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Invertible Toeplitz Products

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We will discuss invertibility of Toeplitz products $T_f T_{\bar{g}}$, for analytic f and g, on the Bergman space and the Hardy space. We will furthermore describe when these Toeplitz products are Fredholm. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Let P^+ denote the Hardy projection from $L^2(\partial\mathbb{D})$ onto the Hardy space H^2, and let $h \in L^2(\partial\mathbb{D})$, define the Toeplitz operator T_h on H^2 by

$$T_h p = P^+(hp)$$

for polynomials p. It is well known that T_h is bounded if and only if h is bounded on the unit circle $\partial\mathbb{D}$. However, Sarason [12, 13] found examples of f and g in H^2 such that the product $T_f T_{\bar{g}}$ is actually a bounded operator on H^2, though neither T_f nor T_g is bounded. Sarason [14] also conjectured that a necessary and sufficient condition for this product to be bounded is

$$\sup_{w \in \mathbb{D}} \widehat{|f|^2}(w)\widehat{|g|^2}(w) < \infty, \tag{1.1}$$

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where $\hat{u}(w)$ denotes the Poisson extension of u over the $\mathbb{D}$:

$$\hat{u}(w) = \int_{\partial\mathbb{D}} u(\zeta) \frac{1 - |w|^2}{|1 - \bar{w}\zeta|^2}\, dm(\zeta).$$

Treil [14] proved that (1.1) is a necessary condition. Zheng [20] showed that (1.1) is sufficient if the exponent 2 on the functions $|f|$ and $|g|$ is replaced by $2 + \varepsilon$ for any $\varepsilon > 0$. A stronger result, utilizing the scale of Orlicz spaces, was found by Treil, Volberg, and Zheng [17]. Another stronger result, using a rigged non-tangential maximal function, was obtained by Xia [19]. However, Nazarov [7] has constructed a counter-example to Sarason's conjecture.

If we consider instead the question of whether the product $T_f T_{\bar{g}}$ is bounded and invertible, then (1.1) provides the correct condition. More precisely, Cruz-Uribe [2] showed that if f and g are outer functions, a necessary and sufficient condition for $T_f T_{\bar{g}}$ to be bounded and invertible is that $(fg)^{-1}$ is bounded and (1.1) holds. A similar, though different, characterization of bounded invertible Toeplitz products on H^2 with outer symbols was obtained by Zheng [20]. At the heart of Cruz-Uribe's [2] proof is a characterization of invertible Toeplitz operators due to Devinatz and Widom, which in turn is closely related to the Helson–Szegö theorem, that characterizes the weights w such that the conjugation operator (or Hilbert transform) is bounded on $L^2(\partial\mathbb{D}, w\,dm)$. See Sarason's book [11] for more on these results. On the other hand, the proof in [20] is based on a distribution function inequality.

The Helson–Szegö theorem relies heavily on complex analytic methods. There is another characterization of the boundedness of the conjugation operator, derived using real-variable techniques, due to Hunt, Muckenhoupt, and Wheeden [5]; this result has led to an extensive theory of weighted norm inequalities. For a good overview with extensive references, see [3, 4, 6]. For new approaches to the theory of weighted norm inequalities, see [8, 9, 10, 18].

In this article, we will give a complete characterization of the bounded invertible Toeplitz products $T_f T_{\bar{g}}$, for analytic f and g, not only on the Hardy space but also on the Bergman space. We will furthermore describe the Fredholm Toeplitz products $T_f T_{\bar{g}}$ on the Hardy or Bergman space, for analytic f and g.

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The *Bergman space* L^2_a is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$.

The orthogonal projection P of $L^2(\mathbb{D}, dA)$ onto L^2_a is easily seen to be given by the formula

$$Pu(w) = \int_{\mathbb{D}} \frac{u(z)}{(1 - w\bar{z})^2}\, dA(z),$$

for $u \in L^2(\mathbb{D}, dA)$ and $w \in \mathbb{D}$.

If g is a bounded analytic function on $\mathbb{D}$, then

$$(T_{\bar{g}}h)(w) = \int_{\mathbb{D}} \frac{\overline{g(z)}h(z)}{(1 - w\bar{z})^2}\, dA(z),$$

for all $h \in L_a^2$ and $w \in \mathbb{D}$. If $g \in L_a^2$ and $h \in L_a^2$, we define $T_{\bar{g}}h$ by the latter integral. If f is furthermore in L_a^2, then the meaning of $T_f T_{\bar{g}}h$ is clear: it is the analytic function $f\ T_{\bar{g}}h$. We will be concerned with the question for which f and g in L_a^2 the Toeplitz product $T_f T_{\bar{g}}$ is invertible on L_a^2.

The question for which f and g in L_a^2 the operator $T_f T_{\bar{g}}$ is bounded on L_a^2 was considered in [16]. The following result was proved in [16]:

THEOREM 1.2. *Let f and g be in L_a^2.*

(i) *If the Toeplitz product $T_f T_{\bar{g}}$ is bounded on L_a^2, then*

$$\sup_{w\in\mathbb{D}} \widetilde{|f|^2}(w)\widetilde{|g|^2}(w) < \infty.$$

(ii) *If*

$$\sup_{w\in\mathbb{D}} \widetilde{|f|^{2+\varepsilon}}(w)\widetilde{|g|^{2+\varepsilon}}(w) < \infty,$$

for some $\varepsilon > 0$, *then the Toeplitz product $T_f T_{\bar{g}}$ is bounded on L_a^2.*

We will show that if f and g are in L_a^2, then the product $T_f T_{\bar{g}}$ is bounded and invertible on L_a^2 if and only if

$$\inf_{w\in\mathbb{D}} |f(w)g(w)| > 0$$

and

$$\sup_{w\in\mathbb{D}} \widetilde{|f|^2}(w)\widetilde{|g|^2}(w) < \infty.$$

Here $\tilde{f}(w)$ is the Berezin transform of a function $f \in L^2(\mathbb{D}, dA)$ defined on $\mathbb{D}$ by

$$\tilde{f}(w) = \int_{\mathbb{D}} f(z)|k_w(z)|^2\, dA(z)$$

and the functions

$$k_w(z) = \frac{1 - |w|^2}{(1 - \bar{w}z)^2}$$

are the normalized reproducing kernels for L^2_a. To prove the above result, using Theorem 1.2, we need to get the reverse Hölder inequality for the so-called invariant A_2 weights. To do so, we extend the basic techniques of the real-variable theory of weighted norm inequalities [1, 3, 4, 6, 15] to the Bergman space. We form a dyadic grid on $\mathbb{D}$, define a dyadic maximal operator, form a Calderón–Zygmund decomposition, and use this to prove an inequality analogous to the so-called "reverse Hölder inequality" of the theory of weighted norm inequalities (Theorem 2.1).

