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Toeplitz algebra and Hankel algebra on the harmonic Bergman space

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Abstract

In this paper we completely characterize compact Toeplitz operators on the harmonic Bergman space. By using this result we establish the short exact sequences associated with the Toeplitz algebra and the Hankel algebra. We show that the Fredholm index of each Fredholm operator in the Toeplitz algebra or the Hankel algebra is zero.

1. Introduction

Let dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. $L^2(D, dA)$ is the Hilbert space of Lebesgue square integrable functions on D with the inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}\, dA(z).$$

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The *harmonic Bergman space* L_h^2 is the closed subspace of $L^2(D, dA)$ consisting of the harmonic functions on D. The *Bergman space* L_a^2 is the closed subspace of $L^2(D, dA)$ consisting of analytic functions on D. Clearly, the Bergman space L_a^2 is a subspace of the harmonic Bergman space L_h^2.

For $\phi \in L^\infty(D, dA)$, the *Toeplitz operator* T_ϕ with symbol ϕ on the Bergman space L_a^2 is the operator defined by

$$T_\phi f = P(\phi f), \quad f \in L_a^2(D),$$

and the *small Hankel operator* Γ_ϕ on the Bergman space L_a^2 is the operator defined by

$$\Gamma_\phi f = P(\phi U f), \quad f \in L_a^2(D),$$

where P is the orthogonal projection from $L^2(D, dA)$ onto L_a^2 and U is the unitary operator on $L^2(D, dA)$, defined by

$$(Uf)(z) = f(\bar{z}).$$

The *Toeplitz operator* $\widetilde{T}_\phi$ with symbol ϕ on L_h^2 is the operator defined by

$$\widetilde{T}_\phi f = Q(\phi f),$$

here Q is the orthogonal projection from $L^2(D, dA)$ onto L_h^2. The *small Hankel operator* h_ϕ with symbol ϕ on L_h^2 is defined by

$$h_\phi f = Q(\phi U f)$$

for $f \in L_h^2$. Clearly, from the above definitions we have that $h_\phi = \widetilde{T}_\phi U$.

This paper will mainly concern on the Toeplitz operators on the harmonic Bergman space L_h^2. We will show that the study of Toeplitz operators on the harmonic Bergman space can be reduced to that of Toeplitz operators and small Hankel operators on the Bergman space.

Recently Sheldon Axler and the second author [2] completely characterized compact Toeplitz operators on the Bergman space L_a^2. In this paper by using their result we obtain a characterization for the compact Toeplitz operators on the harmonic Bergman space. By means of the characterization we study the *Toeplitz algebra* $\mathcal{T}$, the C^*-algebra generated by the Toeplitz operators $\widetilde{T}_\phi$ with symbol continuous on the closure of D, and the *Hankel algebra* $\mathcal{H}$, the C^*-algebra generated by the Hankel operators h_ϕ and the Toeplitz operators $\widetilde{T}_\phi$ with symbol ϕ continuous on the closure of D. The Toeplitz algebra on the Bergman space was studied by Coburn in [6].

In Section 2, we completely characterize compact Toeplitz operators on the harmonic Bergman space, which will be used to study the Toeplitz algebra and Hankel algebra in the subsequent sections. Let $\mathcal{K}$ be the ideal of compact operators on L_h^2 and $C(T)$ the algebra of continuous functions on the unit circle T. On $\mathcal{T}/\mathcal{K}$, define

$$\pi(T_f + \mathcal{K}) = f|_T.$$

In Section 3, we prove that the sequence

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\pi} C(T) \to 0$$

is exact. Furthermore, this sequence is split.

In Section 4, we study the Hankel algebra $\mathcal{H}$ and show that for each Fredholm operator A in $\mathcal{H}$, $\operatorname{ind} A = 0$. Let $C(T) \times_\sigma Z_2$ be the cross product of $C(T)$ and Z_2 which will be defined in Section 4. Finally, we prove that the sequence

$$0 \to \mathcal{K} \to \mathcal{H} \xrightarrow{\tilde{\tau}} C(T) \times_\sigma Z_2 \to 0$$

is exact.

2. Compact Toeplitz operators

In this section we will obtain a criterion for compactness of Toeplitz operators on the harmonic Bergman space, which will be used to study the Toeplitz algebra and the Hankel algebra in the subsequent sections.

For $f \in L^\infty(D, dA)$ we first define

$$\hat{f}(z) = f(\bar{z}) \quad \text{and} \quad f^*(z) = \overline{f(\bar{z})}.$$

Recall that $U : L^2(D, dA) \to L^2(D, dA)$ is the unitary operator

$$(Uf)(z) = f(\bar{z}).$$

Let $\overline{L_a^2} = \{\bar{f}\colon\ f \in L_a^2\}$. Clearly U maps $\overline{L_a^2}$ onto L_a^2 and $U^* = U = U^{-1}$. For each function $f \in L_h^2$, we write

$$f = \big[P(f) - P(f)(0)\big] + \big[(I - P)(f) + P(f)(0)\big].$$

It is easy to see that $[P(f) - P(f)(0)]$ is in zL_a^2 and $[(I - P)(f) + P(f)(0)]$ is in $\overline{L_a^2}$. Moreover, zL_a^2 is perpendicular to $\overline{L_a^2}$. Thus we obtain the decomposition

$$L_h^2 = zL_a^2 \oplus \overline{L_a^2}.$$

Next define the unitary operator

$$\widetilde{U} : L_h^2 = zL_a^2 \oplus \overline{L_a^2} \to zL_a^2 \oplus L_a^2$$

by

$$\widetilde{U} = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}.$$

Clearly, $\widetilde{U}^*$ maps $zL_a^2 \oplus L_a^2$ to $L_h^2 = zL_a^2 \oplus \overline{L_a^2}$ and equals

$$\widetilde{U}^* = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}$$

since $U^* = U$ on L^2.

For f and g in $L^2(D, dA)$, let $f \otimes g$ be the operator defined by

$$(f \otimes g)h = \langle h, g \rangle f,$$

for $h \in L^2(D, dA)$.

