Throughout the paper, we let $P : L^2(\mathbb{C}^n, d\mu) \to H^2(\mathbb{C}^n, d\mu)$ be the orthogonal projection. It is well known that $H^2(\mathbb{C}^n, d\mu)$ is the $L^2(\mathbb{C}^n, d\mu)$-closure of the polynomials in the variables $z_1, \ldots, z_n$. Therefore

$$\{(k_1! \cdots k_n!)^{-1/2} z_1^{k_1} \cdots z_n^{k_n} \mid k_1 \ge 0, \ldots, k_n \ge 0\}$$

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is an orthonormal basis in $H^2(\mathbb{C}^n, d\mu)$. Thus P has $e^{\langle z,w\rangle}$ as its kernel function, i.e.,

$$(P\varphi)(z) = \int e^{\langle z,w\rangle}\varphi(w)\,d\mu(w), \quad \varphi \in L^2(\mathbb{C}^n, d\mu).$$

Here and in what follows, we write

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$$

for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$. The function $K_z(w) = e^{\langle w,z\rangle}$ is called the *reproducing kernel* for the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$, which simply reflects the fact that $(P\varphi)(z) = \langle \varphi, K_z\rangle$, $\varphi \in L^2(\mathbb{C}^n, d\mu)$. Later on we will also need the normalized reproducing kernel $k_z = K_z/\|K_z\|$.

For each $\zeta \in \mathbb{C}^n$, let $\tau_\zeta : \mathbb{C}^n \to \mathbb{C}^n$ be the translation

$$\tau_\zeta(w) = w + \zeta, \quad w \in \mathbf{C}^n. \tag{1.1}$$

Define

$$\mathcal{T}(\mathbb{C}^n) = \{f \in L^2(\mathbb{C}^n, d\mu) \mid f \circ \tau_\zeta \in L^2(\mathbb{C}^n, d\mu) \text{ for every } \zeta \in \mathbb{C}^n\}.$$

It is easy to see that a measurable function f on $\mathbb{C}^n$ belongs to $\mathcal{T}(\mathbb{C}^n)$ if and only if the function $w \mapsto f(w)e^{\langle w,\zeta\rangle}$ belongs to $L^2(\mathbb{C}^n, d\mu)$ for every $\zeta \in \mathbb{C}^n$. This in particular means that, if $f \in \mathcal{T}(\mathbb{C}^n)$, then the set $\{h \in H^2(\mathbb{C}^n, d\mu) \mid fh \in L^2(\mathbb{C}^n, d\mu)\}$ is a dense, linear subspace of $H^2(\mathbb{C}^n, d\mu)$.

Recall that the Hankel operator $H_f : H^2(\mathbb{C}^n, d\mu) \to L^2(\mathbb{C}^n, d\mu) \ominus H^2(\mathbb{C}^n, d\mu)$ with symbol function f is defined by the formula

$$H_f = (1 - P)M_f P.$$

Thus if $f \in \mathcal{T}(\mathbb{C}^n)$, then H_f has at least a dense domain in $H^2(\mathbb{C}^n, d\mu)$. When considering Hankel operators on the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$, one always needs to impose some growth condition on the symbol function. This is due to the fact that, unlike other reproducing-kernel Hilbert spaces, there are no bounded functions in $H^2(\mathbb{C}^n, d\mu)$ other than constants.

There is another kind of Hankel operators, i.e., the so-called *small* Hankel operators $h_f = \bar{P}M_f P$, where $\bar{P}$ is the projection from $L^2(\mathbb{C}^n, d\mu)$ to $\bar{H}^2(\mathbb{C}^n, d\mu) = \{\bar{f} \mid f \in H^2(\mathbb{C}^n, d\mu)\}$. Such operators were studied extensively in, e.g. [6, 12] and are not the concern of the present paper. In other words, in this paper we will deal with H_f only.

The compactness of Hankel operators with bounded symbols on the Segal-Bargmann $H^2(\mathbb{C}^n, d\mu)$ was characterized by C. Berger and L. Coburn in [2] and by K. Stroethoff in [7] (also see [3]). Normally one would expect that a characterization of the membership of such operators in the Schatten p-classes would soon follow. A search of the literature reveals, however, that the latter characterization is still lacking. The purpose of this paper is to fill this void.

Recall that, for any $1 \le p < \infty$, the Schatten class C_p consists of operators T with $\|T\|_p < \infty$, where the p-norm is defined by the formula

$$\|T\|_p = \{\mathrm{tr}(|T|^p)\}^{1/p} = \{\mathrm{tr}((T^*T)^{p/2})\}^{1/p}.$$

In this paper we will completely determine the membership $H_f \in C_p$ and $H_{\bar{f}} \in C_p$ in terms of the symbol function f.

Given a $\varphi \in L^2(\mathbb{C}^n, d\mu)$, let $\mathrm{SD}(\varphi)$ denote its *standard deviation* with respect to the probability measure $d\mu$. Recall that the formula for standard deviation is

$$\begin{aligned}\mathrm{SD}(\varphi) &= \left\{\int \left|\varphi - \int \varphi\, d\mu\right|^2 d\mu\right\}^{1/2} \\ &= \left\{\int |\varphi|^2\, d\mu - \left|\int \varphi\, d\mu\right|^2\right\}^{1/2} = \{\|\varphi\|^2 - |\langle \varphi, 1\rangle|^2\}^{1/2}.\end{aligned}$$

When $f \in L^\infty(\mathbb{C}^n, d\mu)$, H_f is compact if and only if $H_{\bar{f}}$ is compact [2]. It follows from [7, Theorem 12] that the condition $\lim_{|\zeta|\to\infty} \mathrm{SD}(f \circ \tau_\zeta) = 0$ is equivalent to the compactness of H_f and $H_{\bar{f}}$. The following is the main result of the paper.

Theorem 1.1. *Let $1 \le p < \infty$. Let $f \in \mathcal{T}(\mathbb{C}^n)$ and let H_f and $H_{\bar{f}}$ be the corresponding Hankel operators from $H^2(\mathbb{C}^n, d\mu)$ to $\{H^2(\mathbb{C}^n, d\mu)\}^\perp$ defined above. Then we have the simultaneous membership $H_f \in C_p$ and $H_{\bar{f}} \in C_p$ if and only if*

$$\int \{\mathrm{SD}(f \circ \tau_\zeta)\}^p\, dV(\zeta) < \infty, \tag{1.2}$$

where, as we recall, dV denotes the standard volume measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Since $[M_f, P] = H_f - H_{\bar{f}}^*$ and $[M_f, P]P = H_f$, an alternate statement of Theorem 1.1 is that the membership $[M_f, P] \in C_p$ is equivalent to (1.2).

Theorem 1.1 is motivated by K. Zhu's characterization [11] of Schatten class Hankel operators on the *Bergman space* $L^2_a(B_n, dV)$ of the unit ball of $\mathbb{C}^n$. A detailed comparison of our result with Zhu's is appropriate at this point. However, since such a comparison cannot avoid technical definitions associated with the Bergman space, definitions which have nothing to do with the rest of the paper, we will defer it until Section 6 at the end of the paper.