2. A REVERSE HÖLDER INEQUALITY

First, we introduce more notation and discuss some preliminaries needed in the sequel.

For $w \in \mathbb{D}$, the fractional linear transformation φ_w defined by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}$$

is an automorphism of the unit disk, in fact, $\varphi_w^{-1} = \varphi_w$. The real Jacobian for the change of variable $\xi = \varphi_w(z)$ is equal to $|\varphi_w'(z)|^2 = (1 - |w|^2)^2 / |1 - \bar{w}z|^4$, thus we have the change-of-variable formula

$$\int_{\mathbb{D}} h(\varphi_w(z))\, dA(z) = \int_{\mathbb{D}} h(z) \frac{(1 - |w|^2)^2}{|1 - \bar{w}z|^4}\, dA(z).$$

It follows from the above change-of-variable formula that

$$\widetilde{|f|^2}(w) = \|f \circ \varphi_w\|_2^2,$$

for every $f \in L^2(\mathbb{D}, dA)$ and $w \in \mathbb{D}$. The Berezin transform has the following Möbius-invariance:

$$\widetilde{f \circ \varphi_\lambda}(w) = \tilde{f}(\varphi_\lambda(w)),$$

for every $f \in L^2(\mathbb{D}, dA)$, $w \in \mathbb{D}$ and $\lambda \in \mathbb{D}$.

In this section, we will prove a reverse Hölder inequality for f in L^2_a satisfying the following invariant A_2 weight condition:

$$\sup_{w \in \mathbb{D}} \widetilde{|f|^2}(w)\, \widetilde{|f|^{-2}}(w) < \infty. \tag{A$_2$}$$

We will prove that the above condition implies the invariant weight condition:

$$\sup_{w \in \mathbb{D}} \widetilde{|f|^{2+\varepsilon}}(w)\, \widetilde{|f|^{-(2+\varepsilon)}}(w) < \infty,$$

for sufficiently small $\varepsilon > 0$. The above implication will follow once we prove a reverse Hölder inequality analogous to the Coifman–Fefferman theorem [1] (the fundamental property about A_∞ weights):

THEOREM 2.1. *Suppose that $f \in L_a^2$ satisfies condition* (A_2) *with constant*

$$M = \sup_{w \in \mathbb{D}} \widetilde{|f|^2}(w)\, \widetilde{|f|^{-2}}(w) < \infty.$$

There exist constants $\varepsilon_M > 0$ and $C_M > 0$ such that

$$\widetilde{|f|^{2+\varepsilon}}(w) \leqslant C_M (\widetilde{|f|^2}(w))^{(2+\varepsilon)/2}$$

for every $w \in \mathbb{D}$ and $0 < \varepsilon < \varepsilon_M$.

Our proof will make use of dyadic rectangles and the dyadic maximal function. We first discuss the dyadic rectangles and prove some elementary properties related to these rectangles.

Dyadic rectangles. Any set of the form

$Q_{n,m,k} = \{re^{i\theta} : (m-1)2^{-n} \leqslant r < m2^{-n} \text{ and } (k-1)2^{-n+1}\pi \leqslant \theta < k2^{-n+1}\pi\}$,

where n, m and k are positive integers such that $m \leqslant 2^n$ and $k \leqslant 2^n$ is called a dyadic rectangle. The center of the above dyadic rectangle $Q = Q_{n,m,k}$ is the point $z_Q = (m - \frac{1}{2})2^{-n}e^{i\vartheta}$, with $\vartheta = (k - \frac{1}{2})2^{1-n}\pi$. Write $|E|$ to denote the normalized area of a measurable set $E \in \mathbb{D}$. If $d(Q)$ denotes the distance between Q and $\partial\mathbb{D}$, then a simple calculation shows that

$$|Q| = 4|z_Q|(1 - |z_Q| - d(Q))^2.$$

In particular,

$$|Q| \geqslant (1 - |z_Q| - d(Q))^2,$$

whenever $|z_Q| \geqslant 1/4$.

LEMMA 2.2. *Let Q be a dyadic rectangle with center $w = z_Q$. There is a constant $c_1 > 0$ such that*

$$|k_w(z)|^2 \geqslant \frac{c_1}{(1 - |w|)^2},$$

for every $z \in Q$.

Proof. If $z = re^{i\theta}$ and $w = se^{i\vartheta}$, then

$$\begin{aligned} |1 - \bar{w}z|^2 &= 1 + r^2s^2 - 2rs\cos(\theta - \vartheta) \\ &= (1 - rs)^2 + 4rs\sin^2((\theta - \vartheta)/2) \\ &\leqslant (1 - rs)^2 + rs(\theta - \vartheta)^2. \end{aligned}$$

If $z \in Q$ and $Q = Q_{n,m,k}$, then

$$|\theta - \vartheta| \leqslant \pi/2^n = 2\pi/2^{n+1} \leqslant 2\pi(1 - s).$$

Also

$$|r - s| \leqslant 1/2^{n+1} \leqslant 1 - s,$$

thus

$$1 - rs = (1 + s)(1 - s) - (r - s)s < 2(1 - s) + (1 - s)s < 3(1 - s).$$

Hence

$$|1 - \bar{w}z|^2 < 9(1 - s)^2 + 4\pi^2(1 - s)^2 < 50(1 - s)^2,$$

and we obtain

$$|k_w(z)|^2 = \frac{(1 - |w|^2)^2}{|1 - \bar{w}z|^4} \geqslant \frac{(1 - |w|^2)^2}{50^2(1 - |w|)^4} = \frac{(1 + |w|)^2}{2500(1 - |w|)^2} \geqslant \frac{1}{2500(1 - |w|)^2}.$$

This proves the inequality with $c_1 = 1/2500$. ∎

For $w \in \mathbb{D}$ and $0 < s < 1$ let $D(w, s)$ denote the pseudohyperbolic disk with center w and radius $0 < s < 1$, i.e.,

$$D(w, s) = \{z \in \mathbb{C} : |\varphi_w(z)| < s\}.$$

LEMMA 2.3. *Suppose that $f \in L_a^2$ satisfies the invariant weight condition* (A_2) *and let $0 < s < 1$. There is a constant $c_s > 0$ such that*

$$\frac{1}{c_s} \leqslant \frac{|f(z)|}{|f(w)|} \leqslant c_s,$$

whenever $z \in D(w, s)$.

