J. reine angew. Math. **578** (2005), 1—48

Journal für die reine und angewandte Mathematik

Compact perturbations of Hankel operators

By *Xiaoman Chen* and *Kunyu Guo* at Shanghai, *Keiji Izuchi* at Niigata, and *Dechao Zheng* at Nashville

---

Abstract. We give some necessary and sufficient conditions of when the product of two Hankel operators is a compact perturbation of a Hankel operator on the Hardy space. □

For f in L^∞, the Hankel operator with symbol f is the operator H_f on the Hardy space H^2 of the unit circle, defined by $H_f h = P(Ufh)$, for h in H^2. Here P is the orthogonal projection from L^2 onto H^2 and U is a unitary operator on L^2 defined by $Uh(w) = \overline{w}h(\overline{w})$. There are many fascinating problems about the Hankel operator [30], [32]. In this paper we will concentrate on the problem:

Main problem. *For what symbols f, g is the product $H_f H_g$ of two Hankel operators a compact perturbation of a Hankel operator?*

There are many motivations for us to study the above problem. On one hand, the problem involves another important class of operators, Toeplitz operators. The Toeplitz operator induced by the function f in L^∞ is the operator T_f on H^2 defined by $T_f h = P(fh)$. Hankel operators and Toeplitz operators are closely related by the important fact that the product $H_{\tilde{f}} H_g$ of two Hankel operators equals the semicommutator $T_{fg} - T_f T_g$ of two Toeplitz operators. Here $\tilde{f}(w) = f(\overline{w})$. The main problem is more general than and inspired by the problem about semicommutator:

For what symbols f, g is the product $T_f T_g$ of two Toeplitz operators a compact perturbation of a Toeplitz operator?

If $T_f T_g$ is a compact perturbation of the Toeplitz operator T_h, the Douglas symbol mapping [12] gives that h must equal fg. Thus the semicommutator $T_{fg} - T_f T_g$ is compact. The above problem is equivalent to the problem of when the semicommutator is compact, which arose in the Fredholm theory of Toeplitz operators [12], [28], [34]. Fortunately, the semicommutator problem was solved by the combined efforts by Axler, Chang, Sarason and Volberg [2], [37]. They proved the following beautiful result:

$T_{fg} - T_f T_g$ is compact if and only if for each support set S, either $\bar{f}|_S$ or $g|_S$ is in $H^\infty|_S$.

On the other hand, another motivation is the problem of when the product of two Hankel operators equals another Hankel operator. It was shown in [31], [39] that the product of two Hankel operators is rarely a Hankel operator, namely, it is if and only if both operators are scalar multiples of $H_{\overline{\phi_\lambda}}$ for some Blaschke factor $\phi_\lambda = \frac{z-\lambda}{1-\bar{\lambda}z}$ and a number λ in the unit disk D. The product is then also a scalar multiple of $H_{\overline{\phi_\lambda}}$. From the result mentioned above and the Axler-Chang-Sarason-Volberg theorem one may guess that the product of two Hankel operators is a compact perturbation of a Hankel operator if and only if the product is compact. Unfortunately, in Section 3, we will show that there are products of two Hankel operators which are compact perturbations of a noncompact Hankel operator. So the main problem turns out to be quite subtle.

Another motivation is the problem when a Hankel operator is in the Toeplitz algebra, the C^*-algebra generated by bounded Toeplitz operators [5], [6]. The fact that the square of every Hankel operator lies in the Toeplitz C^*-algebra suggests that perhaps the Hankel operators themselves belong. This is the case for positive Hankel operators since they are the unique roots of their squares. So the Hankel operator associated with the Hilbert matrix is in the Toeplitz algebra [6]. But it is not so in general. Axler [31] first observed that it is necessary H_ϕ essentially commutes with the unilateral shift, i.e., $H_\phi T_z - T_z H_\phi$ is compact if H_ϕ is in the Toeplitz algebra. Barría and Halmos [6] asked a natural question whether a Hankel operator is in the Toeplitz algebra if it essentially commutes with the unilateral shift. X. Chen and F. Chen [10] first proved that there are Hankel operators, which essentially commute with the unilateral shift but are not in the Toeplitz algebra. Later such concrete examples of Hankel operators are constructed in [5] and [11]. In Section 4 we will present a concrete example and a short proof of the fact.

In Section 3, we will obtain examples that noncompact Hankel operators are even compact perturbations of a product of two Hankel operators by thin Blaschke products. These examples are inspired by the Volberg solution on Nikolskii's conjectures on bases consisting of rational fractions [37].

For a complex number $z = x + iy$, let us denote by $\Re(z)$ and $\Im(z)$, respectively, the real part x and the imaginary part y of the complex number z. The following theorem is our main result.

Theorem 1. *Suppose that B_1 and B_2 are two inner functions. $H_{\bar{B}_1} H_{\bar{B}_2}$ is a compact perturbation of a Hankel operator if and only if for each support set S_m, one of the following holds*:

(1) *Either $\bar{B}_1|_{S_m}$ or $\bar{B}_2|_{S_m}$ is constant.*

(2) *m is a thin part in the fibre $M_1(H^\infty)$ with the following properties*:

(2a) *m is in the closure of a sequence $\{z_n\}$ in D satisfying*

$$\left|\frac{1-z_n}{1-|z_n|^2}\right| < M$$

for n. Here M is a positive constant.