We close this section with a brief sketch of the ideas involved in the proof of Theorem 1.1. It is fairly easy to show (Lemma 3.1 below) that, when $1 \le p \le 2$, the simultaneous membership $H_f \in C_p$ and $H_{\bar{f}} \in C_p$ implies

$$\sum_{b \in \mathbb{Z}^{2n}} \left\{\int_Q \int_Q |f(z+b) - f(w+b)|^2\, dV(w)\, dV(z)\right\}^{p/2} < \infty, \tag{1.3}$$

where Q is the standard $3\times 3\times \cdots \times 3\times 3$ cube in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The key here is the realization that (1.2) can be deduced from (1.3). And this deduction is the main content of Section 3. Thus, in the case $1 \le p \le 2$, (1.2) is necessary for $H_f \in C_p$ and $H_{\bar{f}} \in C_p$. It is also quite easy to show (Lemma 4.1) that (1.2) implies (1.3). As it turns out, in the case $2 \le p < \infty$, we can deduce $[M_f, P] \in C_p$ from (1.3) by an explicit decomposition $\sum_{v\in\mathbb{Z}^{2n}} Y_v$ of the commutator and by direct estimates of $\|Y_v\|_p$, which are given in Section 4. Thus, when $2 \le p < \infty$, (1.2) is sufficient for $H_f \in C_p$ and $H_{\bar{f}} \in C_p$. The opposite directions in the cases $1 \le p \le 2$ and $2 \le p < \infty$, which will be taken care of in Section 5, are both easy. Indeed the material contained in Section 5 is simply a specialization to the Segal-Bargmann space of what is well known for reproducing-kernel Hilbert spaces [10].

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Note added January 31, 2003. From the conversations mentioned above arose the following problem: *For $f \in L^\infty(\mathbb{C}^n, d\mu)$ and $1 \le p < \infty$, does the membership $H_f \in C_p$ imply $H_{\bar{f}} \in C_p$?* This problem was motivated by a theorem of Berger and Coburn [2] which asserts that, for $f \in L^\infty(\mathbb{C}^n, d\mu)$, H_f is compact if and only if $H_{\bar{f}}$ is compact. We were able to solve the problem for the special case where $p = 2$ *and* $n = 1$. When this paper was first completed in September 2000, it contained Theorem 1.3: *If f is a bounded measurable function on the complex plane $\mathbb{C}$, then $H_f \in C_2$ if and only if $H_{\bar{f}} \in C_2$.* Theorem 1.3 has since been extended to $f \in L^\infty(\mathbb{C}^n, d\mu)$ for all complex dimensions $n \ge 1$ by W. Bauer (*Hilbert-Schmidt Hankel operators on the Segal-Bargmann space*, preprint, April, 2002). In September 2002, the authors received the preprint *Hilbert-Schmidt Hankel operators on the Fock space* by K. Stroethoff, which also contained the same generalization of Theorem 1.3 to all complex dimensions $n \ge 1$. In view of these developments, and of the fact that our method only covered the case $n = 1$ and was rather ad hoc, the authors have decided not to include Theorem 1.3 in the present version of the paper. But we would like to mention that, for all values $p \ne 2$ and $n \ge 1$, the problem stated above remains open and appears to be rather challenging.

2. PRELIMINARIES

We begin with the lattice $\mathbb{Z}^{2n}$ in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. A subset $s = \{p_0, \dots, p_k\}$ of $\mathbb{Z}^{2n}$ with $k \ge 1$ is said to be a *discrete segment* in $\mathbb{Z}^{2n}$ if there exist a $j \in \{1, \dots, 2n\}$ and a $z \in \mathbb{Z}^{2n}$ such that

$$p_i = z + ie_j, \quad 0 \le i \le k,$$

where e_j is the vector in $\mathbb{Z}^{2n}$ whose j-th component is 1 and whose other components are 0. Furthermore, in this setting we will say that p_0 and p_k are the *end points* of s. Also, for this s, we define its *length* $|s|$ to be k, i.e., the distance between the end points p_0 and p_k. In other words, for a discrete segment s, we have $|s| = \text{card}(s) - 1$.

Let $v = (v_1, \ldots, v_{2n}) \in \mathbb{Z}^{2n}$ and suppose that v has at least one nonzero component. Then we can enumerate the integers $\{j \mid v_j \neq 0,\ 1 \leq j \leq 2n\}$ as $j_1, \ldots, j_m$ in ascending order. That is, $j_1 < \cdots < j_m$ in the event $m > 1$. We set $z_0(v) = (0, \ldots, 0)$, the zero vector in $\mathbb{Z}^{2n}$. We then inductively define $z_t(v) = z_{t-1}(v) + v_{j_t} e_{j_t}$ for every $t \in \{1, \ldots, m\}$. Note that $z_m(v) = v$. Let $s_t(v)$ be the discrete segment in $\mathbb{Z}^{2n}$ which has $z_{t-1}(v)$ and $z_t(v)$ as its end points, $t \in \{1, \ldots, m\}$. The union of the discrete segments $s_1(v), \ldots, s_m(v)$ will be denoted by $\Gamma(v)$. We will call $\Gamma(v)$ *the discrete path in* $\mathbb{Z}^{2n}$ *from* 0 *to* v. Furthermore, we define the *length* $|\Gamma(v)|$ of $\Gamma(v)$ to be $|s_1(v)| + \cdots + |s_m(v)|$. In other words, the length of the discrete path $\Gamma(v)$ is simply the sum of the lengths of the discrete segments which make up $\Gamma(v)$. This defines the discrete path $\Gamma(v)$ and its length $|\Gamma(v)|$ in the case $v \neq 0$.

In the case $v = 0$, we define the discrete path form 0 to 0 to be the singleton set $\Gamma(0) = \{0\}$. Naturally, the length $|\Gamma(0)|$ of $\Gamma(0)$ is defined to be 0.

We will consider $\mathbb{Z}^{2n}$ as a lattice in $\mathbb{C}^n$ in the obvious sense. That is, we will identify

$$(k_1, \ell_1, \ldots, k_n, \ell_n) \quad \text{with} \quad (k_1 + i\ell_1, \ldots, k_n + i\ell_n)$$

for any $k_1, \ell_1, \ldots, k_n, \ell_n \in \mathbb{Z}$.

We conclude this section with a few more conventions which will be used in the rest of the paper. The symbol S will denote the unit cube in $\mathbb{C}^n$. That is,

$$S = \{(x_1 + iy_1, \ldots, x_n + iy_n) \mid x_1, y_1, \ldots, x_n, y_n \in [0, 1)\}. \tag{2.1}$$

The symbol Q is also reserved to denote a particular cube in $\mathbb{C}^n$:

$$Q = \{(x_1 + iy_1, \ldots, x_n + iy_n) \mid x_1, y_1, \ldots, x_n, y_n \in [-1, 2)\}. \tag{2.2}$$

For any $f \in L^2_{\text{local}}(\mathbb{C}^n, dV)$, we will write

$$J(f) = \int_Q \int_Q |f(z) - f(w)|^2 \, dV(z) \, dV(w). \tag{2.3}$$

As we will see in the next two sections, this notation saves a lot of writing. For example, with this notation, condition (1.3) is now simply

$$\sum_{b \in \mathbb{Z}^{2n}} \{J(f \circ \tau_b)\}^{p/2} < \infty.$$

Finally, if E is a Borel set with $0 < V(E) < \infty$, we will denote the mean value of f on E by f_E. That is,

$$f_E = \frac{1}{V(E)} \int_E f \, dV. \tag{2.4}$$

In particular, $f_S = \int_S f\,dV$ since $V(S) = 1$. We emphasize that the measure which appears in (2.3) and (2.4) is dV, not $d\mu$.