Proof. Fix $w \in \mathbb{D}$. Let u be in $D(0, s)$. Since f is in L_a^2 we have $f(u) = \langle f, K_u \rangle$. Applying the Cauchy–Schwarz inequality we obtain

$$|f(u)| \leqslant \|f\|_2 \|K_u\|_2 = \frac{\|f\|_2}{1 - |u|^2} \leqslant \frac{\|f\|_2}{1 - s^2},$$

for each u in $D(0,s)$. Now if $z \in D(w,s)$ then $z = \varphi_w(u)$, for some $u \in D(0,s)$. Replacing f by $f \circ \varphi_w$ in the above inequality gives

$$|f(z)| = |(f \circ \varphi_w)(u)| \leqslant \frac{\|f \circ \varphi_w\|_2}{1-s^2} = \frac{1}{1-s^2} \widetilde{|f|^2}(w)^{1/2}.$$

By the Cauchy–Schwarz inequality

$$\frac{1}{|f(w)|} = |(f^{-1} \circ \varphi_w)(0)| \leqslant \|f^{-1} \circ \varphi_w\|_2 = \widetilde{|f^{-1}|^2}(w)^{1/2}.$$

Combining these inequalities we have

$$\frac{|f(z)|}{|f(w)|} \leqslant \frac{1}{1-s^2} \widetilde{|f|^2}(w)^{1/2} \widetilde{|f^{-1}|^2}(w)^{1/2} \leqslant \frac{M^{1/2}}{1-s^2},$$

for all $z \in D(w,s)$. Replacing f by its reciprocal f^{-1} gives the other inequality. ∎

Lemma 2.4. *If $f \in L^2_a$ satisfies the invariant weight condition* (A_2), *then there is a constant $C > 0$ such that*

$$\left(\frac{1}{|Q|}\int_Q |f|^2\, dA\right)\left(\frac{1}{|Q|}\int_Q |f|^{-2}\, dA\right) \leqslant C,$$

for every dyadic rectangle Q.

Proof. Suppose that $\widetilde{|f|^2}(w)\widetilde{|f|^{-2}}(w) \leqslant M$, for all $w \in \mathbb{D}$. Let Q be a dyadic rectangle. We first consider the case that $|z_Q| \geqslant 1/4$. We consider two subcases. First we assume that $|Q| \geqslant d(Q)^2/100$. By Lemma 2.2 we see that

$$\begin{aligned}
\widetilde{|f|^2}(z_Q) &= \int_{\mathbb{D}} |f|^2 |k_{z_Q}|^2\, dA \\
&\geqslant \int_Q |f|^2 |k_{z_Q}|^2\, dA \\
&\geqslant \frac{c_1}{(1-|z_Q|)^2} \int_Q |f|^2\, dA.
\end{aligned}$$

Because $|z_Q| \geqslant 1/4$ we have $1 - |z_Q| \leqslant d(Q) + |Q|^{1/2}$. Thus

$$(1-|z_Q|)^2 \leqslant 2(d(Q)^2 + |Q|) \leqslant 2(100|Q| + |Q|) = 202|Q|.$$

Combining the above two inequalities yields

$$\widetilde{|f|^2}(z_Q) \geqslant \frac{c_2}{|Q|} \int_Q |f|^2 \, dA.$$

A similar inequality holds for f^{-1}. Hence we have

$$\left(\frac{1}{|Q|}\int_Q |f|^2 \, dA\right)\left(\frac{1}{|Q|}\int_Q |f|^{-2} \, dA\right) \leqslant \left(\frac{1}{c_2}\widetilde{|f|^2}(z_Q)\right)\left(\frac{1}{c_2}\widetilde{|f|^{-2}}(z_Q)\right) \leqslant \frac{M}{c_2^2}.$$

Next we assume that $|Q| < d(Q)^2/100$. Suppose that $z = re^{i\theta} \in Q$ and $z_Q = se^{i\vartheta}$. If $Q = Q_{n,m,k}$, then $|r - s| \leqslant 1/2^{n+1}$ and $|\theta - \vartheta| \leqslant \pi/2^n$, thus

$$|z - z_Q|^2 = (r - s)^2 + 4rs \sin^2\left(\frac{\theta - \vartheta}{2}\right) \leqslant \frac{1 + 4\pi^2}{2^{2n+2}} < \frac{49}{2^{2n+2}}.$$

On the other hand,

$$|Q| \geqslant (1 - |z_Q| - d(Q))^2 = \frac{1}{2^{2n+2}}.$$

Thus

$$|z - z_Q| \leqslant 7|Q|^{1/2} \leqslant (7/10)d(Q) \leqslant (7/10)(1 - |z_Q|).$$

This implies

$$\left|\frac{z_Q - z}{1 - \bar{z}_Q z}\right| \leqslant \frac{|z_Q - z|}{1 - |z_Q|} \leqslant 7/10.$$

So Q is a subset of $D(z_Q, 7/10)$. By Lemma 2.3, there is a constant C, which is independent of Q such that

$$C^{-1}|f(z_Q)| \leqslant |f(z)| \leqslant C|f(z_Q)|,$$

for all $z \in Q$. Therefore

$$\left(\frac{1}{|Q|}\int_Q |f|^2 \, dA\right)\left(\frac{1}{|Q|}\int_Q |f|^{-2} \, dA\right) \leqslant C^2|f(z_Q)|^2 C^2|f(z_Q)|^{-2} = C^4.$$

This completes the proof in case $|z_Q| \geqslant 1/4$.

Finally, we consider the case that $|z_Q| \leqslant 1/4$. Then $Q \subset D(0, 1/2)$, and the proof is finished as in the second subcase above. ∎

The following lemma and its proof are adapted from the theory of weighted norm inequalities [1].

Lemma 2.5. *Suppose that $f \in L_a^2$ satisfies the invariant weight condition* (A_2). *For every $w \in \mathbb{D}$ let $d\mu_w = |f \circ \varphi_w|^2\, dA$. If $0 < \gamma < 1$, then there exists a $0 < \delta < 1$ such that*

$$\mu_w(E) \leqslant \delta \mu_w(Q),$$

whenever E a subset of Q with $|E| \leqslant \gamma |Q|$.