(2b) $B_1|_{S_m} = cB_2|_{S_m}$ *and* $B_2 \circ \phi_m(\lambda) = \xi\lambda$ *for some unimodular constants c and* ξ.

(2c) *If m is in the closure of some sequence* $\{w_n\} \subset D$, *then*

$$\rho\big(\Re(w_\alpha), w_\alpha\big) \to 0$$

whenever the subnet $\{w_\alpha\}$ *converges to m.*

(3) *m is a thin part in the fibre* $M_{-1}(H^\infty)$ *with the following properties*:

(3a) *m is in the closure of a sequence* $\{z_n\}$ *in D satisfying*

$$\left|\frac{1+z_n}{1-|z_n|^2}\right| < M$$

for n. Here M is a positive constant.

(3b) $B_1|_{S_m} = cB_2|_{S_m}$ *and* $B_2 \circ \phi_m(\lambda) = \xi\lambda$ *for some unimodular constants c and* ξ.

(3c) *If m is in the closure of some sequence* $\{w_n\} \subset D$, *then*

$$\rho\big(\Re(w_\alpha), w_\alpha\big) \to 0$$

whenever the subnet $\{w_\alpha\}$ *converges to m.*

Fibres $M_1(H^\infty)$ and $M_{-1}(H^\infty)$ play the privilege roles in the above theorem since 1 and -1 are the fixed points of the reflection map $z \to \bar{z}$ and the map is used in the definition of the Hankel operator.

Some notation in the above theorem will be introduced in Section 1. The proof of Theorem 1 is long and so it is divided into two parts, in Sections 5 and 6. We will discuss Theorem 1 in Section 4. Many ideas in [2], [18], [23], [22], [37] and [40] are useful for us to study the main problem. Two important properties of thin Blaschke sequences will play an important role in this paper: (1) Sundberg and Wolff ([36]) proved that a sequence is thin interpolating if and only if it is an interpolating sequence for $QA = H^\infty \cap VMO$, where VMO is the space of functions on the unit circle with vanishing mean oscillation; (2) Volberg [37] proved that $\{z_n\}$ is a thin interpolating sequence if and only if $\{k_{z_n}\}$ is a $\mathcal{U} + K_\infty$ basis where k_{z_n} is the normalized reproducing kernel $\dfrac{\sqrt{1-|z_n|^2}}{1-\bar{z}_n z}$.

1. Some notation

Some notation is needed. The unit disk will be denoted by D and the unit circle by ∂D. We shall regard functions in L^2 as extended harmonically into D by means of Poisson's formula:

$$g(z) = \int_{\partial D} g(e^{i\theta}) \frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2} d\sigma(\theta),$$

for $z \in D$. Thus the Poisson integral gives that for each $z \in D$,

$$g \to g(z)$$

is a bounded linear functional on H^2. So there is a function $K_z(w)$ in H^2 such that

$$g(z) = \langle g, K_z \rangle.$$

$K_z(w)$ is called the reproducing kernel of H^2 at z. We use k_z to denote the normalized reproducing kernel at z. In fact, $k_z(w) = \frac{\sqrt{1-|z|^2}}{1-\bar{z}w}$.

By H^∞ we denote the usual Hardy space on ∂D of boundary functions for bounded holomorphic functions in D. The space of continuous functions on ∂D will be denoted by C. The algebra QC is defined by $QC = (H^\infty + C) \cap (\overline{H^\infty + C})$. By QA we denote $QC \cap H^\infty$.

Let B be a commutative Banach algebra. The Gelfand space (space of nonzero multiplicative linear functionals) of the algebra B will be denoted by $M(B)$.

If we think of $M(H^\infty)$ as a subset of the dual of H^∞ with the weak-star topology, then $M(H^\infty)$ becomes a compact Hausdorff space. Explicitly, a net $\{\phi_\alpha\}$ in $M(H^\infty)$ converges to ϕ in $M(H^\infty)$ if and only if

$$\phi_\alpha(f) \to \phi(f) \quad \text{for every } f \in H^\infty.$$

If z is a point in the unit disk D, then the point evaluation at z is a multiplicative linear functional on H^∞, and so we can think of z as an element of $M(H^\infty)$. Carleson's Corona theorem [9] implies that the unit disk D is a dense subset of $M(H^\infty)$.

The maximal ideal space $M(H^\infty)$ of H^∞ is unraveled by interpolating sequences and their Blaschke products. An interpolating sequence is a sequence $\{z_n\}$ in D such that for every bounded sequence $\{c_n\}$ of complex numbers, there is a function $f \in H^\infty$ such that $f(z_n) = c_n$ for every positive integer n. Carleson [8] proved that a sequence $\{z_n\}$ in D is interpolating if and only if

$$\inf_n \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| > 0.$$

For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$, there corresponds a Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D.$$

Blaschke products play an important role in the study of H^∞. A sequence $\{z_n\}_n$ and an associated Blaschke product are called thin if

$$\lim_{n\to\infty} \prod_{k \neq n} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| = 1.$$

If b is a thin Blaschke product with zeros $\{z_n\}_n$, then $|b(z_j)| \to 1$ for every sequence $\{z_j\}_j$ in D satisfying $\rho(z_j, \{z_n\}_n) \to 1$ as $j \to \infty$.