The following theorem gives a matrix representation of $\widetilde{T}_\phi$. The representation is useful in this paper and shows that the Toeplitz operators on the harmonic Bergman space are closely related to the Toeplitz operators and small Hankel operators on the Bergman space.

Theorem 2.1. *On* $zL_a^2 \oplus L_a^2$

$$\widetilde{U}\widetilde{T}_\phi\widetilde{U}^* = \begin{pmatrix} T_\phi - (1 \otimes \bar{\phi}) & \Gamma_\phi - (1 \otimes \phi^*) \\ \Gamma_{\hat{\phi}} & T_{\hat{\phi}} \end{pmatrix}. \tag{1}$$

Proof. Let P_1 be the projection from $L^2(D, dA)$ onto $\overline{L_a^2}$. Since $\{\sqrt{n+1}\, z^n\}$ is an orthonormal basis for L_a^2, $\{\sqrt{n+1}\, \bar{z}^n\}$ is an orthonormal basis for $\overline{L_a^2}$. Hence for each $f \in L^2(D, dA)$,

$$\begin{aligned} P_1(f)(z) &= \sum_{n=0}^{\infty} \langle f, \sqrt{n+1}\, \overline{w}^n \rangle \sqrt{n+1}\, \bar{z}^n \\ &= \left\langle f, \sum_{n=0}^{\infty} (n+1)\overline{w}^n z^n \right\rangle = \int_D f(w) \frac{1}{(1-\bar{z}w)^2}\, dA(w). \end{aligned}$$

It is known [13] that

$$P(f)(z) = \int_D f(w) \frac{1}{(1 - z\overline{w})^2}\, dA(w).$$

These give

$$UP_1 f(z) = \int_D f(w) \frac{1}{(1-\bar{\bar{z}}w)^2}\, dA(w) = \int_D f(w) \frac{1}{(1-zw)^2}\, dA(w),$$

and

$$PUf(z) = \int_D f(\overline{w}) \frac{1}{(1-z\overline{w})^2}\, dA(w) = \int_D f(\lambda) \frac{1}{(1-z\lambda)^2}\, dA(\lambda).$$

The last equality follows from the change of variable $\lambda = \overline{w}$. Thus we obtain

$$UP_1 = PU.$$

Note $\{\sqrt{n+1}\, z^n\}_{n=0}^{\infty} \cup \{\sqrt{n+1}\, \bar{z}^n\}_{n=1}^{\infty}$ is an orthonormal basis for the harmonic Bergman space L_h^2. For each $f \in L^2(D, dA)$, we have

$$\begin{aligned} Qf(z) &= \sum_{n=0}^{\infty} \langle f, \sqrt{n+1}\, w^n \rangle \sqrt{n+1}\, z^n + \sum_{n=1}^{\infty} \langle f, \sqrt{n+1}\, \overline{w}^n \rangle \sqrt{n+1}\, \bar{z}^n \\ &= P(f)(z) - \langle f, 1 \rangle + P_1(f)(z). \end{aligned}$$

Thus

$$Q = P - (1 \otimes 1) + P_1.$$

If f_1 is in zL_a^2, by the definition of $\widetilde{T}_\phi$ we have

$$\begin{aligned} \widetilde{T}_\phi f_1 &= P(\phi f_1) - P(\phi f_1)(0) + P_1(\phi f_1) = T_\phi f_1 - (1 \otimes \bar{\phi}) f_1 + UPU(\phi f_1) \\ &= T_\phi f_1 - (1 \otimes \bar{\phi}) f_1 + UP(\hat{\phi} U f_1) = \big[T_\phi - (1 \otimes \bar{\phi})\big] f_1 + U\Gamma_{\hat{\phi}} f_1. \end{aligned}$$

The first term in the last equation is in zL_a^2 and the second term is in $\overline{L_a^2}$. The second equation follows from

$$P(\phi f_1)(0) = \langle \phi f_1, 1 \rangle = \langle f_1, \bar{\phi} \rangle = (1 \otimes \bar{\phi})(f_1).$$

If f_2 is in L_a^2, similarly we have that

$$\begin{aligned} \widetilde{T}_\phi U f_2 &= P(\phi U f_2) - (1 \otimes \bar{\phi}) U f_2 + P_1(\phi U f_2) \\ &= \big[\Gamma_\phi - (1 \otimes \phi^*)\big] f_2 + UPU(\phi U f_2) \\ &= \big[\Gamma_\phi - (1 \otimes \phi^*)\big] f_2 + UP\big(\hat{\phi} U^2 f_2\big) \\ &= \big[\Gamma_\phi - (1 \otimes \phi^*)\big] f_2 + UT_{\hat{\phi}} f_2. \end{aligned}$$

The first term in the last equation is in zL_a^2 and the second term is in $\overline{L_a^2}$. Therefore for given $[f_1, f_2]^T$ in $zL_a^2 \oplus L_a^2$ the above calculation gives

$$\begin{aligned} \widetilde{U}\widetilde{T}_\phi \widetilde{U}^* \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \widetilde{U}\widetilde{T}_\phi \begin{pmatrix} f_1 \\ Uf_2 \end{pmatrix} \\ &= \widetilde{U} \begin{pmatrix} \big[T_\phi - (1 \otimes \bar{\phi})\big] f_1 + \big[\Gamma_\phi - (1 \otimes \phi^*)\big] f_2 \\ U\Gamma_{\hat{\phi}} f_1 + UT_{\hat{\phi}} f_2 \end{pmatrix} \\ &= \begin{pmatrix} \big[T_\phi - (1 \otimes \bar{\phi})\big] f_1 + \big[\Gamma_\phi - (1 \otimes \phi^*)\big] f_2 \\ \Gamma_{\hat{\phi}} f_1 + T_{\hat{\phi}} f_2 \end{pmatrix}. \end{aligned}$$

This gives (1) to complete the proof. □

For $z \in D$, the *Bergman reproducing kernel* is the function $\in L_a^2$ such that

$$f(z) = \langle f, K_z \rangle$$

for every $f \in L_a^2$. The *normalized Bergman reproducing kernel* k_z is the function $K_z/\|K_z\|_2$.