Proof. Suppose that $\widetilde{|f|^2}(w)\widetilde{|f|^{-2}}(w) \leqslant M$, for all $w \in \mathbb{D}$. Let g be locally integrable and let Q a dyadic rectangle. We use g_Q to denote the average value of g over Q. If g is non-negative, then

$$g_Q^2 = \frac{1}{|Q|^2}\left(\int_Q g|f|\,|f|^{-1}\, dA\right)^2.$$

Applying the Cauchy–Schwarz inequality yields

$$\begin{aligned} g_Q^2 &\leqslant \frac{1}{|Q|^2}\left(\int_Q g^2 |f|^2\, dA\right)\left(\int_Q |f|^{-2}\, dA\right) \\ &= \frac{1}{|Q|^2 \mu_0(Q)}\left(\int_Q g^2 |f|^2\, dA\right)\left(\int_Q |f|^2\, dA\right)\left(\int_Q |f|^{-2}\, dA\right). \end{aligned}$$

By Lemma 2.4 we have

$$g_Q^2 \leqslant \frac{C}{\mu_0(Q)}\left(\int_Q g^2 |f|^2\, dA\right),$$

where C is the constant in Lemma 2.4.

Let F be a subset of Q. Taking $g = \chi_F$ in the last inequality gives

$$\left(\frac{|F|}{|Q|}\right)^2 \leqslant C\frac{\mu_0(F)}{\mu_0(Q)}.$$

Let E be a subset of Q with $|E| \leqslant \gamma |Q|$ for $0 < \gamma < 1$. Let F be the complement of E in Q. Thus

$$\frac{|F|}{|Q|} \geqslant (1 - \gamma).$$

So

$$\frac{\mu_0(F)}{\mu_0(Q)} \geqslant \frac{(1-\gamma)^2}{C}.$$

Note that $\mu_0(E) = \mu_0(Q) - \mu_0(F)$. The last inequality yields

$$\mu_0(E) \leqslant \left(1 - \frac{(1-\gamma)^2}{C}\right)\mu_0(Q).$$

So, putting $\delta = 1 - (1-\gamma)^2/C$, for each fixed $w \in \mathbb{D}$ applying the above argument to $|f \circ \varphi_w|^2$ leads to

$$\mu_w(E) \leqslant \delta \mu_w(Q),$$

whenever E a subset of Q with $|E| \leqslant \gamma |Q|$ for $0 < \gamma < 1$. ∎

The dyadic maximal function. The dyadic maximal operator M^{Δ} is defined by

$$(M^{\Delta} f)(w) = \sup_{w \in Q} \frac{1}{|Q|} \int_Q |f| \, dA,$$

where the supremum is over all dyadic rectangles Q that contain w. The maximal function is of weak-type $(1,1)$ (see [3] or [15]) and the maximal function is greater than the dyadic maximal function, so the dyadic maximal function of any continuous integrable function is finite on $\mathbb{D}$. In particular, if $f \in L^2_a$ satisfies the invariant A_2-condition, then the dyadic maximal function $M^{\Delta}|f|^2$ is always finite. This can also be seen directly as follows. Given a point $w \in \mathbb{D}$, there is a number $0 < R < 1$ such that all but a finite number of dyadic rectangles containing the point w lie inside the closed disk $\bar{D}(0, R) = \{z \in \mathbb{C} : |z| \leqslant R\}$. If $f \in L^2_a$ and Q is a dyadic rectangle containing w inside the disk $\bar{D}(0, R)$, then

$$\frac{1}{|Q|} \int_Q |f(z)|^2 \, dA(z) \leqslant \max\{|f(z)|^2 : |z| \leqslant R\}.$$

If $Q_1, \ldots, Q_m$ are dyadic rectangles containing w not contained in the disk $\bar{D}(0, R)$, then

$$M^{\Delta}|f|^2(w) \leqslant \max\{|f(z)|^2 : |z| \leqslant R\} + \max_{1 \leqslant j \leqslant m} \frac{1}{|Q_j|} \int_{Q_j} |f(z)|^2 \, dA(z) < \infty.$$

This proves that the dyadic function of $|f|^2$ is finite on $\mathbb{D}$.

The principal fact about the dyadic maximal function is the Calderon–Zygmund decomposition formulated in the next theorem. We will need the notion of "doubling" of dyadic rectangles in its proof. Suppose that $n \geqslant 1$ and m, k are positive integers such that $m, k \leqslant 2^n$. The double of $Q = Q_{n,m,k}$, denoted by $2Q$, is defined by

$$2Q = Q_{n-1,[(m+1)/2],[(k+1)/2]},$$

where $[\ell]$ denotes the greatest integer less than or equal to ℓ. An elementary calculation shows that

$$\frac{|2Q|}{|Q|} \leqslant 8,$$

for every proper dyadic rectangle Q in the unit disk.

The following theorem and proof should be compared with Lemma 1 in Section lV.3 (p. 150) of Stein's book [15].

Calderon–Zygmund decomposition theorem. Let f be locally integrable on $\mathbb{D}$, let $t > 0$, and suppose that $\Omega = \{z \in \mathbb{D} : M^{\Delta} f(z) > t\}$ is not equal to $\mathbb{D}$. Then Ω may be written as the disjoint union of dyadic rectangles $\{Q_j\}$ with

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f|\, dA < 8t.$$

Proof. Suppose that $w \in \Omega$, that is, $M^{\Delta} f(w) > t$. Then there exists a dyadic rectangle Q containing w such that

$$\frac{1}{|Q|} \int_{Q} |f|\, dA > t.$$

Now, if $z \in Q$, then

$$M^{\Delta} f(z) \geqslant \frac{1}{|Q|} \int_{Q} |f|\, dA > t.$$

It follows $z \in \Omega$. Thus $Q \subset \Omega$. It follows that $\Omega = \bigcup_j Q_j$. We may assume that the Q_j are *maximal* dyadic rectangles. Since $Q = Q_j$ is not equal to $\mathbb{D}$, by maximality its double $2Q$ is not contained in Ω. This means that $2Q$ contains a point z which is not in Ω. Since $M^{\Delta} f(z) \leqslant t$, we obtain

$$\frac{1}{|2Q|} \int_{2Q} |f|\, dA \leqslant M^{\Delta} f(z) \leqslant t,$$

and hence

$$\int_{Q} |f|\, dA \leqslant \int_{2Q} |f|\, dA \leqslant t|2Q|.$$

It follows that

$$\frac{1}{|Q|} \int_{Q} |f|\, dA \leqslant t \frac{|2Q|}{|Q|} \leqslant 8t,$$

completing the proof. ∎

Before we prove the reverse Hölder inequality (Theorem 2.1), we need one more preliminary result for the dyadic maximal function:

PROPOSITION 2.6. *If $f \in L_a^2$, then*

(i) $|f|^2 \leqslant M^{\Delta}|f|^2$ *on* $\mathbb{D}$, *and*
(ii) $\|f\|_2^2 \leqslant M^{\Delta}|f|^2(0) \leqslant 2\|f\|_2^2$.