3. NECESSITY IN THE CASE $1 \le p \le 2$

The purpose of this section is to establish that the simultaneous membership $H_f \in C_p$ and $H_{\bar f} \in C_p$ implies (1.2) in the case $1 \le p \le 2$.

Lemma 3.1. *Suppose that $1 \le p \le 2$. If $f \in \mathcal{T}(\mathbb{C}^n)$ is such that $H_f \in C_p$ and $H_{\bar f} \in C_p$, then $\sum_{v\in\mathbb{Z}^{2n}}\{J(f\circ\tau_v)\}^{p/2} < \infty$, where τ_v is the translation defined by* (1.1).

Proof. Given f as above, for each $v \in \mathbb{Z}^{2n}$ we define the operator

$$(K_v\psi)(z) = \chi_{Q+v}(z)\int_{Q+v}(\bar f(w)-\bar f(z))e^{\langle z,w\rangle}\psi(w)\,d\mu(w), \quad \psi \in L^2(\mathbb{C}^n, d\mu).$$

Let us first consider the case $1 < p \le 2$. In this case we set $q = p/(p-1)$. It is obvious that $K_v \in C_2$ for every v. Since $q \ge 2$, we have

$$\begin{aligned}(3.1)\quad \|K_v\|_q^2 \le \|K_v\|_2^2 &= \int_{Q+v}\int_{Q+v}|f(w)-f(z)|^2\,|e^{\langle z,w\rangle}|^2\,d\mu(w)\,d\mu(z)\\ &= \pi^{-2n}\int_{Q+v}\int_{Q+v}|f(w)-f(z)|^2e^{-|w-z|^2}\,dV(w)\,dV(z)\\ &= \pi^{-2n}\int_Q\int_Q|f(w+v)-f(z+v)|^2e^{-|w-z|^2}\,dV(w)\,dV(z)\\ &\le J(f\circ\tau_v).\end{aligned}$$

For each $v \in \mathbb{Z}^{2n}$, define $\alpha_v = \{J(f\circ\tau_v)\}^{(p-2)/2}$ in the case $J(f\circ\tau_v) > 0$ and define $\alpha_v = 0$ in the case $J(f\circ\tau_v) = 0$. Given any finite subset $F \subset \mathbb{Z}^{2n}$, we define $K_F = \sum_{v\in F}\alpha_vK_v$. We claim that

$$(3.2)\qquad \|K_F\|_q \le 3^{2n}\Big\{\sum_{v\in F}(J(f\circ\tau_v))^{p/2}\Big\}^{1/q}.$$

To prove (3.2), let us introduce the subset

$$\Lambda = \{(\lambda_1,\dots,\lambda_{2n}) \mid \lambda_j \in \{0,1,2\},\ 1 \le j \le 2n\}$$

of the additive group $\mathbb{Z}^{2n}$. It is clear that $\bigcup_{\lambda\in\Lambda}\{(3\mathbb{Z})^{2n}+\lambda\} = \mathbb{Z}^{2n}$ and that $\{(3\mathbb{Z})^{2n}+\lambda\}\cap\{(3\mathbb{Z})^{2n}+\lambda'\} = \emptyset$ for $\lambda \ne \lambda'$ in Λ. Thus we can write

$$(3.3)\qquad K_F = \sum_{\lambda\in\Lambda}K_{F,\lambda}, \quad \text{where } K_{F,\lambda} = \sum_{v\in F\cap\{(3\mathbb{Z})^{2n}+\lambda\}}\alpha_vK_v.$$

Note that each K_v maps the subspace $L^2(Q+v,d\mu)$ to itself. Because the length of each side of Q is 3, for any fixed $\lambda \in \Lambda$, the families of subspaces $\{L^2(Q+v,d\mu) \mid v \in (3\mathbb{Z})^{2n}+\lambda\}$ are pairwise orthogonal. Thus, recalling (3.1), for any $\lambda \in \Lambda$, we obtain

$$\begin{aligned}(3.4)\qquad \|K_{F,\lambda}\|_q &= \Big\{\sum_{v\in F\cap\{(3\mathbb{Z})^{2n}+\lambda\}} \alpha_v^q \|K_v\|_q^q\Big\}^{1/q} \\ &\le \Big\{\sum_{v\in F}\alpha_v^q (J(f\circ\tau_v))^{q/2}\Big\}^{1/q} = \Big\{\sum_{v\in F}(J(f\circ\tau_v))^{p/2}\Big\}^{1/q},\end{aligned}$$

where the second $=$ follows from the definition of α_v and from the fact that $(q(p-2)/2)+(q/2)=p/2$. Therefore (3.2) follows from (3.3), (3.4) and the fact that $\mathrm{card}(\Lambda)=3^{2n}$.

Now $[M_f,P]=[M_f,P]P+[M_f,P](1-P)=H_f-(H_{\bar f})^* \in C_p$ under our assumption. Since $[M_f,P]$ has $(f(z)-f(w))e^{\langle z,w\rangle}$ as its kernel function, for any $v\in\mathbb{Z}^{2n}$,

$$\begin{aligned}\mathrm{tr}([M_f,P]K_v) &= \int_{Q+v}\int_{Q+v}|f(w)-f(z)|^2\,|e^{\langle z,w\rangle}|^2\,d\mu(w)\,d\mu(z)\\ &= \pi^{-2n}\int_Q\int_Q|f(w+v)-f(z+v)|^2 e^{-|(w+v)-(z+v)|^2}\,dV(w)\,dV(z)\\ &\ge \pi^{-2n}e^{-18n}J(f\circ\tau_v),\end{aligned}$$

where the $\ge$ follows from the fact that $|w-z|\le 3\sqrt{2n}$ for any $z,w\in Q$. Hence

$$\begin{aligned}\|[M_f,P]\|_p\,\|K_F\|_q \ge \mathrm{tr}([M_f,P]K_F) &\ge \pi^{-20n}\sum_{v\in F}\alpha_v J(f\circ\tau_v)\\ &= \pi^{-20n}\sum_{v\in F}(J(f\circ\tau_v))^{p/2}.\end{aligned}$$

Combining this with (3.2), we now have

$$\sum_{v\in F}(J(f\circ\tau_v))^{p/2} \le \pi^{22pn}\|[M_f,P]\|_p^p.$$

Since this holds for any finite subset $F\subset\mathbb{Z}^{2n}$, we conclude that, in the case $1<p\le 2$, we have $\sum_{v\in\mathbb{Z}^{2n}}(J(f\circ\tau_v))^{p/2}<\infty$.