Let ϕ be in $L^2(D, dA)$. The Berezin transform of ϕ is defined by

$$\tilde{\phi}(z) = \int_D \phi(w)\left|k_z(w)\right|^2 dA.$$

Because $\|k_z\|_2 = 1$, the Berezin transform of ϕ is a weighted average of ϕ. If ϕ is bounded, Axler and Zheng [2] proved that the Toeplitz operator T_ϕ on the Bergman space is compact if and only if $\tilde{\phi}(z) \to 0$ as $|z| \to 1$. We will use this result to obtain a complete characterization of the compact Toeplitz operators on the harmonic Bergman space. First we show that the Berezin transform commutes with U.

Lemma 2.2. *On $L^1(D, dA)$, the Berezin transform commutes with U. That is, for each $\phi \in L^1(D, dA)$, $\widetilde{U\phi} = U\tilde{\phi}$.*

Proof. Let ϕ be in $L^1(D, dA)$. Recall that the Berezin transform of ϕ is

$$\tilde{\phi}(z) = \int_D \phi(w)\left|k_z(w)\right|^2 dA,$$

for $z \in D$. Since

$$\begin{aligned}
\widetilde{U\phi}(z) &= \int_D (U\phi)(w)\left|k_z(w)\right|^2 dA \\
&= \int_D \phi(\overline{w})\left|k_z(w)\right|^2 dA = \int_D \phi(w)\left|k_z(\overline{w})\right|^2 dA \\
&= \int_D \phi(w)\left|k_{\bar{z}}(w)\right|^2 dA = \tilde{\phi}(\bar{z}) = U\tilde{\phi}(z).
\end{aligned}$$

This gives the desired result to complete the proof. □

Our main result in this section is the following theorem.

Theorem 2.3. *Suppose that ϕ is in $L^\infty(D, dA)$. On the harmonic Bergman space L_h^2, the Toeplitz operator $\widetilde{T}_\phi$ is compact if and only if $\tilde{\phi}(z) \to 0$ as $|z| \to 1$, and*

$$\lim_{|z|\to 1}\left(1 - |z|^2\right)\left(\left|\frac{\partial\psi}{\partial z}\right| + \left|\frac{\partial\psi}{\partial\bar{z}}\right|\right) = 0.$$

Here $\psi = Q\phi$ is the harmonic part of ϕ.

Proof. First we assume that $\widetilde{T}_\phi$ is compact. The matrix representation (1) of the Toeplitz operator $\widetilde{T}_\phi$ gives that the operators Γ_ϕ and $\Gamma_{\hat{\phi}}$ are compact on the Bergman space L_a^2.

From [13] and [10], we have that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\frac{\partial(P(\phi))}{\partial z}\right| = 0$$

and

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\frac{\partial(P(\hat{\phi}))}{\partial z}\right| = 0.$$

Note that the harmonic part ψ of ϕ is

$$\psi = Q(\phi) = P_1(\phi) + P(\phi) - P(\phi)(0)$$

where P_1 is the projection from $L^2(D, dA)$ onto $\overline{L^2_a}$. Thus

$$\psi = UPU(\phi) + P(\phi) - P(\phi)(0) = UP(\hat{\phi}) + P(\phi) - P(\phi)(0).$$

So

$$\frac{\partial\psi}{\partial z} = \frac{\partial(P(\phi))}{\partial z}$$

and

$$\frac{\partial\psi}{\partial \bar{z}} = U\frac{\partial(P(\hat{\phi}))}{\partial z}.$$

Hence we have that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left(\left|\frac{\partial\psi}{\partial z}\right| + \left|\frac{\partial\psi}{\partial \bar{z}}\right|\right) = 0.$$

Now we need only to prove that the limit of Berezin transform on the boundary of the unit disk is zero.

The matrix representation (1) of $\widetilde{T}_\phi$ gives that both T_ϕ and $T_{\hat{\phi}}$ are compact on the Bergman space. The main result in [2] implies that both $\tilde{\phi}(z)$ and $\tilde{\hat{\phi}}(z)$ converge to zero as $|z| \to 1$.

Conversely, suppose that the harmonic part ψ of ϕ satisfies

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left(\left|\frac{\partial\psi}{\partial z}\right| + \left|\frac{\partial\psi}{\partial \bar{z}}\right|\right) = 0.$$

As we show as above we have that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\frac{\partial(P(\phi))}{\partial z}\right| = 0$$

and

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\frac{\partial(P(\hat{\phi}))}{\partial z}\right| = 0.$$

It follows from [13] and [10] that both Γ_ϕ and $\Gamma_{\hat{\phi}}$ are compact. If $\tilde{\phi}(z) \to 0$ as $|z| \to 1$, Lemma 2.2 implies that $\tilde{\hat{\phi}}(z) = \tilde{\phi}(\bar{z}) \to 0$ as $|z| \to 1$. Then by the main result in [2], we have that both T_ϕ and $T_{\hat{\phi}}$ are compact. The matrix representation (1) of the Toeplitz operator on the harmonic Bergman space gives that $\widetilde{T}_\phi$ is compact. This finishes the proof of the theorem. □

As a consequence of Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let ϕ be a bounded harmonic function. Then $\widetilde{T}_\phi$ is compact if and only if $\phi = 0$.*

Proof. Obviously, if ϕ is zero, then $\widetilde{T}_\phi$ is zero. Conversely suppose that $\widetilde{T}_\phi$ is compact. Since ϕ is harmonic on the unit disk, the mean value theorem gives that

$$\tilde{\phi}(z) = \int_D \phi(w)\left|k_z(w)\right|^2 dA = \int_D \phi \circ \phi_z \, dA = \phi(z).$$

The second equality above follows from the change of variable $\lambda = \phi_z(w)$ and the fact that

$$\left|k_z(w)\right|^2 = \left|\phi_z'(w)\right|^2,$$

where $\phi_z(w)$ is the Móbius transform

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

By Theorem 2.3, the compactness of $\widetilde{T}_\phi$ implies that $\tilde{\phi}$ vanishes on the boundary of the unit disk. The above equality gives that ϕ also vanishes on the unit circle. Because ϕ is harmonic on the unit disk, ϕ identically equals zero on the unit disk. □

3. The Toeplitz algebra

Recall that $\mathcal{T}$ is the C^*-algebra generated by Toeplitz operators on the harmonic Bergman space with their symbols in $C(\overline{D})$. The set of all compact operators on the harmonic Bergman space will be denoted by $\mathcal{K}$.