Two important properties of thin Blaschke sequences will play an important role in the paper:

(1) Sundberg and Wolff ([36]) proved that a sequence $\{z_n\}$ is thin interpolating if and only if it is an interpolating sequence for QA, i.e, for each sequence $\{w_n\} \in l^\infty$, there is a function $h \in QA$ such that

$$h(z_n) = w_n.$$

(2) Volberg [37] proved that $\{z_n\}$ is a thin interpolating sequence if and only if $\{k_{z_n}\}$ is a $\mathscr{U} + K_\infty$ basis, i.e., $\{k_{z_n}\}$ is near an orthogonal basis in the following sense:

$$k_{z_n} = (V + K)e_n,$$

where $\{e_n\}$ is the standard orthogonal basis of l^2, V is a unitary operator and K is a compact operator.

A Douglas algebra is, by definition, a closed subalgebra of L^∞ which contains H^∞. If B is a Douglas algebra, then $M(B)$ can be identified with the set of nonzero linear functionals in $M(H^\infty)$ whose representing measures (on $M(L^\infty)$) are multiplicative on B, and we identify the function f with its Gelfand transform on $M(B)$. In particular, $M(H^\infty + C) = M(H^\infty) - D$, and a function $f \in H^\infty$ may be thought of as a continuous function on $M(H^\infty + C)$. A subset of $M(L^\infty)$ is called a support set if it is the (closed) support of the representing measure for a functional in $M(H^\infty + C)$. For each m in the maximal ideal space $M(H^\infty + C)$, we use S_m to denote the support set for m. The fiber of $M(H^\infty)$ above the point λ of ∂D is the set $\{x \in M(H^\infty) : x(z) = \lambda\}$ and will be denoted by $M_\lambda(H^\infty)$.

The pseudohyperbolic distance between two points m_1 and m_2 in $M(H^\infty)$ is given by

$$\rho(m_1, m_2) = \sup\{|f(m_2)| : f \in H^\infty, \|f\| \leqq 1, f(m_1) = 0\}.$$

The Gleason part of a point $m_1 \in M(H^\infty)$, denoted by $P(m_1)$ is given by

$$P(m_1) = \{m : \rho(m, m_1) < 1\}.$$

It is well known that each Gleason part of $M(H^\infty)$ is either one point or an analytic disc. When the Gleason part of m consists of one point, m is said to be a trivial point. Otherwise m is a nontrivial point.

A continuous mapping $F : D \to M(H^\infty)$ is analytic if $f \circ F$ is analytic on D whenever $f \in H^\infty$. An analytic disk in $M(H^\infty)$ is the image $F(D)$ where F is a one-to-one analytic map from D to $M(H^\infty)$. A theorem from the general theory of logmodular algebras implies that each Gleason part of $M(H^\infty)$ is either a one-point part or an analytic disk [25]. For each nontrivial point m, Hoffman [26] constructed a canonical map ϕ_m of the disk D onto the part $P(m)$. This map is defined by taking a net $\{z_\alpha\} \in D$ such that $z_\alpha \to m$, and defining

$$\big(\phi_m(z)\big)(f) = \lim_{\alpha} f\left(\frac{z_\alpha - z}{1 - \overline{z_\alpha} z}\right)$$

for $z \in D$ and $f \in H^\infty$. The above limit exists and is independent of the net $\{z_\alpha\}$, provided that $z_\alpha \to m$. Hoffman [26] showed the following remarkable properties of ϕ_m and analytic disks:

(H0) Let b be an interpolating Blaschke product with the zero sequence $\{z_n\}$ in D. Then m is in $Z_{H^\infty + C}(b)$ if and only if m lies in the closure $\{z_n\}$. Here $Z_{H^\infty + C}(b)$ denotes the zero set $\{m \in M(H^\infty + C) : b(m) = 0\}$ of b in $M(H^\infty + C)$.

(H1) For m in $M(H^\infty) \backslash D$, $P(m)$ is an analytic disk if and only if there is an interpolating sequence whose closure contains m.

(H2) If $z_\alpha \to m$, then for each bounded harmonic function h on D,

$$h \circ \phi_{z_\alpha}(z) \to h \circ \phi_m(z)$$

uniformly on each compact subset of D where $\phi_{z_\alpha}(z) = \dfrac{z_\alpha - z}{1 - \overline{z_\alpha} z}$.

Recall some notation and facts about abstract H^p-theory on a support set. Let m be in $M(H^\infty + C)$ and let $d\mu_m$ denote the unique representing measure for m with support S. That is,

(i) $m(fg) = \int_S fg \, d\mu_m = \int_S f \, d\mu_m \int_S g \, d\mu_m$ for all $f, g \in H^\infty$.

(ii) If h is an a.e. nonnegative function in $L^1(d\mu_m)$ such that $\int_S fh \, d\mu_m = \int_S f \, d\mu_m$ for all $f \in H^\infty$, then $h = 1$ a.e. $d\mu_m$.

Define $H^p(m)$ to be the closure of H^∞ in $L^p(d\mu_m)$. Let $H_m^\infty = \{f \in H^\infty : m(f) = 0\}$ and $H_0^2(m) = \left\{f \in H^2(m) : \int_S f \, d\mu_m = 0\right\}$. Hoffman ([25], page 289) proved that:

(H3) $H^\infty + \overline{H_m^\infty}$ is dense in $L^2(d\mu_m)$.

(H4) $L^2(d\mu_m) = H^2(m) \oplus \overline{H_0^2(m)}$.

2. Hankel operators which are products of two Hankel operators

In this section we present a proof of the result of when the product of two Hankel operators equals a Hankel operator [31], [39].