Let us now consider the case $p=1$. In this case $[M_f,P]\in C_1$. For any $1<t\le 2$, since $\|[M_f,P]\|_t\le\|[M_f,P]\|_1$, the above provides

$$\sum_{v\in F}(J(f\circ\tau_v))^{t/2}\le \pi^{22tn}\|[M_f,P]\|_1^t.$$

Taking the limit $t\downarrow 1$, we obtain the finiteness of $\sum_{v\in\mathbb{Z}^{2n}}(J(f\circ\tau_v))^{1/2}$ under the assumption $H_f\in C_1$ and $H_{\bar f}\in C_1$. □

Lemma 3.2. *For any* $f \in L^2_{\text{local}}(\mathbb{C}^n, dV)$ *and* $v \in \mathbb{Z}^{2n}$*, we have*

$$\int_S |f \circ \tau_v - f_S|^2 \, dV \le (1 + 2|\Gamma(v)|) \sum_{a \in \Gamma(v)} J(f \circ \tau_a), \tag{3.5}$$

where $\Gamma(v)$ *is the discrete path in* $\mathbb{Z}^{2n}$ *from* 0 *to* v *and* $|\Gamma(v)|$ *is the length of* $\Gamma(v)$ *as defined in Section 2.*

Proof. The case $v = 0$ is trivial. Suppose that $v \neq 0$. Then by Section 2 we can enumerate the points in $\Gamma(v)$ as $a_0, a_1, \dots, a_\ell$ in such a way that $\ell = |\Gamma(v)|$, $a_0 = 0$, $a_\ell = v$, and

$$\{S + a_{j-1}\} \cup \{S + a_j\} \subset Q + a_{j-1}, \quad 1 \le j \le \ell.$$

Obviously,

$$\begin{aligned} |(f \circ \tau_{a_j})_S - (f \circ \tau_{a_{j-1}})_S| \le {} & |(f \circ \tau_{a_j})_S - (f \circ \tau_{a_{j-1}})_Q| \\ & + |(f \circ \tau_{a_{j-1}})_Q - (f \circ \tau_{a_{j-1}})_S| \end{aligned} \tag{3.6}$$

for any $1 \le j \le \ell$. Since $V(S + a_j) = 1$ and $S + a_j \subset Q + a_{j-1}$, we have

$$\begin{aligned} |(f \circ \tau_{a_j})_S - (f \circ \tau_{a_{j-1}})_Q|^2 &= \left| \int_{S + a_j} \{f - f_{Q + a_{j-1}}\} \, dV \right|^2 \\ &\le \int_{S + a_j} |f - f_{Q + a_{j-1}}|^2 \, dV \\ &\le \int_{Q + a_{j-1}} |f - f_{Q + a_{j-1}}|^2 \, dV \\ &= \int_Q |f \circ \tau_{a_{j-1}} - (f \circ \tau_{a_{j-1}})_Q|^2 \, dV \\ &= \frac{J(f \circ \tau_{a_{j-1}})}{2V(Q)}. \end{aligned}$$

Similarly, $|(f \circ \tau_{a_{j-1}})_Q - (f \circ \tau_{a_{j-1}})_S|^2 \le J(f \circ \tau_{a_{j-1}})/2V(Q)$. Thus, by (3.6),

$$|(f \circ \tau_{a_j})_S - (f \circ \tau_{a_{j-1}})_S|^2 \le J(f \circ \tau_{a_{j-1}}), \quad 1 \le j \le \ell. \tag{3.7}$$

Now

$$\int_S |f \circ \tau_v - f_S|^2 \, dV \le 2 \int_S \{|f \circ \tau_v - (f \circ \tau_v)_S|^2 + |(f \circ \tau_v)_S - f_S|^2\} \, dV. \tag{3.8}$$

We have

$$2\int_S |f\circ\tau_v - (f\circ\tau_v)_S|^2\,dV = \int_S\int_S |f(w+v)-f(z+v)|^2\,dV(w)\,dV(z)$$
$$\le J(f\circ\tau_v) = J(f\circ\tau_{a_\ell})$$

and, by (3.7),

$$\begin{aligned}(3.9)\qquad |(f\circ\tau_v)_S - f_S|^2 &= |(f\circ\tau_{a_\ell})_S - (f\circ\tau_{a_0})_S|^2\\ &\le \Big\{\sum_{j=1}^{\ell} |(f\circ\tau_{a_j})_S - (f\circ\tau_{a_{j-1}})_S|\Big\}^2\\ &\le \ell\sum_{j=1}^{\ell} |(f\circ\tau_{a_j})_S - (f\circ\tau_{a_{j-1}})_S|^2\\ &\le \ell\sum_{j=1}^{\ell} J(f\circ\tau_{a_{j-1}}).\end{aligned}$$

Thus, upon noting that $\ell = |\Gamma(v)|$, (3.5) follows from (3.8) and (3.9). ◻

Lemma 3.3. *There exists a constant* $0 < C_{3.3} < \infty$ *such that*

$$(3.10)\qquad \sup_{z\in S}\int\left|f\circ\tau_z - \int f\circ\tau_z\,d\mu\right|^2 d\mu \le C_{3.3}\sum_{v\in\mathbb{Z}^{2n}}\sum_{a\in\Gamma(v)} e^{-|v|^2/3}J(f\circ\tau_a)$$

for any $f\in\mathcal{T}(\mathbb{C}^n)$*, where* $\Gamma(v)$ *is the discrete path in* $\mathbb{Z}^{2n}$ *from* 0 *to* v *as in Lemma 3.2.*

Proof. For any $z\in S$, we have

$$\begin{aligned}(3.11)\qquad &\int\left|f\circ\tau_z - \int f\circ\tau_z\,d\mu\right|^2 d\mu\\ &\le \int |f\circ\tau_z - f_S|^2\,d\mu\\ &= \sum_{v\in\mathbb{Z}^{2n}}\frac{1}{\pi^n}\int_{S+v}|f(w)-f_S|^2 e^{-|w-z|^2}\,dV(w)\\ &= \sum_{v\in\mathbb{Z}^{2n}}\frac{1}{\pi^n}\int_S |(f\circ\tau_v)(w)-f_S|^2 e^{-|(w-z)+v|^2}\,dV(w)\\ &\le \sum_{v\in\mathbb{Z}^{2n}} d(v)\int_S |f\circ\tau_v - f_S|^2\,dV,\end{aligned}$$

where $d(v) = \exp\{-\inf_{w,\zeta\in S} |(w-\zeta)+v|^2\}$. Since $|(w-\zeta)+v|^2 \ge |v|^2 + |w-\zeta|^2 - 2|w-\zeta|\,|v| \ge (|v|^2/2) - |w-\zeta|^2$, there is a $B = B(n) > 0$ such that $d(v) \le Be^{-|v|^2/2}$. Obviously, $|v|$ dominates the length of every segment in $\Gamma(v)$. This means that $|\Gamma(v)| \le 2n|v|$. Therefore

$$(1+2|\Gamma(v)|)d(v) \le B(1+4n|v|)e^{-|v|^2/2} \le C_{3.3}e^{-|v|^2/3},$$

where $C_{3.3}$ depends only on n. Applying (3.5) and this inequality in (3.11), (3.10) follows. □

The main result of the section is the following.