Coburn [6] showed that on the Bergman space L_a^2, a Toeplitz operator T_ϕ with ϕ in $C(\overline{D})$ is Fredholm if and only if ϕ has no zeros on the circle T. Note that Γ_ϕ is compact if ϕ is in $C(\overline{D})$ [13]. Hence from the matrix representation (1) of Toeplitz operators on the harmonic Bergman space, a Toeplitz operator $\widetilde{T}_\phi$ with its symbol in $C(\overline{D})$ is Fredholm if and only if ϕ has no zeros on the circle T. Thus for $\phi \in C(\overline{D})$, the essential spectrum of $\widetilde{T}_\phi$ equals

$$\sigma_e\left(\widetilde{T}_\phi\right) = \phi(T) = \left\{\phi(\lambda)\colon \lambda \in T\right\}.$$

For ϕ in $L^\infty(D, dA)$, define the essential norm of the Toeplitz operator $\widetilde{T}_\phi$ by

$$\left\|\widetilde{T}_\phi\right\|_e = \inf_{K \in \mathcal{K}} \left\|\widetilde{T}_\phi + K\right\|.$$

The following lemma gives a formula about the essential norm of a Toeplitz operator on the harmonic Bergman space analogous to the result in classical Hardy space [8].

Lemma 3.1. *Let ϕ be in $C(\overline{D})$. Then*

$$\|\widetilde{T}_\phi\|_e = \|\phi|_T\|_\infty.$$

Proof. Since ϕ is in $C(\overline{D})$, it follows from [13] that both Γ_ϕ and $\Gamma_{\hat{\phi}}$ are compact. From the matrix representation (1) of Toeplitz operators on L_h^2, we have that

$$\|\widetilde{T}_\phi\|_e = \max\{\|T_\phi\|_e, \|T_{\hat{\phi}}\|_e\} = \|T_\phi\|_e.$$

The last equality follows from the fact that the essential norms of T_ϕ and $T_{\hat{\phi}}$ are the same.

Let ϕ_1 denote the harmonic extension of ϕ on D given by

$$\phi_1(z) = \int\limits_T \phi(e^{i\theta})\frac{(1-|z|^2)^2}{|1-\bar{z}e^{i\theta}|^2}\,d\theta.$$

Clearly, $\phi - \phi_1$ vanishes on the unit circle and is continuous on the closure of the unit disk. Thus

$$\|T_\phi\|_e = \|T_{\phi_1}\|_e.$$

Choose a point z_0 on the unit circle such that $|\phi(z_0)| = \|\phi|_T\|_\infty$. Note that for each compact operator K,

$$\begin{aligned}\|T_{\phi_1} + K\| &\geqslant \lim_{z\to z_0}\left|\langle (T_{\phi_1}+K)k_z, k_z\rangle\right| = \lim_{z\to z_0}\left|\phi_1(z) + \langle Kk_z, k_z\rangle\right| \\ &= \left|\phi_1(z_0)\right| = \left|\phi(z_0)\right| = \|\phi|_T\|_\infty.\end{aligned}$$

The second equality comes from that ϕ_1 is harmonic and the third equality follows from that for a compact operator K, Kk_z converges to zero as $|z| \to 1$. This leads to the inequality

$$\|T_\phi\|_e \geqslant \|\phi|_T\|_\infty.$$

On the other hand,

$$\|T_\phi\|_e = \|T_{\phi_1}\|_e \leqslant \|\phi_1\|_\infty = \|\phi|_T\|_\infty.$$

So we conclude that

$$\|\widetilde{T}_\phi\|_e = \|\phi|_T\|_\infty.$$

In [9] it was proved that the commutator ideal of the Toeplitz algebra $\mathcal{T}$ equals the ideal of compact operators. We obtain a short exact sequence about the Toeplitz algebra.

The basic fact about the Toeplitz algebra $\mathcal{T}$ is contained in the following theorem which is analogous to the classical result of Coburn [7,8]. □

Theorem 3.2. *The sequence*

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\pi} C(T) \to 0$$

is a short exact sequence; that is, the quotient algebra $\mathcal{T}/\mathcal{K}$ is $*$-isometrically isomorphic to $C(T)$, where π is the symbol map which maps $\widetilde{T}_\phi + \mathcal{K}$ to $\phi|_T$.

Proof. First we prove that $\mathcal{K} \subset \mathcal{T}$. By Theorem 5 in [5], the commutator $\widetilde{T}_z\widetilde{T}_{z^2} - \widetilde{T}_{z^2}\widetilde{T}_z$ is not zero. By a result [6], the Hankel operator on the Bergman space with continuous symbol on the closure of the unit disk is compact. Since for two functions f and g in $C(\overline{D})$, $T_fT_g - T_{fg}$ is compact on the Bergman space, the commutator $T_fT_g - T_gT_f$ of two Toeplitz operators is compact on the Bergman space. By the matrix representation (1) we have that $\widetilde{T}_z\widetilde{T}_{z^2} - \widetilde{T}_{z^2}\widetilde{T}_z$ is compact. So $\mathcal{T}$ contains a nonzero compact operator. A theorem in [8] states that the commutator ideal of every irreducible algebra contains the ideal of compact operators if it contains a nontrivial compact operator. Thus we need only to prove that $\mathcal{T}$ is irreducible.