The relationship between Hankel operators and Toeplitz operators is not just formal but, in fact, rather intimate. To get the relationship, we consider the multiplication operator M_ϕ on L^2 for $\phi \in L^\infty$, defined by

$$M_\phi h = \phi h$$

for $h \in L^2$.

By the property that U is a unitary operator which maps H^2 onto $[H^2]^\perp$ and $UP = (1-P)U$, if M_ϕ is expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus [H^2]^\perp$, the result is of the form

$$M_\phi = \begin{pmatrix} T_\phi & H_{\tilde{\phi}}U \\ UH_\phi & UT_{\tilde{\phi}}U \end{pmatrix}. \tag{1}$$

If f and g are in L^∞, then $M_{fg} = M_f M_g$, and therefore (multiply matrices and compare upper or lower left corners)

$$T_{fg} = T_f T_g + H_{\tilde{f}} H_g \tag{2}$$

and

$$H_{\tilde{f}g} = T_f H_g + H_{\tilde{f}} T_g. \tag{3}$$

The second equality gives that if $\tilde{f}$ is in H^∞, then

$$T_f H_g = H_g T_{\tilde{f}}, \tag{4}$$

for $g \in L^\infty$. The above Hankel and Toeplitz relations have been known before [6], [7], [13] and [31].

Let x and y be two functions in L^2. $x \otimes y$ is the operator of rank one defined by

$$(x \otimes y)(f) = \langle f, y \rangle x,$$

for $f \in L^2$.

Now we are ready to present a proof of the following theorem [31], [39].

Theorem 2. *For three functions f, g, and h in L^∞, $H_f H_g = H_h$ if and only if $H_f = c_f H_{\overline{\phi_\lambda}}$, $H_g = c_g H_{\overline{\phi_\lambda}}$, and $H_h = c_h H_{\overline{\phi_\lambda}}$ for some constants c_f, c_g and c_h and a point $\lambda \in D$.*

Proof. For a fixed λ in D, the long division for the rational function $\phi_\lambda(z) = \dfrac{\lambda - z}{1 - \bar{\lambda} z}$ and zK_λ gives

$$\phi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z} = \frac{1}{\bar{\lambda}} + \left(\lambda - \frac{1}{\bar{\lambda}}\right) K_\lambda$$

and

$$zK_\lambda = \frac{-1}{\bar{\lambda}} + \frac{1}{\bar{\lambda}} K_\lambda.$$

It is easy to verify that $H_{\overline{\phi_\lambda}} = c_\lambda H_{\overline{zK_\lambda}}$. So proving this theorem is equivalent to proving that $H_f H_g = H_h$ if and only if $H_f = c_f H_{\overline{zK_\lambda}}$, $H_g = c_g H_{\overline{zK_\lambda}}$, and $H_h = c_h H_{\overline{zK_\lambda}}$ for some constants c_f, c_g and c_h and a fixed point $\lambda \in D$.

To do this, simply compute to verify that

$$H_{\overline{zK_\lambda}} = K_{\bar{\lambda}} \otimes K_\lambda.$$

Take product of both sides of the above equality to obtain

$$H_{\overline{zK_\lambda}} H_{\overline{zK_\lambda}} = 1/[1-\lambda^2] H_{\overline{zK_\lambda}}.$$

Now we will show that it is only the above case if the product of two Hankel operators is a Hankel operator. Let f, g be co-analytic such that $f(0) = g(0) = 0$, and

$$H_f H_g = H_h.$$

Noting that the commutator $I - T_z T_{\bar{z}}$ of the unilateral shift equals the rank one operator $1 \otimes 1$, we have

$$\begin{aligned} H_f(1 \otimes 1)H_g &= \bar{z}\tilde{f} \otimes \bar{z}\bar{g} \\ &= H_h - H_f T_z T_{\bar{z}} H_g \\ &= H_{h(1-z^2)}. \end{aligned}$$

Thus the Hankel operator $H_{h(1-z^2)}$ is of rank one, and so $\ker H_{h(1-z^2)}$ is an invariant subspace with codimension 1. The Beurling theorem [12] gives that for some $\lambda \in D$,

$$\ker H_{h(1-z^2)} = \{K_\lambda\}^\perp,$$

to obtain

$$\bar{g} = c_1 z K_\lambda.$$

Taking adjoint of $H_{h(1-z^2)}$ gives

$$\bar{z}\bar{g} \otimes \bar{z}\tilde{f} = H_{h^*(1-z^2)},$$

to obtain that for some $\mu \in D$,

$$\bar{f} = c_2 z K_\mu.$$

Use

$$H_{\overline{zK_\lambda}} = K_{\bar{\lambda}} \otimes K_\lambda,$$

to get

$$H_f H_g = \frac{\overline{c_1 c_2}}{1 - \lambda\mu} K_{\bar{\mu}} \otimes K_\lambda.$$

Noting that $H_f H_g = H_h$, we have

$$T_{\bar{z}}[K_{\bar{\mu}} \otimes K_\lambda] = [K_{\bar{\mu}} \otimes K_\lambda] T_z,$$

getting

$$\mu K_{\bar{\mu}} = \lambda K_{\bar{\mu}}.$$

Hence $\lambda = \mu$, to complete the proof.

Remark. An analogous result to the above theorem was obtained in [21] for small Hankel operators on the Bergman space.