Proof. (i) In fact, we will prove that if g is continuous on $\mathbb{D}$, then $|g(w)| \leqslant M^{\Delta}g(w)$ for every $w \in \mathbb{D}$. Fix $w \in \mathbb{D}$. Let Q_0 be any dyadic rectangle containing w. Since $\bar{Q}_0$ is a compact subset of $\mathbb{D}$, function g is uniformly continuous on Q_0. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $|g(z) - g(w)| < \varepsilon$ whenever $z, w \in Q_0$ are such that $|z - w| < \delta$. Subdividing Q_0 a number of times there exists a dyadic rectangle Q containing w with diameter less than δ. Then

$$|g(w)| \leqslant |g(z)| + |g(w) - g(z)| \leqslant |g(z)| + \varepsilon$$

for all $z \in Q$. This implies that

$$|g(w)| \leqslant \frac{1}{|Q|}\int_Q |g(z)|\, dA(z) + \varepsilon \leqslant M^{\Delta}g(w) + \varepsilon.$$

Therefore

$$|g(w)| \leqslant M^{\Delta}g(w),$$

as desired.

(ii) Since $\mathbb{D}$ is a dyadic rectangle we have

$$M^{\Delta}|f|^2(0) \geqslant \frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |f|^2\, dA = \|f\|_2^2.$$

Suppose that Q is a dyadic rectangle containing 0. Then $m = 1$, so $Q = Q_{n,1,k}$. It follows that

$$\begin{aligned}\int_Q |f|^2\, dA &= \frac{1}{\pi}\sum_{j=0}^{\infty} |a_j|^2 \int_0^{1/2^n}\int_{2(k-1)\pi/2^n}^{2k\pi/2^n} 2rr^{2j}\, dr\, d\theta \\ &= \frac{1}{2^{n-1}}\sum_{j=0}^{\infty} |a_j|^2 \frac{(1/4^n)^{j+1}}{j+1}.\end{aligned}$$

Using that $|Q| = 2^{-3n}$ we get

$$\frac{1}{|Q|}\int_Q |f|^2\, dA = 2\sum_{j=0}^{\infty} |a_j|^2 \frac{4^{-nj}}{j+1} \leqslant 2\sum_{j=0}^{\infty} \frac{|a_j|^2}{j+1} = 2\|f\|_2^2.$$

Hence

$$M^{\Delta}|f|^2(0) \leqslant 2\|f\|_2^2,$$

as desired. ∎

We are now ready to prove the reverse Hölder inequality contained in Theorem 2.1. The following proof is analogous to the proof about A_∞ weights in [1, 4, 15].

Proof of Theorem 2.1. First we prove that for some constant $C_M > 0$,

$$\int_{\mathbb{D}} |f|^{2+\varepsilon}\, dA \leqslant C_M \left(\int_{\mathbb{D}} |f|^2\, dA \right)^{(2+\varepsilon)/2}.$$

For each integer $k \geqslant 0$, set

$$E_k = \{ z \in \mathbb{D} : M^{\Delta}|f|^2(z) > 2^{4k+1} \|f\|_2^2 \}.$$

Since $M^{\Delta}|f|^2(0) \leqslant 2\|f\|_2^2 \leqslant 2^{4k+1}\|f\|_2^2$, it follows from Proposition 2.6(ii) that for every positive integer k the set E_k is not equal to $\mathbb{D}$. Fix $k \geqslant 1$. By the Calderon–Zygmund decomposition theorem, $E_k = \bigcup_j Q_j$, where Q_j are disjoint dyadic rectangles in E_k that satisfy

$$2^{4k+1}\|f\|_2^2 < \frac{1}{|Q_j|} \int_{Q_j} |f|\, dA < 8 \times 2^{4k+1} \|f\|_2^2,$$

thus

$$|Q_j| \leqslant 2^{-4k-1} \|f\|_2^{-2} \int_{Q_j} |f|\, dA \qquad \text{and} \qquad \int_{Q_j} |f|\, dA < 8 \times 2^{4k+1} \|f\|_2^2\, |Q_j|.$$

Let Q be a maximal dyadic rectangle in E_{k-1}. Summing over all such $Q_j \subset Q$ gives that

$$|E_k \cap Q| = \sum_{j\,:\,Q_j \subset Q} |Q_j| \leqslant 2^{-4k-1} \|f\|_2^{-2} \int_Q |f|^2\, dA,$$

since the Q_j are disjoint and their union is E_k. On the other hand,

$$\int_Q |f|^2\, dA \leqslant 8 \times 2^{4(k-1)+1} \|f\|_2^2\, |Q| = 2^{4k} \|f\|_2^2\, |Q|.$$

Hence

$$|E_k \cap Q| \leqslant \tfrac{1}{2} |Q|.$$

Now by Lemma 2.5 there exists a $0<\delta<1$ such that

$$\mu(E_k \cap Q) \leqslant \delta\mu(Q),$$

where $d\mu = |f|^2\, dA$. Taking the union over all maximal dyadic rectangles Q in E_{k-1} gives

$$\mu(E_k) \leqslant \delta\mu(E_{k-1}),$$

and therefore

$$\mu(E_k) \leqslant \delta^k \mu(E^0) \leqslant \delta^k \|f\|_2^2.$$

Now, using Proposition 2.6, we have

$$\begin{aligned}
\int_{\mathbb{D}} |f|^{2+\varepsilon}\, dA &\leqslant \int_{\mathbb{D}} (M^{\Delta}|f|^2)^{\varepsilon/2} |f|^2\, dA \\
&= \int_{\{M^{\Delta}|f|^2 \leqslant \|f\|_2^2\}} (M^{\Delta}|f|^2)^{\varepsilon/2} |f|^2\, dA \\
&\quad + \sum_{k=0}^{\infty} \int_{E_k \setminus E_{k+1}} (M^{\Delta}|f|^2)^{\varepsilon/2} |f|^2\, dA \\
&\leqslant \|f\|_2^{\varepsilon} \|f\|_2^2 + \sum_{k=0}^{\infty} 2^{(4(k+1)+1)\varepsilon/2} \|f\|_2^{\varepsilon} \mu(E_k) \\
&\leqslant \|f\|_2^{2+\varepsilon} + \sum_{k=0}^{\infty} 2^{(2k+5/2)\varepsilon} \delta^k \|f\|_2^{2+\varepsilon} \\
&\leqslant \left(1 + \frac{2^{5\varepsilon/2}}{1 - 2^{2\varepsilon}\delta}\right) \|f\|_2^{2+\varepsilon},
\end{aligned}$$

if $2^{2\varepsilon}\delta < 1$. Put $\varepsilon_M = \ln(2/(1+\delta))/\ln 4$. If $0<\varepsilon<\varepsilon_M$, then $2^{2\varepsilon} < 2/(1+\delta)$, so that

$$\frac{2^{5\varepsilon/2}}{1-2^{2\varepsilon}\delta} < \frac{(2/(1+\delta))^{5/4}}{1-2\delta/(1+\delta)} = \frac{2^{5/4}}{(1-\delta)(1+\delta)^{1/4}} < \frac{3}{1-\delta}.$$