Lemma 3.4. *Suppose that* $1 \le p \le 2$. *If* $f \in \mathcal{T}(\mathbb{C}^n)$ *is such that*

$$\sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2} < \infty,$$

then $\int \{\mathrm{SD}(f\circ\tau_\zeta)\}^p\, dV(\zeta) < \infty$.

Proof. Since $\bigcup\{S+u \mid u\in\mathbb{Z}^{2n}\} = \mathbb{C}^n$ and $V(S+u) = 1$, it suffices to prove

$$\sum_{u\in\mathbb{Z}^{2n}} \sup_{\zeta\in S+u} \{\mathrm{SD}(f\circ\tau_\zeta)\}^p = \sum_{u\in\mathbb{Z}^{2n}} \sup_{z\in S}\{\mathrm{SD}(f\circ\tau_u\circ\tau_z)\}^p < \infty \tag{3.12}$$

under the assumptions of the lemma. Since $p/2 \le 1$, $(\sum t_\nu)^{p/2} \le \sum t_\nu^{p/2}$ if every t_ν is positive. Applying this elementary inequality to the sum in (3.10), we obtain

$$\sup_{z\in S}\{\mathrm{SD}(f\circ\tau_u\circ\tau_z)\}^p \le C_{3.3}^{p/2} \sum_{v\in\mathbb{Z}^{2n}} \sum_{a\in\Gamma(v)} e^{-p|v|^2/6}\{J(f\circ\tau_u\circ\tau_a)\}^{p/2},$$

where $\Gamma(v)$ is the discrete path from 0 to v. Since $\tau_u\circ\tau_a = \tau_{u+a}$, we have

$$\begin{aligned}
&\sum_{u\in\mathbb{Z}^{2n}} \sup_{z\in S}\{\mathrm{SD}(f\circ\tau_u\circ\tau_z)\}^p \\
&\qquad \le C_{3.3}^{p/2} \sum_{u\in\mathbb{Z}^{2n}} \sum_{v\in\mathbb{Z}^{2n}} \sum_{a\in\Gamma(v)} e^{-p|v|^2/6}\{J(f\circ\tau_{u+a})\}^{p/2} \\
&\qquad = C_{3.3}^{p/2} \sum_{v\in\mathbb{Z}^{2n}} e^{-p|v|^2/6} \sum_{a\in\Gamma(v)} \sum_{u\in\mathbb{Z}^{2n}} \{J(f\circ\tau_{u+a})\}^{p/2} \\
&\qquad = C_{3.3}^{p/2} \sum_{v\in\mathbb{Z}^{2n}} e^{-p|v|^2/6}\,\mathrm{card}(\Gamma(v)) \sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2}.
\end{aligned}$$

Since $\mathrm{card}(\Gamma(v)) = 1 + |\Gamma(v)| \le 1 + 2n|v|$, (3.12) follows. □

4. Sufficiency in the Case $2 \le p < \infty$

In this section we prove that (1.2) implies $H_f \in C_p$ and $H_{\bar f} \in C_p$ in the case $2 \le p < \infty$.

Lemma 4.1. *Let* $1 \le p < \infty$ *and let* $f \in \mathcal{T}(\mathbb{C}^n)$*. If* $\int \{\mathrm{SD}(f \circ \tau_\zeta)\}^p \, dV(\zeta) < \infty$*, then* $\sum_{v \in \mathbb{Z}^{2n}} \{J(f \circ \tau_v)\}^{p/2} < \infty$*.*

Proof. We first show that there is a $c > 0$ such that

$$\{\mathrm{SD}(f \circ \tau_\zeta)\}^2 \ge c J(f \circ \tau_v) \quad \text{for any } f \in \mathcal{T}(\mathbb{C}^n),\ v \in \mathbb{Z}^{2n},\ \zeta \in S + v. \tag{4.1}$$

For this we recall the well-known fact that $\int |\varphi - \int \varphi \, d\nu|^2 \, d\nu = \inf_{\alpha \in \mathbb{C}} \int |\varphi - \alpha|^2 \, d\nu$ if $d\nu$ is a probability measure and $\varphi \in L^2(d\nu)$. By the definition (2.4), we have

$$\begin{aligned}
J(f \circ \tau_v) &= \int_Q \int_Q |f(z+v) - f(w+v)|^2 \, dV(z) \, dV(w) \\
&= 2\left(V(Q) \int_{Q+v} |f|^2 \, dV - \left| \int_{Q+v} f \, dV \right|^2 \right) \\
&= 2(V(Q))^2 \frac{1}{V(Q)} \int_{Q+v} |f - f_{Q+v}|^2 \, dV \\
&\le 2(V(Q))^2 \frac{1}{V(Q)} \int_{Q+v} \left| f - \int f \circ \tau_\zeta \, d\mu \right|^2 dV \\
&= 2V(Q) \int_{Q+v-\zeta} \left| f \circ \tau_\zeta - \int f \circ \tau_\zeta \, d\mu \right|^2 dV.
\end{aligned}$$

If $\zeta \in S + v$, then $Q + v - \zeta \subset Q - S = \{x - y \mid x \in Q,\ y \in S\}$. Hence

$$\frac{J(f \circ \tau_v)}{2V(Q)} \le \int_{Q-S} \left| f \circ \tau_\zeta - \int f \circ \tau_\zeta \, d\mu \right|^2 dV, \quad \text{whenever } \zeta \in S + v.$$

Obviously there is a $\delta = \delta(n) > 0$ such that $e^{-|w|^2} \ge \delta$ for every $w \in Q - S$. Since $d\mu(w) = \pi^{-n} e^{-|w|^2} \, dV(w)$, we see that $c = \delta / 2\pi^n V(Q)$ will do for (4.1). But from (4.1) it follows immediately that

$$\begin{aligned}
\int \{\mathrm{SD}(f \circ \tau_\zeta)\}^p \, dV(\zeta) &= \sum_{v \in \mathbb{Z}^{2n}} \int_{S+v} \{\mathrm{SD}(f \circ \tau_\zeta)\}^p \, dV(\zeta) \\
&\ge c^{p/2} \sum_{v \in \mathbb{Z}^{2n}} \{J(f \circ \tau_v)\}^{p/2}.
\end{aligned}$$

□

Lemma 4.2. *For each* $2 \le p < \infty$, *there exists a constant* $0 < C_{4.2} < \infty$ *such that*

$$\|[M_f, P]\|_p \le C_{4.2}\Big(\sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2}\Big)^{1/p} \quad \text{for all } f \in L^2_{\text{local}}(\mathbb{C}^n, dV).$$