Suppose that $\mathcal{T}$ is reducible. Then there exists a nontrivial orthogonal projection P_0 which commutes with each $\widetilde{T}_\phi$ for all ϕ in $C(\overline{D})$. Set $f = P_0 1$. For any function p analytic on D and continuous on the closure of D,

$$\widetilde{T}_p P_0 = P_0\widetilde{T}_p \quad \text{and} \quad \widetilde{T}_{\bar{p}} P_0 = P_0\widetilde{T}_{\bar{p}},$$

thus

$$P_0 p = P_0\widetilde{T}_p 1 = \widetilde{T}_p P_0 1 = Q(pf) = \widetilde{T}_f p$$

and

$$P_0\bar{p} = P_0\widetilde{T}_{\bar{p}} 1 = \widetilde{T}_{\bar{p}} P_0 1 = Q(\bar{p}f) = \widetilde{T}_f(\bar{p}).$$

In particular, letting $p = k_\lambda$ for $\lambda \in D$ we have

$$P_0 k_\lambda = \widetilde{T}_f k_\lambda.$$

Hence

$$\langle P_0 k_\lambda, k_\lambda\rangle = \left\langle \widetilde{T}_f k_\lambda, k_\lambda\right\rangle = \langle f k_\lambda, k_\lambda\rangle.$$

Applying the Cauchy–Schwartz inequality to the first left term in the above equalities gives

$$\left|\langle P_0 k_\lambda, k_\lambda\rangle\right| \leqslant \|P_0 k_\lambda\|\,\|k_\lambda\| \leqslant 1.$$

Then

$$\left|\langle f k_\lambda, k_\lambda\rangle\right| \leqslant 1.$$

But f is in L_h^2, the mean value property for harmonic functions leads to

$$\langle f k_\lambda, k_\lambda\rangle = f(\lambda).$$

This implies that f is in $L^\infty(D, dA)$. Noting that the set of functions $p+\bar{q}$ for polynomials p and q is dense in L_h^2, we obtain

$$P_0 h = \widetilde{T}_f h, \quad \forall h \in L_h^2.$$

This gives that $P_0 = \widetilde{T}_f$. Since $\widetilde{T}_z \widetilde{T}_f = \widetilde{T}_f \widetilde{T}_z$, Theorem 10 in [5] implies that $f = \alpha z + \beta$ for some constants α, β. From $P_0^* = P_0$, we have that $f = \bar{f}$, and hence f is a constant. But this contradicts the assumption that P_0 is a nontrivial projection. Thus we finish the proof of that $\mathcal{T}$ is irreducible.

Let us define the symbol map π given by

$$\pi\bigl(\widetilde{T}_\phi + \mathcal{K}\bigr) \to \phi|_T.$$

By Lemma 3.1, π is well-defined. By the matrix representation (1) of the Toeplitz operator we have that $\widetilde{T}_f \widetilde{T}_g - \widetilde{T}_{fg}$ is compact for f and g in $C(\overline{D})$. Thus Lemma 3.1 implies that π is a $*$-isometrically isomorphic from $\mathcal{T}/\mathcal{K}$ to $C(T)$. For each ϕ in $C(T)$, let ϕ_1 denote the harmonic extension of ϕ on D as in the proof of Lemma 3.1. Define

$$\xi(\phi) = \widetilde{T}_{\phi_1}$$

from $C(T)$ to $\mathcal{T}$. It is easy to check that ξ is an isometric cross section for the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\pi} C(T) \to 0.$$

This gives that the sequence is split to complete the proof of the theorem. □

Lemma 3.3. *Let m be a nonnegative integer. Then $\widetilde{T}_{z^m}$ is an invertible operator on L_h^2.*

Proof. Since $\widetilde{T}_{z^m}$ is Fredholm, we need only to prove

$$\ker \widetilde{T}_{\bar{z}^m} = \ker \widetilde{T}_{z^m} = 0.$$

Let h be in $\ker \widetilde{T}_{\bar{z}^m}$. For any $f \in L_a^2$, we have

$$\langle z^m f, h\rangle = \bigl\langle f, \widetilde{T}_{\bar{z}^m} h\bigr\rangle = 0.$$

This implies that h can be decomposed as $h_1 + \bar{h}_2$, where h_1 is a polynomial with the degree less than m, and h_2 is in zL_a^2. Write $h_1 = \sum_{k=0}^{m-1} a_k z^k$. Note that for $k < m$,

$$\bigl\langle \widetilde{T}_{\bar{z}^m} z^k, z^l\bigr\rangle = \bigl\langle Q(\bar{z}^m z^k), z^l\bigr\rangle = \langle \bar{z}^m z^k, z^l\rangle = \langle z^k, z^{l+m}\rangle = 0$$

and

$$\bigl\langle \widetilde{T}_{\bar{z}^m} z^k, \bar{z}^l\bigr\rangle = \bigl\langle Q(\bar{z}^m z^k), \bar{z}^l\bigr\rangle = \langle \bar{z}^m z^k, \bar{z}^l\rangle = \langle z^{k+l}, z^m\rangle.$$

Thus

$$\widetilde{T}_{\bar{z}^m} z^k = \frac{m-k+1}{m+1} \bar{z}^{m-k}.$$

In addition, $\bar{z}^m \bar{h}_2$ is in L_h^2, so we obtain

$$\widetilde{T}_{\bar{z}^m}(h_1 + \bar{h}_2) = \sum_{k=0}^{m-1} a_k \frac{m-k+1}{m+1} \bar{z}^{m-k} + \bar{z}^m \bar{h}_2.$$

Since $\widetilde{T}_{\bar{z}^m}(h_1 + \bar{h}_2) = 0$, we have $h_1 = h_2 = 0$. This gives that $\ker \widetilde{T}_{\bar{z}^m}$ consists of only zero. Similarly, we have that $\ker \widetilde{T}_{z^m} = \{0\}$. Thus $\widetilde{T}_{z^m}$ is invertible, completing the proof. □

Clearly, Lemma 3.3 implies that $\widetilde{T}_{\bar{z}^m}$ is invertible for nonnegative integer m.

By Theorem 3.2, each operator A in the Toeplitz algebra $\mathcal{T}$ is of the form

$$A = \widetilde{T}_\psi + K,$$

where ψ is a harmonic function in $C(\overline{D})$ and K is compact.