3. Noncompact Hankel operators in the Toeplitz algebra

Clearly, compact Hankel operators are in the Toeplitz algebra [12]. In this section we will construct a Hankel operator which is in the Toeplitz algebra but not compact. In fact, we will construct concrete examples that the Hankel operator is a compact perturbation of the product of two Hankel operators. In other words, we obtain examples that the product of two Hankel operators is a compact perturbation of a Hankel operator.

To do this, let $\{x_n\}$ be a thin interpolating sequence on the x-axis such that

$$0 < \prod_{n=1}^{\infty} \delta_n.$$

Here

$$\delta_n = \left| \prod_{m \neq n} \frac{x_m - x_n}{1 - x_m x_n} \right|.$$

Let B be the Blaschke product associated with the sequence $\{x_n\}$. Because those numbers x_n are real numbers, we see that

$$\tilde{B} = \bar{B}.$$

By the interpolating theorem [36], there is a function h in QA such that

$$h(x_n) = \frac{\prod_{m \neq n} \frac{x_m - x_n}{1 - x_m x_n}}{\delta_n},$$

for all n.

Theorem 3. *Suppose that B is the thin Blaschke product defined above. Then $H_{\bar{B}}$ is in the Toeplitz algebra.*

Proof. First we show that $H_{\bar{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{h}}$ is compact.

To estimate $\|[H_{\bar{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{h}}] k_{x_n}\|_2$, we have

$$H_{\bar{B}}T_{\bar{h}}k_{x_n} = H_{\overline{h(x_n)}\bar{B}}k_{x_n} = \bar{h}(x_n)U[\bar{B}k_{x_n}]$$

and

$$H_{\bar{B}}H_{\bar{B}}k_{x_n} = k_{x_n},$$

to obtain

$$[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}]k_{x_n} = k_{x_n} - \bar{h}(x_n)U[\bar{B}k_{x_n}].$$

Thus

$$\begin{aligned}\|[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}]k_{x_n}\|_2^2 &= \|k_{x_n} - \bar{h}(x_n)U[\bar{B}k_{x_n}]\|_2^2 \\ &= 2 - 2h(x_n)\Re(\langle k_{x_n}, U[\bar{B}k_{x_n}]\rangle) \\ &= 2\big(1 - h(x_n)\Re(1 - x_n^2)B'(x_n)\big) \\ &= 2(1 - \delta_n).\end{aligned}$$

The last equality follows from

$$h(x_n) = \frac{\prod_{m \neq n} \frac{x_m - x_n}{1 - x_m x_n}}{\delta_n},$$

and

$$(1 - x_n^2)B'(x_n) = \prod_{m \neq n} \frac{x_m - x_n}{1 - x_m x_n},$$

for all n.

By Theorem 3 in [37], $\{k_{x_n}\}$ is a $\mathscr{U} + K_2$-Riesz basis, that is, there are a unitary operator V and a Hilbert-Schmidt operator K such that

$$k_{x_n} = (V + K)e_n$$

where $\{e_n\}$ is the standard orthogonal basis of l^2. Thus $\{k_{z_n}\}$ is a basis for the kernel of $T_{\bar{B}}$. So for each f in the kernel of $T_{\bar{B}}$, there is a sequence $\{a_n\}$ in l^2 such that

$$f = \sum_{n=1}^{\infty} a_n k_{x_n}$$

and

$$\|f\|_2 \simeq \left[\sum_{n=1}^{\infty} |a_n|^2\right]^{1/2}.$$

Let P_n be the projection from the kernel of $T_{\bar{B}}$ onto the subspace spanned by $\{k_{x_i}\}_{i=1}^n$. Clearly P_n is a compact operator on the kernel of $T_{\bar{B}}$. Now we have that for each $f \in \operatorname{Ker} T_{\bar{B}}$

$$\begin{aligned}\|[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}][I - P_n]f\|_2 &\leqq \sum_{i=n+1}^{\infty} |a_i| \, \|[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}]k_{x_i}\|_2 \\ &\leqq \left[\sum_{i=n+1} |a_i|^2\right]^{1/2} \left[\sum_{i=n+1}^{\infty} \|[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}]k_{x_i}\|_2^2\right]^{1/2} \\ &\leqq C\|f\|_2 \left[\sum_{i=n+1}^{\infty} (1 - \delta_n)\right]^{1/2},\end{aligned}$$

to obtain

$$\|[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}][I - P_n]\| \leqq C\left[\sum_{i=n+1}^{\infty} (1 - \delta_n)\right]^{1/2} \to 0.$$

This shows that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}$ is compact on the kernel $\operatorname{Ker} T_{\bar{B}}$ of $T_{\bar{B}}$.

In order to prove that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}$ is compact, we need only to show that $[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}]$ is compact on BH^2. To do so, letting f_n be a weak convergence sequence in BH^2, we write

$$f_n = Bg_n.$$

Thus g_n is also a weak convergence sequence in H^2. An easy calculation gives that

$$\begin{aligned}[][H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}]f_n &= -H_{\bar{B}}T_{\bar{h}}Bg_n \\ &= -H_{\bar{B}}T_BT_{\bar{h}}g_n - H_{\bar{B}}H_{\tilde{B}}H_{\bar{h}}g_n \\ &= -H_{\bar{B}}H_{\tilde{B}}H_{\bar{h}}g_n \to 0,\end{aligned}$$

to obtain that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}}$ is compact. The last limit comes from that $H_{\bar{h}}$ is compact on H^2. The second equality follows from (2).