So, if $C_M = (4-\delta)/(1-\delta)$, then for $0<\varepsilon<\varepsilon_M$ we have shown that

$$\int_{\mathbb{D}} |f|^{2+\varepsilon}\, dA \leqslant C_M \left(\int_{\mathbb{D}} |f|^2\, dA\right)^{(2+\varepsilon)/2}.$$

Observe that C_M depends only on M. For a fixed $w \in \mathbb{D}$ by Möbius-invariance of the Berezin transform we also have

$$M = \sup_{\lambda \in \mathbb{D}} |\widetilde{f \circ \varphi_w|^2}(\lambda)\ |\widetilde{f \circ \varphi_w|^{-2}}(\lambda).$$

Let $|f \circ \varphi_w|^2$ in the above argument. We obtain

$$\int_{\mathbb{D}} |f \circ \varphi_w|^{2+\varepsilon}\, dA \leqslant C_M \left(\int_{\mathbb{D}} |f \circ \varphi_w|^2\, dA \right)^{(2+\varepsilon)/2}$$

that is,

$$\widetilde{|f|^{2+\varepsilon}}(w) \leqslant C_M (\widetilde{|f|^2}(w))^{(2+\varepsilon)/2},$$

as desired. ∎

3. INVERTIBLE TOEPLITZ PRODUCTS

In this section, we will completely characterize the bounded invertible Toeplitz products $T_f T_{\bar{g}}$ on L_a^2. We have the following result:

THEOREM 3.1. *Let $f, g \in L_a^2$. Then: $T_f T_{\bar{g}}$ is bounded and invertible on L_a^2 if and only if* $\sup\{\widetilde{|f|^2}(w)\widetilde{|g|^2}(w) : w \in \mathbb{D}\} < \infty$ *and* $\inf\{|f(w)||g(w)| : w \in \mathbb{D}\} > 0$.

Proof. $\Rightarrow$: Suppose that $T_f T_{\bar{g}}$ is bounded and invertible on L_a^2. By Theorem 1.2 there exists a constant M such that

$$\widetilde{|f|^2}(w)\ \widetilde{|g|^2}(w) \leqslant M, \tag{3.2}$$

for all $w \in \mathbb{D}$. Note that

$$T_f T_{\bar{g}} k_w = \overline{g(w)} f k_w.$$

Thus

$$\|T_f T_{\bar{g}} k_w\|_2^2 = |g(w)|^2 \|f k_w\|_2^2 = |g(w)|^2 \widetilde{|f|^2}(w),$$

so the invertibility of $T_f T_{\bar{g}}$ yields

$$|g(w)|^2 \widetilde{|f|^2}(w) \geqslant \delta_2 > 0, \tag{3.3}$$

for some constant δ_1 and for all $w \in \mathbb{D}$. Since also $T_g T_{\bar{f}} = (T_f T_{\bar{g}})^*$ is bounded and invertible, there also is a constant δ_2 such that

$$|f(w)|^2 \widetilde{|g|^2}(w) \geqslant \delta_2 > 0, \tag{3.4}$$

for all $w \in \mathbb{D}$. Putting $\delta = \delta_1 \delta_2$, it follows from (3.2) to (3.4) that

$$\delta \leqslant |f(w)|^2 |g(w)|^2 \widetilde{|f|^2}(w) \widetilde{|g|^2}(w) \leqslant M |f(w)|^2 |g(w)|^2,$$

and thus

$$|f(w)||g(w)| \geqslant \frac{\delta^{1/2}}{M^{1/2}},$$

for all $w \in \mathbb{D}$.

$\Leftarrow$: Suppose that

$$M = \sup\{\widetilde{|f|^2}(w)\widetilde{|g|^2}(w) : w \in \mathbb{D}\} < \infty$$

and

$$\eta = \inf\{|f(w)||g(w)| : w \in \mathbb{D}\} > 0.$$

By the inequality of Cauchy–Schwarz,

$$|f(w)|^2 \leqslant \widetilde{|f|^2}(w),$$

for all $w \in \mathbb{D}$, thus $|f(w)||g(w)| \leqslant M^{1/2}$, for all $w \in \mathbb{D}$. So, fg is a bounded function on $\mathbb{D}$. Note that f and g cannot have zeros in $\mathbb{D}$. Since $|g(z)|^2 \geqslant \eta^2 |f(z)|^{-2}$, for all $z \in \mathbb{D}$, we have

$$\widetilde{|g|^2}(w) \geqslant \eta^2 \widetilde{|f|^{-2}}(w),$$

for all $w \in \mathbb{D}$. Consequently

$$M \geqslant \widetilde{|f|^2}(w)\widetilde{|g|^2}(w) \geqslant \eta^2 \widetilde{|f|^2}(w)\widetilde{|f|^{-2}}(w)$$

so that

$$\widetilde{|f|^2}(w)\widetilde{|f|^{-2}}(w) \leqslant M/\eta^2,$$

for all $w \in \mathbb{D}$. This means that f satisfies the (A_2) condition. By the reverse Hölder inequality, for some $\varepsilon > 0$,

$$\sup_{w \in \mathbb{D}} \widetilde{|f|^{2+\varepsilon}}(w)\widetilde{|f|^{-(2+\varepsilon)}}(w) < \infty.$$

By Theorem 1.2, $T_f T_{\overline{f^{-1}}}$ is bounded on L_a^2. Since fg is bounded on $\mathbb{D}$, the operator $T_{\overline{fg}}$ is bounded on L_a^2. That $T_f T_{\bar{g}}$ is bounded follows from the fact that $T_f T_{\overline{f^{-1}}} T_{\overline{fg}}$ is bounded on L_a^2 and the claim that $T_f T_{\bar{g}} = T_f T_{\overline{f^{-1}}} T_{\overline{fg}}$ on a dense subset of L_a^2.