Theorem 3.4. *Let $A \in \mathcal{T}$. If A is Fredholm, then* $\operatorname{ind} A = 0$.

Proof. Suppose that $A = \widetilde{T}_\psi + K$, where ψ is a harmonic function in $C(\overline{D})$ and K is compact. Then $\operatorname{ind} A = \operatorname{ind} \widetilde{T}_\psi$. Since $\widetilde{T}_\psi$ is Fredholm, $\psi|_T$ does not have zero on the unit circle T.

Let $\mathcal{F}$ denote the set of functions in $C(T)$ which $1/f$ is also in $C(T)$. For ϕ_1, ϕ_2 in $\mathcal{F}$, we say that ϕ_1 is homotopic to ϕ_2 if there exists a continuous function

$$F : T \times [0,1] \to C_* \bigl(= C - \{0\}\bigr)$$

such that $F(z,0) = \phi_1(z)$ and $F(z,1) = \phi_2(z)$ for z in T. It is well known that the group of homotopic classes of $\mathcal{F}$ is the integer group Z, and each ϕ is homotopic to some z^n, where n is an integer [8]. To complete the proof, we may assume that $\psi|_T$ is homotopic to z^m for some nonnegative integer m. Otherwise we may consider the case that $\psi|_T$ is homotopic to $\bar{z}^m$ for some nonnegative integer m. Therefore there exists a continuous function

$$F : T \times [0,1] \to C_*$$

such that $F(z,0) = \psi(z)$ and $F(z,1) = z^m$ for z in T. Taking the harmonic extension of $F(z,t)$ with respect to z we may extend F to a continuous function $\widetilde{F} : \overline{D} \times [0,1] \to C$ such that $\widetilde{F}(z,0) = \psi(z)$ and $\widetilde{F}(z,1) = z^m$ for z in $\overline{D}$. Since the Fredholm index ind is integer-valued and continuous, we have

$$\operatorname{ind} A = \operatorname{ind} \widetilde{T}_\psi = \operatorname{ind} \widetilde{T}_{z^m}.$$

By Lemma 3.3, we conclude that $\operatorname{ind} A = 0$. This completes the proof of the theorem. □

In terms of BDF-theory [4], we see that the split short exact sequence in Theorem 3.2 is an extension of $\mathcal{K}$ by $C(T)$. Combining the BDF-theory with

Theorem 3.4, this extension is trivial. The proof of Theorem 3.2 says that $\pi \circ \xi = I$. So, $\mathcal{T}$ can be written as the direct sum of $\mathcal{K}$ and a commutative C^*-subalgebra of $\mathcal{T}$.

4. The Hankel algebra

Recall that the Hankel algebra $\mathcal{H}$ is generated by all Toeplitz operators and little Hankel operators with symbols in $C(\overline{D})$ on the harmonic Bergman space. Since $\mathcal{T} \subset \mathcal{H}$, the ideal $\mathcal{K}$ of compact operators is contained in $\mathcal{H}$.

The next proposition gives the form of operators in $\mathcal{H}$.

Proposition 4.1. *Let $A \in \mathcal{H}$. Then there are functions ϕ_1 and ϕ_2 in $C(\overline{D})$ and a compact operator K such that*

$$A = \widetilde{T}_{\phi_1} + \widetilde{T}_{\phi_2} U + K.$$

Proof. For $f, g \in C(\overline{D})$, first we prove that

$$\big\|\widetilde{T}_f + \widetilde{T}_g U\big\|_e \geqslant \max\big\{\big\|\widetilde{T}_f\big\|_e, \big\|\widetilde{T}_g\big\|_e\big\} = \max\big\{\big\|f|_T\big\|_\infty, \big\|g|_T\big\|_\infty\big\}. \tag{2}$$

In fact, for each compact operator K', we have

$$\begin{aligned}\big\|\widetilde{T}_f + \widetilde{T}_g U + K'\big\| &\geqslant \big\|P\big(\widetilde{T}_f + \widetilde{T}_g U + K'\big)P\big\| \\ &\geqslant \big\|(T_f + \Gamma_g + PK'P)\big\| \\ &\geqslant \|T_f\|_e = \big\|f|_T\big\|_\infty = \big\|\widetilde{T}_f\big\|_e.\end{aligned}$$

The third inequality follows from that Γ_g is compact and the last equality follows from Lemma 3.1. Therefore,

$$\big\|\widetilde{T}_f + \widetilde{T}_g U\big\|_e \geqslant \big\|\widetilde{T}_f\big\|_e.$$

Since

$$\widetilde{T}_f + \widetilde{T}_g U = \big(\widetilde{T}_g + \widetilde{T}_f U\big)U,$$

we obtain

$$\big\|\widetilde{T}_f + \widetilde{T}_g U\big\|_e \geqslant \big\|\widetilde{T}_g\big\|_e.$$

Note that $\widetilde{T}_f U = U\widetilde{T}_{\hat{f}}$, $h_f = \widetilde{T}_f U$, and the commutator of two Toeplitz operators with continuous symbols on L_h^2 is compact. Thus we see that the set $\{\widetilde{T}_f + \widetilde{T}_g U + K\colon f, g \in C(\overline{D}), K \text{ is compact}\}$ is dense in $\mathcal{H}$. For each operator A in the Hankel algebra $\mathcal{H}$, there are functions f_n and g_n in $C(\overline{D})$, and compact operators K_n such that

$$\big\|\widetilde{T}_{f_n} + \widetilde{T}_{g_n} U + K_n - A\big\| \to 0$$

as $n \to \infty$. Thus $\{\widetilde{T}_{f_n} + \widetilde{T}_{g_n}U + K_n\}$ is a Cauchy sequence. The inequality (2) implies that $\{f_n|_T\}$ and $\{g_n|_T\}$ are Cauchy sequences of functions in $C(T)$. Thus f_n and g_n converge to some functions f and g in $C(T)$, respectively. We use ϕ_1 and ϕ_2 to denote the harmonic extensions of f and g on D, respectively. So

$$\left\|\widetilde{T}_{f_n} + \widetilde{T}_{g_n}U - \widetilde{T}_{\phi_1} - \widetilde{T}_{\phi_2}U\right\|_e \to 0.$$

Note that

$$\left\|\widetilde{T}_{f_n} + \widetilde{T}_{g_n}U - A\right\|_e \to 0.$$

Thus we conclude that

$$\left\|\widetilde{T}_{\phi_1} + \widetilde{T}_{\phi_2}U - A\right\|_e = 0.$$

So there is a compact operator K such that

$$A = \widetilde{T}_{\phi_1} + \widetilde{T}_{\phi_2}U + K.$$

This completes the proof of the proposition. □

The proof of Proposition 4.1 also gives that $\widetilde{T}_f + \widetilde{T}_g U$ is compact if and only if both $f|_T = 0$ and $g|_T = 0$.