Second we show that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}\bar{h}}$ is compact on the Hardy space. Since h is in QA, the Hankel operator $H_{\bar{h}}$ is compact. Using (3), we have

$$H_{\bar{B}\bar{h}} = H_{\bar{B}}T_{\bar{h}} + T_BH_{\bar{h}},$$

getting that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}\bar{h}}$ is compact.

Finally, we show that $H_{\bar{B}} - H_{\bar{B}}H_{\bar{B}h}$ is compact. Noting that $h(x_n)^2 = 1$ for each n, we have that $|h(m)|^2 - 1 = 0$ for each m in the zero set

$$Z_{H^\infty+C}(B) = \{m \in M(H^\infty + C) : B(m) = 0\},$$

to obtain that for such m, $(|h|^2 - 1)|_{S_m} = 0$. By the fact that for each $m \in Z_{H^\infty + C}(B)$, and each $\tilde{m} \in P(m)$, m and $\tilde{m}$ have the same support set, we have

$$\bar{B}(|h|^2 - 1)|_{S_{\tilde{m}}} = \bar{B}(|h|^2 - 1)|_{S_m} = 0.$$

For each $\tilde{m} \in M(H^\infty + C) / \Big[\bigcup_{m \in Z_{H^\infty + C}(B)} P(m)\Big]$, the Hedenmalm result [24] gives that $\bar{B}|_{S_{\tilde{m}}}$ is constant. Since h is in QA, $(|h|^2 - 1)|_{S_m}$ is also constant. Thus $\bar{B}(|h|^2 - 1)|_{S_{\tilde{m}}}$ is constant. So we have proved that for each support set S, $\bar{B}(|h|^2 - 1)|_S$ is constant, getting that $\bar{B}(|h|^2 - 1)$ is in QC. Hence $H_{\bar{B}(|h|^2 - 1)}$ is compact.

On the other hand, by (4), we have

$$H_{\bar{B}|h|^2} = H_{\bar{B}\bar{h}} T_h,$$

to conclude

$$\begin{aligned} H_{\bar{B}} &= -H_{\bar{B}(|h|^2-1)} + H_{\bar{B}|h|^2} \\ &= -H_{\bar{B}(|h|^2-1)} + H_{\bar{B}\bar{h}} T_h \\ &= -H_{\bar{B}(|h|^2-1)} + [H_{\bar{B}\bar{h}} - H_{\tilde{B}} H_{\bar{B}}] T_h + H_{\tilde{B}} H_{\bar{B}} T_h \\ &= H_{\tilde{B}} H_{\bar{B}h} + K \end{aligned}$$

for some compact operator K. This implies that $H_{\bar{B}}$ is in the Toeplitz algebra since $H_{\tilde{B}} H_{\bar{B}h}$ is a semicommutator of two Toeplitz operators and the ideal of compact operators is contained in the Toeplitz algebra to complete the proof.

Remark. From the last part of the above proof, we see that the product $H_{\tilde{B}} H_{\bar{B}h}$ of two Hankel operators is the compact perturbation of the Hankel operator $H_{\bar{B}}$.

4. Discussion on Theorem 1

In this section we first give a proof that there is a Hankel operator not in the Toeplitz algebra even if it essentially commutes with the unilateral shift, which was first shown in [10] and constructed in [5] and [11].

Recall that the Toeplitz algebra is the C^*-algebra generated by bounded Toeplitz operators. It is well known [12] that the ideal $\mathscr{K}$ of compact operators on the Hardy space H^2 is contained in the Toeplitz algebra. First we state some facts, which are known before, e.g., [10].

Fact 1. For $f \in L^\infty$ and $g \in QC$, $T_f T_g - T_g T_f$ is compact.

Hartman's theorem gives that both H_g and $H_{\tilde{g}}$ are compact. By (2), we have

$$T_f T_g - T_g T_f = H_{\tilde{g}} H_f - H_{\tilde{f}} H_g,$$

to obtain that the commutator $T_f T_g - T_g T_f$ is compact.

Fact 2. For each $g \in QC$, $H_f T_g - T_g H_f$ is compact if H_f is in the Toeplitz algebra.

Since the Toeplitz algebra is the C^*-algebra generated by bounded Toeplitz operators, we see that if T is in the Toeplitz algebra, then $TT_g - T_g T$ is compact. This leads to that if the Hankel operator H_f is in the Toeplitz algebra, then $H_f T_g - T_g H_f$ is compact.

Fact 3. For $f \in L^\infty$ and $g \in QC$, the function $f(g - \tilde{g})$ is in $H^\infty + C$ if and only if $H_f T_g - T_g H_f$ is compact.

To do this, use (3) to obtain

$$H_f T_g - T_g H_f = H_{f(g-\tilde{g})} + H_{\tilde{g}} T_f - T_{\tilde{f}} H_g. \tag{5}$$

The Hartman theorem gives that the second and third terms on the right hand side of the above equality are compact, so $H_f T_g - T_g H_f$ is compact if and only if the Hankel operator $H_{f(g-\tilde{g})}$ is compact. By the Hartman theorem again, we have that $H_f T_g - T_g H_f$ is compact if and only if the function $f(g - \tilde{g})$ is in $H^\infty + C$.

Fact 4. For $f \in L^\infty$, put

$$A(f) = \{x \in M(H^\infty + C) : f|S_x \notin H^\infty|S_x\}.$$

Then $H_f T_z - T_z H_f$ is compact if and only if $A(f) \subset M_1(H^\infty + C) \cup M_{-1}(H^\infty + C)$.

This follows from Fact 3.