Proof. If $f \in L^2_{\text{local}}(\mathbb{C}^n, dV)$, than $M_{\chi_E}[M_f, P]M_{\chi_E} \in C_2$ whenever E is a bounded Borel set in $\mathbb{C}^n$. Therefore it suffices to produce a constant $0 < C_{4.2} < \infty$, which depends only on n and p, such that

$$\|M_{\chi_E}[M_f, P]M_{\chi_E}\|_p \le C_{4.2}\Big(\sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2}\Big)^{1/p} \tag{4.2}$$

for any bounded Borel set $E \subset \mathbb{C}^n$. Given such an E, there is a finite subset $F = F(E) \subset \mathbb{Z}^{2n}$ such that $\bigcup\{S + v \mid v \in F\} \supset E$. Since $\|XYX\|_p \le \|X\|\,\|Y\|_p\,\|X\|$ and since $M_{\chi_E}M_{\chi_{\bigcup\{S+v|v\in F\}}} = M_{\chi_E}$, it suffices to estimate the p-norm of

$$\begin{aligned} Y &= \sum_{u,u'\in F} M_{\chi_{S+u}}[M_f, P]M_{\chi_{S+u'}} = \sum_{v\in\mathbb{Z}^{2n}} Y_v, \quad \text{where} \\ Y_v &= \sum_{u\in\mathbb{Z}^{2n}} \chi_{F\times F}(u, u+v)M_{\chi_{S+u}}[M_f, P]M_{\chi_{S+u+v}}. \end{aligned}$$

Note that, for each given v, the families $\{L^2(S + u + v, d\mu) \mid u \in \mathbb{Z}^{2n}\}$ of subspaces are pairwise orthogonal. Since $\|T\|_p \le \|T\|_2$ when $p \ge 2$, for each $v \in \mathbb{Z}^{2n}$, we have

$$\begin{aligned} ||Y_v||_p^p &= \sum_{u\in\mathbb{Z}^{2n}} \chi_{F\times F}(u, u+v)||M_{\chi_{S+u}}[M_f, P]M_{\chi_{S+u+v}}||_p^p \\ &\le \sum_{u\in\mathbb{Z}^{2n}} ||M_{\chi_{S+u}}[M_f, P]M_{\chi_{S+u+v}}||_2^p. \end{aligned} \tag{4.3}$$

Since $[M_f, P]$ has $(f(z) - f(w))e^{\langle z,w\rangle}$ as its kernel function, we have

$$\begin{aligned} &||M_{\chi_{S+u}}[M_f, P]M_{\chi_{S+u+v}}||_2^2 \\ &\quad = \int_{S+u}\int_{S+u+v} |(f(z) - f(w))e^{\langle z,w\rangle}|^2\, d\mu(w)\, d\mu(z) \\ &\quad = \pi^{-2n}\int_{S+u}\int_{S+u+v} |f(z) - f(w)|^2 e^{-|w-z|^2}\, dV(w)\, dV(z) \\ &\quad \le d(v)\int_S\int_{S+v} |(f\circ\tau_u)(z) - (f\circ\tau_u)(w)|^2\, dV(w)\, dV(z), \end{aligned} \tag{4.4}$$

where $d(v) = \exp(-\inf_{w,z\in S}|(w - z) + v|^2) \le Be^{-|v|^2/2}$ as in the proof of Lemma 3.3.

Because $V(S)=1$, for any $g \in L^2_{\text{local}}(\mathbb{C}^n, dV)$,

$$\begin{aligned}\int_S \left\{ \int_{S+v} |g(z)-g(w)|^2\, dV(w) \right\} dV(z) &\le \int_S \int_S 2(|g(z)-g_S|^2 + |g_S - (g\circ\tau_v)(w)|^2)\, dV(w)\, dV(z) \\ &= 2\int_S |g-g_S|^2\, dV + 2\int_S |g\circ\tau_v - g_S|^2\, dV\end{aligned}$$

(also see (2.4)). Clearly, $\int_S |g-g_S|^2\, dV \le \int_S |g-g_Q|^2\, dV \le \frac{1}{2}J(g)$ by (2.3). Thus, applying Lemma 3.2 to $\int_S |g\circ\tau_v - g_S|^2\, dV$, we obtain

$$\begin{aligned}\int_S \int_{S+v} |g(z)-g(w)|^2\, dV(w)\, dV(z) &\le J(g) + 4|\Gamma(v)| \sum_{a\in\Gamma(v)} J(g\circ\tau_a) \\ &\le (1+4|\Gamma(v)|) \sum_{a\in\Gamma(v)} J(g\circ\tau_a),\end{aligned}$$

where $\Gamma(v)$ is the discrete path in $\mathbb{Z}^{2n}$ from 0 to v. Letting $g = f\circ\tau_u$ in the above and recalling (4.4), we now have

$$||M_{\chi_{S+u}}[M_f,P]M_{\chi_{S+u+v}}||_2^2 \le Be^{-|v|^2/2}(1+4|\Gamma(v)|) \sum_{a\in\Gamma(v)} J(f\circ\tau_u\circ\tau_a).$$

Since $p/2 \ge 1$, we can apply Hölder's inequality in the above to obtain

$$||M_{\chi_{S+u}}[M_f,P]M_{\chi_{S+u+v}}||_2^p \le h(v) \sum_{a\in\Gamma(v)} \{J(f\circ\tau_u\circ\tau_a)\}^{p/2},$$

where $h(v) = (Be^{-|v|^2/2})^{p/2}(1+8n|v|)^{(p/2)+\{(p-2)/2\}}$. (Recall that $|\Gamma(v)| \le 2n|v|$.) Thus, combining this with (4.3), we find that

$$\begin{aligned}||Y_v||_p^p &\le \sum_{u\in\mathbb{Z}^{2n}} h(v) \sum_{a\in\Gamma(v)} \{J(f\circ\tau_{u+a})\}^{p/2} \\ &= h(v) \sum_{a\in\Gamma(v)} \sum_{u\in\mathbb{Z}^{2n}} \{J(f\circ\tau_{u+a})\}^{p/2} \\ &= h(v)\,\text{card}(\Gamma(v)) \sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2} \\ &\le h(v)(1+2n|v|) \sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2}.\end{aligned}$$

Consequently

$$\|Y\|_p \le \sum_{v\in\mathbb{Z}^{2n}} \|Y_v\|_p \le \sum_{v\in\mathbb{Z}^{2n}} \{h(v)(1+2n|v|)\}^{1/p} \Big(\sum_{b\in\mathbb{Z}^{2n}} \{J(f\circ\tau_b)\}^{p/2} \Big)^{1/p}.$$

By the definition of $h(v)$, the constant $C_{4.2} = \sum_{v\in\mathbb{Z}^{2n}}\{h(v)(1+2n|v|)\}^{1/p}$ is obviously finite. With this $C_{4.2}$, (4.2) holds for any bounded Borel set $E \subset \mathbb{C}^n$. □

5. THE EASY DIRECTIONS

The substantive work for the proof of Theorem 1.1 has already been done in the previous sections. The rest of the proof follows well-established routines for reproducing-kernel Hilbert spaces, which we carry out below in the interest of completeness. We start with an adaptation of a well-known result. See, e.g., [10, Proposition 7.3.5].