Theorem 4.2. *Let $f, g \in C(\overline{D})$. Then $\widetilde{T}_f + \widetilde{T}_g U$ is Fredholm if and only if $f\hat{f} - g\hat{g}$ has no zeros on T and in this case we have*

$$\operatorname{ind}\big(\widetilde{T}_f + \widetilde{T}_g U\big) = 0.$$

Proof. Recall the unitary operator

$$\widetilde{U} : L_h^2 = zL_a^2 \oplus \overline{L_a^2} \to zL_a^2 \oplus L_a^2$$

given by

$$\widetilde{U} = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}.$$

Via the unitary operator $\widetilde{U}$, $\widetilde{T}_f + \widetilde{T}_g U$ is unitarily equivalent to the operator:

$$\widetilde{U}\big(\widetilde{T}_f + \widetilde{T}_g U\big)\widetilde{U}^* = \begin{pmatrix} T_f + \Gamma_g & T_g + \Gamma_f \\ T_{\hat{g}} + \Gamma_{\hat{f}} & T_{\hat{f}} + \Gamma_{\hat{g}} \end{pmatrix} + \text{a finite rank operator.}$$

Since $\Gamma_f, \Gamma_g, \Gamma_{\hat{f}}$, and $\Gamma_{\hat{g}}$ are compact, we have that

$$\widetilde{U}\big(\widetilde{T}_f + \widetilde{T}_g U\big)\widetilde{U}^* = \begin{pmatrix} T_f & T_g \\ T_{\hat{g}} & T_{\hat{f}} \end{pmatrix} + \text{a compact operator.}$$

Note that zL_a^2 is a subspace of L_a^2 with codimension one. Hence $\widetilde{T}_f + \widetilde{T}_g U$ is Fredholm if and only if on $L_a^2(D) \oplus L_a^2(D)$, the operator

$$\begin{pmatrix} T_f & T_g \\ T_{\hat{g}} & T_{\hat{f}} \end{pmatrix}$$

is Fredholm. This is equivalent to that the function $f\hat{f} - g\hat{g}$ is invertible on the unit circle T, that is, $f\hat{f} - g\hat{g}$ has no zeros on T.

Assume that the operator $\widetilde{T}_f + \widetilde{T}_g U$ is Fredholm. Then by [11] or [12], we have

$$\operatorname{ind}\big(\widetilde{T}_f + \widetilde{T}_g U\big) = \operatorname{ind}\begin{pmatrix} T_f & T_g \\ T_{\hat{g}} & T_{\hat{f}} \end{pmatrix} = \operatorname{ind} T_{f\hat{f}-g\hat{g}}.$$

Set $\phi = f\hat{f} - g\hat{g}$. Then $\phi(z) = \phi(\bar{z})$. By Theorem 2.3 in [10], we conclude that

$$\operatorname{ind} T_{f\hat{f}-g\hat{g}} = 0,$$

to complete the proof of the theorem. □

From Proposition 4.1 and Theorem 4.2, we see that $\operatorname{ind} A = 0$ for each Fredholm operator A in the Hankel algebra. In fact, this is true in matrix case too. Let M_n denote the algebra of complex $n \times n$ matrices. $\mathcal{H} \otimes M_n$ is the tensor product of the Hankel algebra $\mathcal{H}$ and M_n.

Proposition 4.3. *Let A be in $\mathcal{H} \otimes M_n$. If A is Fredholm, then* $\operatorname{ind} A = 0$.

Proof. By Proposition 4.1, there exists a compact operator K on the Hilbert space $L_h^2 \oplus L_h^2 \oplus \cdots \oplus L_h^2$ (n-direct sum) such that A can be written as

$$A = \widetilde{T}_F + \widetilde{T}_G V + K,$$

where $F = (f_{ij})$, $G = (g_{ij})$ are matrices with elements in $C(\overline{D})$, and $V = \operatorname{diag}(U, U, \ldots, U)$. As the same as the proof of Theorem 4.2, there exists a compact operator K' on $L_a^2 \oplus L_a^2 \oplus \cdots \oplus L_a^2$ ($2n$-direct sum) such that A is unitarily equivalent to $T_L + K'$, where L is a $2n \times 2n$ matrix, $L = (L_{ij})$, and

$$L_{ij} = \begin{pmatrix} f_{ij} & g_{ij} \\ \hat{g}_{ij} & \hat{f}_{ij} \end{pmatrix}.$$

Therefore, A is Fredholm if and only if T_L is Fredholm, and in this case,

$$\operatorname{ind} A = \operatorname{ind} T_L = \operatorname{ind} T_{\det L},$$

where $\det L$ denotes the determinant of the matrix L (see [11] or [12]). So, A is Fredholm if and only if $\det L$ has no zeros on T. It is easy to see $\det L(z) = \det L(\bar{z})$. From [10], we obtain that if A is Fredholm, then

$$\operatorname{ind} A = \operatorname{ind} T_{\det L} = 0.$$

This completes the proof of the proposition. □

To understand the structure of the Hankel algebra better, let us first consider the quotient algebra $\mathcal{H}/\mathcal{K}$. In the proof of Proposition 4.1 we have that $\mathcal{H}/\mathcal{K}$ is generated by operators $\widetilde{T}_f + \mathcal{K}$ and $U + \mathcal{K}$ for $f \in C(\overline{D})$, and $U\widetilde{T}_f = \widetilde{T}_{\hat{f}}U$. Thus the quotient algebra is non-commutative. We will show that the quotient algebra is a crossed product C^*-algebra.