To prove the claim, it suffices to show

$$T_f T_{\bar{g}} k_w = T_f T_{\overline{f^{-1}}} T_{\overline{fg}} k_w,$$

for each $w \in \mathbb{D}$, since the linear span of the set $\{k_w : w \in \mathbb{D}\}$ is dense in L_a^2. For $h \in L_a^2$ and a polynomial p, an easy calculation gives

$$\begin{aligned}\langle (\bar{h} - \overline{h(w)}) k_w, p\rangle &= \langle k_w, (h - h(w))p\rangle \\ &= (1 - |w|^2)^2 \overline{(h(w) - h(w))p(w)} = 0.\end{aligned}$$

Thus $(\bar{h} - \overline{h(w)}) k_w$ is in $[L_a^2]^{\perp}$, so

$$T_{\bar{h}} k_w = \overline{h(w)} k_w.$$

Since $\overline{f^{-1}}$, $\bar{g}$ and $\overline{fg}$ are in L_a^2, we obtain

$$T_f T_{\bar{g}} k_w = f T_{\bar{g}} k_w = \overline{g(w)} f k_w$$

and

$$\begin{aligned}T_f T_{\overline{f^{-1}}} T_{\overline{fg}} k_w &= \overline{f(w) g(w)} T_f T_{\overline{f^{-1}}} k_w \\ &= f \overline{(w) g(w) f^{-1}(w)} T_f k_w \\ &= \overline{g(w)} f k_w.\end{aligned}$$

This gives

$$T_f T_{\bar{g}} k_w = T_f T_{\overline{f^{-1}}} T_{\overline{fg}} k_w$$

to complete the proof of the above claim.

The function $\psi = 1/(f\bar{g})$ is bounded on $\mathbb{D}$, so that the operator T_ψ is bounded on L_a^2. Using that

$$T_f T_{\bar{g}} T_\psi = I = T_\psi T_f T_{\bar{g}},$$

we conclude that $T_f T_{\bar{g}}$ is invertible on L_a^2. ∎

Using Theorem 8 of [20] for boundedness of Toeplitz products on the Hardy space, by essentially the same argument as above we obtain the following characterization of bounded invertible Toeplitz products on the Hardy space.

THEOREM 3.5. *Let $f, g \in H^2$. Then: $T_f T_{\bar{g}}$ is bounded and invertible on H^2 if and only if* $\sup\{\widehat{|f|^2}(w)\widehat{|g|^2}(w) : w \in \mathbb{D}\} < \infty$ *and* $\inf\{|f(w)||g(w)| : w \in \mathbb{D}\} > 0$.

This generalizes the main result of David Cruz-Uribe [2]: if f and g are outer functions and

$$\sup_{w \in \mathbb{D}} \widehat{|f|^2}(w)\widehat{|g|^2}(w) < \infty,$$

then it follows from the above theorem that $T_f T_{\bar{g}}$ is bounded and invertible on H^2 if and only if

$$\inf\{|f(w)||g(w)| : w \in \mathbb{D}\} > 0.$$

4. FREDHOLM TOEPLITZ PRODUCTS

In this section, we will completely characterize the bounded Fredholm Toeplitz products $T_f T_{\bar{g}}$ on L_a^2. We have the following result:

THEOREM 4.1. *Let f and g be in L_a^2. Then: $T_f T_{\bar{g}}$ is a bounded Fredholm operator on L_a^2 if and only if $\widetilde{|f|^2}\,\widetilde{|g|^2}$ is bounded on $\mathbb{D}$ and the function $|f||g|$ is bounded away from zero near $\partial\mathbb{D}$.*

The latter condition simply means that there exists a number r with $0 < r < 1$ such that $\inf\{|f(z)||g(z)| : r < |z| < 1\} > 0$.

In the proof of the above theorem we will need the following lemma.

LEMMA 4.2. *Suppose that f is an analytic function on $\mathbb{D}$ with a finite number of zeros. Let B denote the Blaschke product of the zeros of f and $F = f/B$. Then there exists a constant C such that*

$$\widetilde{|F|^2}(w) \leqslant C\widetilde{|f|^2}(w),$$

for all w in $\mathbb{D}$.

Proof. Choose $0<R<1$ such that $|B(z)|>1/\sqrt{2}$, for all $R<|z|<1$. Suppose $w\in\mathbb{D}$. Then

$$\begin{aligned}\widetilde{|f|^2}(w) &= \int_{\mathbb{D}} |f(\varphi_w(z))|^2\,dA(z)\\ &= \int_{\mathbb{D}} |B(\varphi_w(z))|^2|F(\varphi_w(z))|^2\,dA(z)\\ &\geqslant \frac{1}{2}\int_{R<|\varphi_w(z)|<1} |F(\varphi_w(z))|^2\,dA(z).\end{aligned}$$

By a change-of-variable,

$$\int_{R<|\varphi_w(z)|<1} |F(\varphi_w(z))|^2\,dA(z) = \int_{R<|z|<1} |F(z)|^2\frac{(1-|w|^2)^2}{|1-\bar{w}z|^4}\,dA(z).$$

Now, if h is analytic on $\mathbb{D}$, then using power series it is easily shown that

$$\int_{\mathbb{D}} |h(z)|^2\,dA(z)\leqslant\frac{1}{1-R^2}\int_{R<|z|<1} |h(z)|^2\,dA(z).$$

Applying the above estimate to the function

$$h(z) = F(z)\frac{1-|w|^2}{(1-\bar{w}z)^2},$$

we see that

$$\begin{aligned}\int_{R<|z|<1} |F(z)|^2\frac{(1-|w|^2)}{|1-\bar{w}z|^4}\,dA(z) &\geqslant (1-R^2)\int_{\mathbb{D}} |F(z)|^2\frac{(1-|w|^2)}{|1-\bar{w}z|^4}\,dA(z)\\ &\geqslant (1-R^2)\widetilde{|F|^2}(w).\end{aligned}$$