Lemma 5.1. *For any $g \in L^2(\mathbb{C}^n, d\mu)$, we have*

$$\|(1-P)g\|^2 \le \{\mathrm{SD}(g)\}^2 \le \|(1-P)g\|^2 + \|(1-P)\bar{g}\|^2.$$

Proof. Since $\{\mathrm{SD}(g)\}^2 = \|g\|^2 - |\langle g,1\rangle|^2$ and $|\langle g,1\rangle|^2 \le \|Pg\|^2$, we have

$$\{\mathrm{SD}(g)\}^2 \ge \|g\|^2 - \|Pg\|^2 = \|(1-P)g\|^2.$$

To prove the other inequality, let us write $\{\mathrm{SD}(g)\}^2 = \|(1-P)g\|^2 + \{\|Pg\|^2 - |\langle g,1\rangle|^2\}$. It suffices to show that

$$\|Pg\|^2 - |\langle g,1\rangle|^2 \le \|(1-P)\bar{g}\|^2. \tag{5.1}$$

Let $\bar{P}$ (respectively P_0) denote the orthogonal projection from $L^2(\mathbb{C}^n, d\mu)$ onto $\bar{H}^2(\mathbb{C}^n, d\mu) = \{\bar{\varphi} \mid \varphi \in H^2(\mathbb{C}^n, d\mu)\}$ (respectively $\mathbb{C}$, the subspace of constant functions in $L^2(\mathbb{C}^n, d\mu)$). It is obvious that $\bar{P}\bar{g}$ is the complex conjugate of Pg. That is, $\|Pg\|^2 = \|\bar{P}\bar{g}\|^2$. Also, $|\langle g,1\rangle|^2 = |\langle \bar{g},1\rangle|^2 = \|P_0\bar{g}\|^2$. Therefore $\|Pg\|^2 - |\langle g,1\rangle|^2 = \|(\bar{P}-P_0)\bar{g}\|^2$. Thus (5.1) follows from the obvious fact that $\bar{P} - P_0 \le 1 - P$. □

Recall that $K_z(w) = e^{\langle w,z\rangle}$ is the reproducing kernel for $H^2(\mathbb{C}^n, d\mu)$, i.e.,

$$(P\varphi)(z) = \langle \varphi, K_z\rangle, \quad \varphi \in L^2(\mathbb{C}^n, d\mu).$$

Let k_z denote the normalized reproducing kernel $K_z/\|K_z\|$. Since $\|K_z\| = e^{|z|^2/2}$, we have

$$k_z(w) = e^{\langle w,z\rangle}e^{-|z|^2/2}.$$

Since $K_z(z)\,d\mu(z) = \pi^{-n}\,dV(z)$, Lemma 5.2 below is well-known. In fact its proof is simply an adaptation of the proof of, e.g., [10, Proposition 6.3.2] to the Segal-Bargmann space in the obvious way.

Lemma 5.2. *If A is a trace class operator or a positive operator on $H^2(\mathbb{C}^n, d\mu)$, then*

$$\mathrm{tr}(A) = \frac{1}{\pi^n}\int \langle Ak_z, k_z\rangle\, dV(z).$$

It is straightforward to verify that, for any given $z \in \mathbb{C}^n$, the formula

$$(U_z\varphi)(w) = (\varphi \circ \tau_{-z})(w)k_z(w) = \varphi(w - z)k_z(w)$$

defines a unitary operator on $L^2(\mathbb{C}^n, d\mu)$. It is also easy to verify that $[U_z, P] = 0$ for every $z \in \mathbb{C}^n$. In particular this means

$$\begin{aligned} \|(1-P)(f\circ\tau_z)\| &= \|U_z(1-P)(f\circ\tau_z)\| \\ &= \|(1-P)U_z(f\circ\tau_z)\| = \|(1-P)fk_z\| = \|H_f k_z\| \end{aligned} \tag{5.2}$$

for all $f \in \mathcal{T}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$.

Proof of Theorem 1.1. Let us first consider the case $1 \le p \le 2$. By Lemmas 3.1 and 3.4, the simultaneous membership $H_f \in C_p$ and $H_{\bar f} \in C_p$ implies (1.2). To prove the converse in this case, let us assume that f is a function in $\mathcal{T}(\mathbb{C}^n)$ for which (1.2) holds. To prove that $H_f \in C_p$, we first need to establish the existence of $(H_f^* H_f)^{p/2}$ as a positive operator. By Lemma 4.1, (1.2) implies that

$$\sum_{v\in\mathbb{Z}^{2n}} \{J(f\circ\tau_v)\}^{p/2} < \infty.$$

Since $p/2 \le 1$, this in turn implies $\sum_{v\in\mathbb{Z}^{2n}} J(f\circ\tau_v) < \infty$. Thus, by Lemma 4.2, $[M_f, P] \in C_2$. This in particular tells us that H_f is bounded and, therefore, the positive operator $(H_f^* H_f)^{p/2}$ is well defined. Using the condition $p/2 \le 1$ once more, we have $\langle (H_f^* H_f)^{p/2} k_z, k_z\rangle \le \langle H_f^* H_f k_z, k_z\rangle^{p/2}$. But $\langle H_f^* H_f k_z, k_z\rangle^{p/2} = \|H_f k_z\|^p = \|U_z(1-P)(f\circ\tau_z)\|^p \le \{\mathrm{SD}(f\circ\tau_z)\}^p$, where we used (5.2) and Lemma 5.1 for the last two steps. Therefore, by Lemma 5.2,

$$\begin{aligned} ||H_f||_p^p = \mathrm{tr}((H_f^* H_f)^{p/2}) &= \frac{1}{\pi^n}\int \langle (H_f^* H_f)^{p/2} k_z, k_z\rangle\, dV(z) \\ &\le \frac{1}{\pi^n}\int \{\mathrm{SD}(f\circ\tau_z)\}^p\, dV(z) < \infty \end{aligned}$$

whenever (1.2) holds. Since $\mathrm{SD}(\bar f\circ\tau_z) = \mathrm{SD}(f\circ\tau_z)$, we also have $H_{\bar f} \in C_p$ if (1.2) holds. This completes the proof in the case $1 \le p \le 2$.

Let us now assume $2 \le p < \infty$. Then it follows from Lemmas 4.1 and 4.2 that $H_f \in C_p$ and $H_{\bar f} \in C_p$ whenever (1.2) holds. Conversely, let us suppose that $H_f \in C_p$ and $H_{\bar f} \in C_p$. Since $p/2$ is now at least 1, we have $\langle (H_f^* H_f)^{p/2} k_z, k_z\rangle \ge \langle H_f^* H_f k_z, k_z\rangle^{p/2} = \|H_f k_z\|^p$ and the same holds with $\bar f$

in place of f. Therefore, by Lemma 5.2,

$$\begin{aligned}\int(\|H_f k_z\|^p + \|H_{\bar f} k_z\|^p)\,dV(z) &\le \int\langle\{(H_f^* H_f)^{p/2} + (H_{\bar f}^* H_{\bar f})^{p/2}\}k_z, k_z\rangle\,dV(z)\\ &= \pi^n(\|H_f\|_p^p + \|H_{\bar f}\|_p^p).\end{aligned}$$