Consider C^*-dynamical system $(C(T), Z_2, \sigma)$, where the action of Z_2 on $C(T)$ is given by

$$\sigma(f)(z) = f(\bar{z}).$$

Then a covariant representation for $(C(T), Z_2, \sigma)$ is a C^*-representation π of $C(T)$ on a Hilbert space H and an idempotent unitary operator U on the same space such that

$$U\pi(f)U^* = \pi\big(\sigma(f)\big), \quad \forall f \in C(T).$$

The crossed product $C(T) \times_\sigma Z_2$ is then defined as the universal C^*-algebra for all covariant representations. Hence, $C(T) \times_\sigma Z_2$ contains a canonical idempotent unitary element δ such that $\delta f = \sigma(f)\delta$, and the linear space

$$\mathcal{D} = \big\{ f + g\delta\colon\ f, g \in C(T) \big\}$$

is a dense $*$-subalgebra of $C(T) \times_\sigma Z_2$.

Theorem 4.4. *The map*

$$\tau : \mathcal{H}/\mathcal{K} \to C(T) \times_\sigma Z_2, \quad \tau\big(\widetilde{T}_f + \widetilde{T}_g U + \mathcal{K}\big) = f|_T + g|_T\delta$$

is an isomorphism of C^*-algebras.

Proof. On the Hilbert space $L^2(T, d\theta)$, let $\mathcal{L}$ denote the C^*-algebra generated by all multiplication operators M_ϕ with continuous symbols and the unitary operator U given by

$$(Uf)(z) = f(\bar{z}).$$

Define the map

$$\gamma : \mathcal{H}/\mathcal{K} \to \mathcal{L}$$

by

$$\gamma\big(\widetilde{T}_f + \widetilde{T}_g U + \mathcal{K}\big) = M_{f|_T} + M_{g|_T}U.$$

By the remark after Proposition 4.1, this map is well defined. It is easy to see that the map is an injective $*$-homomorphism. Therefore by [1, Theorem 1.3.2], γ is isometric. Since the set $\{M_\phi + M_\psi U\colon\ \phi, \psi \in C(T)\}$ is a dense $*$-subalgebra of $\mathcal{L}$, we conclude that the map γ is an isomorphism of C^*-algebras.

Now we consider the covariant representation (π, U) of $(C(T), Z_2, \sigma)$ on $L^2(T, d\theta)$, where $\pi(f) = M_f$. Then there is a canonical surjective C^*-homomorphism

$$\alpha : C(T) \times_\sigma Z_2 \to \mathcal{L},$$

where α maps $f + g\delta$ to $M_f + M_g U$.

First we are going to prove that this homomorphism is injective. Note that

$$\|M_f + M_g U\| \geqslant \max\bigl\{\|f\|_\infty, \|g\|_\infty\bigr\}.$$

The above inequality holds because

$$\|M_f + M_g U\| = \bigl\|\widetilde{T}_{\dot f} + \widetilde{T}_{\dot g} U\bigr\|_e \geqslant \max\bigl\{\|f\|_\infty, \|g\|_\infty\bigr\},$$

where $\dot f$ and $\dot g$ denote the harmonic extensions of f and g to $\overline{D}$, respectively. Now assume that there is a $\zeta \in C(T) \times_\sigma Z_2$ such that $\alpha(\zeta) = 0$. Then there exist a sequence $\{f_n + g_n\delta\}$ converging to ζ, and hence

$$\alpha(f_n + g_n\delta) = M_{f_n} + M_{g_n} U \to 0 \quad (\text{as } n \to \infty).$$

This implies that $\max\{\|f_n\|_\infty, \|g_n\|_\infty\} \to 0$. Since

$$\|f_n + g_n\delta\| \leqslant \|f_n\|_\infty + \|g_n\|_\infty,$$

we have that $\zeta = 0$, and α is injective. This means that α is a C^*-isomorphism. Since $\tau = \alpha^{-1} \circ \gamma$, this insures that

$$\tau : \mathcal{H}/\mathcal{K} \to C(T) \times_\sigma Z_2, \quad \tau\bigl(\widetilde{T}_f + \widetilde{T}_g U + \mathcal{K}\bigr) = f|_T + g|_T\delta$$

is an isomorphism of C^*-algebras. The proof is complete. □

As an immediate corollary of Theorem 4.4, we have a short exact sequence associated with the Hankel algebra.

Corollary 4.5. *The sequence*

$$0 \to \mathcal{K} \to \mathcal{H} \xrightarrow{\tilde\tau} C(T) \times_\sigma Z_2 \to 0$$

is exact, where $\tilde\tau$ maps $\widetilde{T}_f + \widetilde{T}_g U + compact$ to $f|_T + g|_T\delta$.

The above exact sequence enables us to compute K-groups of the Hankel algebra. From [3, (10.11.5) and (6.10.4)], we see that

$$K_0\bigl(C(T) \times_\sigma Z_2\bigr) \cong Z^3 \quad \text{and} \quad K_1\bigl(C(T) \times_\sigma Z_2\bigr) = 0.$$

Noting the fact that

$$K_0(\mathcal{K}) \cong Z \quad \text{and} \quad K_1(\mathcal{K}) = 0,$$

and applying six-term exact sequence (Theorem 9.3.1 on p. 77 in [3]) to

$$0 \to \mathcal{K} \to \mathcal{H} \xrightarrow{\tilde\tau} C(T) \times_\sigma Z_2 \to 0,$$

we have

$$0 \to K_1(\mathcal{H}) \xrightarrow{\tilde\tau_*} 0$$

and

$$0 \to Z \to K_0(\mathcal{H}) \xrightarrow{\tilde{\tau}_*} Z^3 \to 0.$$

So we conclude

$$K_0(\mathcal{H}) \cong Z^4 \quad \text{and} \quad K_1(\mathcal{H}) = 0.$$

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