Fact 5. Let b be a Blaschke product with zeros $\{z_n\}_n$ in D such that $|z_n| \to 1$ as $n \to \infty$. Then $H_{\bar{b}} T_z - T_z H_{\bar{b}}$ is compact if and only if cluster points of $\{z_n\}_n$ in $\overline{D}$ are 1 or -1.

Since $A(\bar{b}) \subset M_1(H^\infty + C)$, this follws from Fact 4.

Fact 6. There exists a function g in QC such that $g - \tilde{g}$ does not vanish on $M_1(H^\infty + C)$.

Lemma 4. *Let $\{z_n\}_n$ be thin. Suppose that $\Im z_n > 0$, $z_n \to 1$, and $\rho(z_n, \bar{z}_n) \to 1$. Then $\{z_n, \bar{z}_n\}_n$ is thin.*

Proof. Write $z_n = x_n + iy_n$ for real numbers x_n and y_n. Then $y_n > 0$. Using

$$\rho(z_n, z_m) = \left|\frac{z_n - z_m}{1 - \bar{z}_n z_m}\right| = \sqrt{\frac{(x_n - x_m)^2 + (y_n - y_m)^2}{(1 - x_n x_m - y_n y_m)^2 + (x_n y_m - y_n x_m)^2}},$$

and

$$\rho(z_n, \bar{z}_m) = \left|\frac{z_n - \bar{z}_m}{1 - \bar{z}_n \bar{z}_m}\right| = \sqrt{\frac{(x_n - x_m)^2 + (y_n + y_m)^2}{(1 - x_n x_m + y_n y_m)^2 + (x_n y_m + y_n x_m)^2}},$$

simply compute to verify

$$\rho(z_n, \bar{z}_m)^2 - \rho(z_n, z_m)^2$$
$$= \frac{4y_n y_m(1 - x_n^2 - y_n^2)(1 - x_m^2 - y_m^2)}{[(1 - x_n x_m - y_n y_m)^2 + (x_n y_m - y_n x_m)^2][(1 - x_n x_m + y_n y_m)^2 + (x_n y_m + y_n x_m)^2]}.$$

Thus

$$\rho(z_n, z_m) \leqq \rho(z_n, \bar{z}_m) \quad \text{for } n \neq m,$$

and so

$$\left|\left(\prod_{k \neq n} \frac{z_k - z_n}{1 - \bar{z}_k z_n}\right)\left(\prod_{k=1}^{\infty} \frac{\bar{z}_k - z_n}{1 - z_k z_n}\right)\right|$$
$$= \left(\prod_{k:k \neq n} \rho(z_n, z_k)\right)\left(\prod_{k:k \neq n} \rho(z_n, \bar{z}_k)\right)\rho(z_n, \bar{z}_n) \geqq \left(\prod_{k:k \neq n} \rho(z_n, z_k)\right)^2 \rho(z_n, \bar{z}_n)$$
$$\to 1$$

as $n \to \infty$. Similarly,

$$\left(\prod_{k:k \neq n} \rho(\bar{z}_n, \bar{z}_k)\right)\left(\prod_{k:k \neq n} \rho(\bar{z}_n, z_k)\right)\rho(\bar{z}_n, z_n) \to 1.$$

Hence $\{z_n, \bar{z}_n\}_n$ is thin.

Example. Let $\{z_n\}_n$ be a sequence given in Lemma 4. By the Sundberg-Wolff interpolation theorem, there is a function g in QA such that $g(z_n) = 1$ and $g(\bar{z}_n) = 0$ for every n. Let m be a cluster point of $\{z_n\}_n$. Then $m \in M_1(H^\infty + C)$ and there exists a subnet $\{z_{n_\alpha}\}_\alpha$ in $\{z_n\}_n$ such that $z_{n_\alpha} \to m$ as $\alpha \to \infty$. We have $g(m) = 1$ and

$$\tilde{g}(m) = \lim_{\alpha \to \infty} \tilde{g}(z_{n_\alpha}) = \lim_{\alpha \to \infty} g(\bar{z}_{n_\alpha}) = 0.$$

Hence $(g - \tilde{g})(m) = 1$.

The above example and facts suggest the following result.

Theorem 5. *There is an interpolating Blaschke product B such that $H_{\bar{B}}$ not only essentially commutes with T_z, but is also not in the Toeplitz algebra.*

Proof. By Fact 6, there exists $g \in QC$ such that $g - \tilde{g}$ does not vanish on $M_1(H^\infty + C)$. Then there exists an interpolating sequence $\{z_n\}_n$ in D and $\delta > 0$ such that $z_n \to 1$ as $n \to \infty$ and $|(g - \tilde{g})(z_n)| \geqq \delta$ for every n. Let B be the Blaschke product with zeros $\{z_n\}_n$. By Fact 5, $H_{\bar{B}}T_z - T_z H_{\bar{B}}$ is compact. Let m be a cluster point of $\{z_n\}_n$. Then $(g - \tilde{g})(m) \neq 0$. Since $\bar{B}|_{S_m} \notin H^\infty|_{S_m}$ and $(g - \tilde{g})|_{S_m}$ is nonzero constant, $\bar{B}(g - \tilde{g}) \notin H^\infty|_{S_m}$. Hence $\bar{B}(g - \tilde{g}) \notin H^\infty + C$. By Fact 3, $H_{\bar{B}}T_g - T_g H_{\bar{B}}$ is not compact. Thus by Fact 2, $H_{\bar{B}}$ is not in the Toeplitz algebra.