Since $p/2 \ge 1$, Hölder's inequality yields

$$(\|H_f k_z\|^2 + \|H_{\bar f} k_z\|^2)^{p/2} \le 2^{(p-2)/2}(\|H_f k_z\|^p + \|H_{\bar f} k_z\|^p).$$

Now, by (5.2) and Lemma 5.1,

$$\begin{aligned}(\|H_f k_z\|^2 + \|H_{\bar f} k_z\|^2)^{p/2} &= (\|(1-P)(f\circ\tau_z)\|^2 + \|(1-P)(\bar f\circ\tau_z)\|^2)^{p/2}\\ &\ge \{\mathrm{SD}(f\circ\tau_z)\}^p.\end{aligned}$$

Thus the condition $H_f \in C_p$ and $H_{\bar f} \in C_p$ implies (1.2). ◻

6. A COMPARISON WITH THE BERGMAN SPACE

Let B_n denote the unit ball in $\mathbb{C}^n$. Let dv denote the volume measure on B_n normalized in such a way that $v(B_n) = 1$. Recall that the Bergman space $L_a^2(B_n, dv)$ is the collection of analytic functions on B_n which belong to $L^2(B_n, dv)$. Let $P^{\mathrm{Berg}} : L^2(B_n, dv) \to L_a^2(B_n, dv)$ be the orthogonal projection. Then P^{Berg} is given by the formula

$$(P^{\mathrm{Berg}}\varphi)(z) = \int_{B_n}\frac{\varphi(w)\,dv(w)}{(1-\langle z,w\rangle)^{n+1}}, \quad \varphi \in L^2(B_n, dv).$$

The Bergman space Hankel operator H_f^{Berg} is defined by the formula

$$H_f^{\mathrm{Berg}}\psi = (1-P^{\mathrm{Berg}})(f\psi), \quad \psi \in L_a^2(B_n, dv).$$

Recall that, for any $\varphi \in L^1(B_n, dv)$, its *Berezin transform* $\tilde\varphi$ is defined by the formula

$$\tilde\varphi(z) = \int_{B_n}\varphi(w)\frac{(1-|z|^2)^{n+1}}{|1-\langle z,w\rangle|^{2n+2}}\,dv(w), \quad z \in B_n.$$

For any $f \in L^2(B_n, dv)$, its *mean oscillation* $\mathrm{MO}(f)$ is defined by the formula

$$\begin{aligned}\mathrm{MO}(f)(z) &= \{\widetilde{|f|^2}(z) - |\tilde f(z)|^2\}^{1/2}\\ &= \left\{\int_{B_n}|f(w)-\tilde f(z)|^2\frac{(1-|z|^2)^{n+1}}{|1-\langle z,w\rangle|^{2n+2}}\,dv(w)\right\}^{1/2}.\end{aligned}$$

What originally led to Theorem 1.1 was the following result due to Zhu.

Theorem 6.1 ([11]). *Suppose that $2 \le p < \infty$ and f is in $L^2(B_n, dv)$. Then we have the simultaneous membership $H_f^{\mathrm{Berg}} \in C_p$ and $H_{\bar f}^{\mathrm{Berg}} \in C_p$ if and only if*

$$\int_{B_n} \{\mathrm{MO}(f)(z)\}^p \frac{dv(z)}{(1-|z|^2)^{n+1}} < \infty. \tag{6.1}$$

Note that $(1-|z|^2)^{-(n+1)}\, dv(z)$ is the Möbius invariant measure on B_n. On $\mathbb{C}^n$, what corresponds to the Möbius group is the group $\{\tau_\zeta \mid \zeta \in \mathbb{C}^n\}$. Therefore (1.2) is just the Segal-Bargmann space version of (6.1) and, conversely, (6.1) is the Bergman space version of (1.2). Thus, given Theorem 6.1, it does not take too much imagination to conceive Theorem 1.1. But, given that Theorem 6.1 was published in 1991, it is somewhat surprising that Theorem 1.1 has not appeared in the literature until the present paper. There are, however, some not-too-subtle differences between Theorem 6.1 and Theorem 1.1, which might help explain the chronological gap between the two theorems.

First of all, a quick glance of our paper and [11] reveals that the techniques used in the two papers are completely different. The techniques we use in this paper take full advantage of the fact that, as a reproducing-kernel space, the structure of $H^2(\mathbb{C}^n, d\mu)$ is completely flat. This is why the proof of Lemma 4.2 is much shorter and much more straightforward than its counter part on the Bergman space.

But the most obvious difference between Theorem 6.1 and Theorem 1.1 is the range of validity with respect to p: whereas Theorem 1.1 holds true for all $1 \le p < \infty$, in its generality Theorem 6.1 was proved only for $2 \le p < \infty$. Only recently was Theorem 6.1 extended to the case $2n/(n+1) < p < 2$ [8, 9]. It is easy to see that, for any $n \ge 1$, if $1 \le p \le 2n/(n+1)$, then the condition (6.1) is sufficient but *not* necessary for the simultaneous membership $H_f^{\mathrm{Berg}} \in C_p$ and $H_{\bar f}^{\mathrm{Berg}} \in C_p$, both for trivial reasons. Indeed, we have the elementary estimate

$$\begin{aligned}\mathrm{MO}(f)(z) &\ge \left\{\int_{B_n} |f - \tilde f(z)|^2 \frac{(1-|z|^2)^{n+1}}{(1+|z|^2)^{2n+2}}\, dv\right\}^{1/2} \\ &\ge (1-|z|^2)^{(n+1)/2} \cdot 2^{-n-1} \cdot \inf_{\alpha\in\mathbb{C}} \left\{\int_{B_n} |f-\alpha|^2\, dv\right\}^{1/2}.\end{aligned}$$

Thus there are only two possible ways for (6.1) to hold: Either

$$\frac{p(n+1)}{2} - (n+1) > -1,$$

which translates to $p > 2n/(n+1)$, or, failing that, we must have

$$\inf_{\alpha\in\mathbb{C}} \int_{B_n} |f-\alpha|^2\, dv = 0,$$

which forces f to be a constant on B_n. But there are non-constant functions f on B_n for which both H_f^{Berg} and $H_{\bar{f}}^{\text{Berg}}$ belong to the trace class. For example, if f is bounded and vanishes outside $\{z \in B_n \colon |z| < r\}$ for some $0 < r < 1$, then $H_f^{\text{Berg}} \in C_1$ and $H_{\bar{f}}^{\text{Berg}} \in C_1$. (This is because for any $0 < r < 1$, there is an $F \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ such that $(1 - \langle z, w\rangle)^{-n-1}\chi_{[0,r)}(|z|) = F(z,w)\chi_{[0,r)}(|z|)$ when $z, w \in B_n$.) Therefore, when $1 \le p \le 2n/(n+1)$, (6.1) is sufficient but not necessary for $H_f^{\text{Berg}} \in C_p$ and $H_{\bar{f}}^{\text{Berg}} \in C_p$. This comparison between Theorem 1.1 and Theorem 6.1 seems to suggest that the fundamental difference between the domains $\mathbb{C}^n$ and B_n somehow shows up at the level of operator theory.

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