Thus

$$\widetilde{|f|^2}(w)\geqslant\tfrac{1}{2}(1-R^2)\widetilde{|F|^2}(w)$$

so that

$$\widetilde{|F|^2}(w)\leqslant C\widetilde{|f|^2}(w),$$

with $C=2/(1-R^2)$, for all $w\in\mathbb{D}$. ∎

Proof of Theorem 4.1. $\Rightarrow$: If $T_fT_{\bar{g}}$ is bounded, then there is an M such that $\widetilde{|f|^2}\,\widetilde{|g|^2}\leqslant M$ on $\mathbb{D}$. If $T_fT_{\bar{g}}$ is Fredholm, then $T_fT_{\bar{g}}+\mathscr{K}$ is invertible in the Calkin algebra. Thus there exist a bounded operator S and a compact operator A such that

$$ST_f T_{\bar{g}} = I + A.$$

Using that $T_f T_{\bar{g}} k_w = \overline{g(w)} f k_w$ we have

$$\begin{aligned}
\|S\| \, |g(w)| \, \widetilde{|f|^2}(w)^{1/2} &= \|S\| \, \|T_f T_{\bar{g}} k_w\|_2 \\
&\geqslant \|ST_f T_{\bar{g}} k_w\|_2 \\
&\geqslant \|k_w\|_2 - \|Ak_w\|_2 \\
&= 1 - \|Ak_w\|_2.
\end{aligned}$$

Since A is compact, $\|Ak_w\|_2 \to 0$ as $|w| \to 1^-$, so there exists an $0 < r_1 < 1$ such that $\|Ak_w\|_2 < 1/2$, for all $r_1 < |w| < 1$. The above inequality shows that

$$|g(w)|^2 \widetilde{|f|^2}(w) \geqslant M_1 \ \left(= \tfrac{1}{2} \|S\|^{-1}\right),$$

for all $r_1 < |w| < 1$. Since also $T_g T_{\bar{f}} = (T_f T_{\bar{g}})^*$ is Fredholm, there is a positive constant M_2 and a number r_2 with $0 < r_2 < 1$ such that

$$|f(w)|^2 \widetilde{|g|^2}(w) \geqslant M_2,$$

for all $r_2 < |w| < 1$. Thus

$$M_1 M_2 \leqslant |f(z)|^2 |g(z)|^2 \widetilde{|f|^2}(z) \widetilde{|g|^2}(z) \leqslant M |f(z)|^2 |g(z)|^2$$

and hence

$$|f(z)|^2 |g(z)|^2 \geqslant M_1 M_2 / M,$$

for all $\max\{r_1, r_2\} < |z| < 1$.

$\Leftarrow$: Suppose that

$$|f(z)||g(z)| \geqslant \delta > 0, \tag{$*$}$$

for all $0 < r < |z| < 1$. Inequality (*) implies that f and g have no zeros in the annulus $\{z : r < |z| < 1\}$. Let B_1 and B_2 denote the (finite) Blaschke products of the zeros of f and g, respectively. Then $F = f/B_1$ and $G = g/B_2$ are zero free, and by (*) we have

$$|F(z)||G(z)| \geqslant \delta |B_1(z)||B_2(z)|,$$

for all $r<|z|<1$. The function on the right is positive and continuous on annulus $\{z : \frac{1}{2}(1+r) \leqslant |z| \leqslant 1\}$, thus has a positive minimum. So putting $\rho = \frac{1}{2}(1+r)$, we have

$$|F(z)||G(z)| \geqslant \eta',$$

for all $\rho < |z| < 1$. Then

$$|G(z)| \geqslant \eta' |F(z)|^{-1},$$

for all $\rho < |z| < 1$. Note that

$$\eta'' = \inf\{|F(z)||G(z)| : |z| \leqslant \rho\} > 0.$$

If we take $\eta = \min\{\eta', \eta''\}$, then

$$|G(z)| \geqslant \eta |F(z)|^{-1},$$

for all $z \in \mathbb{D}$. By Lemma 4.2, there exist constants C_1 and C_2 such that

$$\widetilde{|F|^2}(z) \leqslant C_1 \widetilde{|f|^2}(z)$$

and

$$\widetilde{|G|^2}(z) \leqslant C_2 \widetilde{|g|^2}(z),$$

for all $z \in \mathbb{D}$. Thus

$$\widetilde{|F|^2}(z) \widetilde{|G|^2}(z) \leqslant M',$$

for all $z \in \mathbb{D}$. As before we conclude that

$$\widetilde{|F|^2}(z) \widetilde{|F|^{-2}}(z) \leqslant \frac{M'}{\eta^2},$$

for all $z \in \mathbb{D}$, so F satisfies condition (A_2). Combining Theorem 2.1 with Theorem 1.2 we see that $T_F T_{1/\bar{F}}$ is bounded. As in the proof of Theorem 3.1 if follows that $T_F T_{\bar{G}}$ is bounded. This implies that

$$T_f T_{\bar{g}} = T_{B_1} T_F T_{\bar{G}} T_{\bar{B}_2}$$

is bounded.

Since $1/(F\bar{G})$ is bounded, the Toeplitz operator $T_{1/(F\bar{G})}$ is bounded, and it follows that $T_F T_{\bar{G}}$ is invertible. Since $T_{\bar{B}_2}$ is Fredholm, there are bounded and compact operator R_2 and K_2 such that $T_{\bar{B}_2} R_2 = I + K_2$. It follows that

$$T_f T_{\bar{g}} R_2 = T_{B_1} T_F T_{\bar{G}} + T_{B_1} T_F T_{\bar{G}} K_2,$$

thus

$$T_f T_{\bar{g}} R_2 (T_F T_{\bar{G}})^{-1} = T_{B_1} + T_{B_1} T_F T_{\bar{G}} K_2 (T_F T_{\bar{G}})^{-1}.$$

Using that also T_{B_1} is Fredholm, there are bounded and compact operator R_1 and K_1 such that $T_{B_1} R_1 = I + K_1$. Then

$$T_f T_{\bar{g}} R_2 (T_F T_{\bar{G}})^{-1} R_1 = I + K_1 + T_{B_1} T_F T_{\bar{G}} K_2 (T_F T_{\bar{G}})^{-1}.$$

Hence $T_f T_{\bar{g}} + \mathscr{K}$ is right-invertible in the Calkin algebra. Similarly $T_f T_{\bar{g}} + \mathscr{K}$ is left-invertible in the Calkin algebra, so that $T_f T_{\bar{g}}$ is Fredholm. ∎

By essentially the same argument as above we obtain the following characterization of Fredholm Toeplitz products on the Hardy space. This theorem generalizes the main result (Theorem 1.2) of David Cruz-Uribe [2].

Theorem 4.3. *Let f and g be in H^2. Then: $T_f T_{\bar{g}}$ is a bounded Fredholm operator on H^2 if and only if $\widehat{|f|^2}\,\widehat{|g|^2}$ is bounded on $\mathbb{D}$ and the function $|f||g|$ is bounded away from zero $\partial\mathbb{D}$.*

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