Now we discuss the reduction of our main result. By making use of results in [38], Guillory and Sarason [19] proved that for each inner function u, there are a Blaschke product B and an invertible function u_1 in QC such that

$$u = Bu_1.$$

Noting that $u_1|_{S_m}$ is a unimodular constant on each support set S_m and $T_{u_1}T_{\bar{u}_1} - I$, $T_{\bar{u}_1}T_{u_1} - I$, and $H_{\bar{u}} - H_{\bar{B}}T_{\bar{u}_1}$ are compact, we see that it suffices to prove Theorem 1 in the special case that both B_1 and B_2 are Blaschke products. So we assume that B_1 and B_2 are Blaschke products in Sections 5 and 6.

By the Axler, Chang, Sarason and Volberg Theorem, condition (1) in Theorem 1 is just the necessary condition for $H_{\bar{B}_1}H_{\bar{B}_2}$ to be compact.

Axler [31] first observed that it is necessary H_ϕ essentially commutes with the unilateral shift, i.e., $H_\phi T_z - T_z H_\phi$ is compact if H_ϕ is in the Toeplitz algebra. But this commutator is compact only when $H_{\phi(1-z^2)}$ is compact. By Hartman's theorem [34], this occurs only when $(z^2 - 1)\phi$ is in $H^\infty + C$ and this need not hold in general. This observation implies the following two lemmas.

Lemma 6. *Suppose $H_f H_g - H_h$ is compact. Then $H_{(1-z^2)h}$ is compact.*

Proof. By the relationship between the Hankel operators and Toeplitz operators

$$H_f H_g = T_{\tilde{f}g} - T_{\tilde{f}}T_g,$$

we see that the operator T_z^* essentially commutes with $H_f H_g$ because every Toeplitz operators essentially commute with T_z^*. This implies that $T_z^* H_h - H_h T_z^*$ is compact. Using the identity

$$[T_z^* H_h - H_h T_z^*]T_z = H_h(T_z - T_{\bar{z}})T_z = -H_{(1-z^2)h},$$

we obtain the desired result.

Lemma 7. *Suppose m is not in $M_1(H^\infty)$ or $M_{-1}(H^\infty)$. If $H_f H_g - H_h$ is compact, then $h|_{S_m}$ is in $H^\infty|_{S_m}$ and either $g|_{S_m}$ is in $H^\infty|_{S_m}$ or $f^*|_{S_m}$ is in $H^\infty|_{S_m}$.*

Proof. For each point m in neither $M_1(H^\infty)$ nor $M_{-1}(H^\infty)$, we see that $(1 - z^2)|_{S_m}$ is a nonzero constant. Suppose that $H_f H_g - H_h$ is compact. By Lemma 6, $H_{(1-z^2)h}$ is compact. Thus $[H_f H_g - H_h]T_{(1-z^2)}$ is compact, and so $H_f H_{g(1-z^2)}$ is compact. By the Axler-Chang-Sarason-Volberg theorem ([2], [37]), the compactness of $H_f H_{g(1-z^2)}$ implies that either $f^*|_{S_m}$ or $g(1 - z^2)|_{S_m}$ is in $H^\infty|_{S_m}$. Hence either $f^*|_{S_m}$ or $g|_{S_m}$ is in $H^\infty|_{S_m}$. The compactness of $H_{(1-z^2)h}$ implies that $(1 - z^2)h|_{S_m}$ is in $H^\infty|_{S_m}$, to conclude that $h|_{S_m}$ is in $H^\infty|_{S_m}$.

The examples in [5] are based on the following lemma.

Lemma 8 ([5]). *Let $\{a_n\}$ be a Blaschke sequence in the unit disk such that*

$$\lim_{n\to\infty} a_n = 1 \quad \left(\lim_{n\to\infty} a_n = -1\right)$$

and

$$\frac{|1-a_n|}{1-|a_n|} \geqq 2^n \quad \left(\frac{|1+a_n|}{1-|a_n|} \geqq 2^n\right).$$

There is a function f such that:

(A) *f is in QC.*

(B) *$\tilde{f} = -f$.*

(C) *$f(a_n) \to 1$.*

The following lemma gives a necessary condition for $H_f H_g - H_h$ to be compact.

Lemma 9. *If $H_f H_g - H_h$ is compact, then for each support set S and F in QC, $[F - \tilde{F}]h|_S$ is in $H^\infty|_S$ and either $[F - \tilde{F}]g|_S$ or $([F - \tilde{F}]f)^*|_S$ is in $H^\infty|_S$.*

Proof. Let S be a support set and F in QC. The Hartman theorem gives that both H_F and $H_{\tilde{F}}$ are compact. By (3), we have

$$T_F H_f + H_{\tilde{F}} T_f = H_{\tilde{F}f} = T_{\tilde{f}} H_{\tilde{F}} + H_f T_{\tilde{F}},$$

to obtain $T_F H_f - H_f T_{\tilde{F}}$ is compact. Similarly $T_{\tilde{F}} H_g - H_g T_F$ is also compact. Thus

$$T_F H_f H_g - H_f H_g T_F$$

is compact. By the compactness of $H_f H_g - H_h$, we have that $T_F H_h - H_h T_F$ is compact. From (3), we see that both $T_F H_h - H_{\tilde{F}h}$ and $H_h T_F - H_{Fh}$ are compact, getting that $H_{(F-\tilde{F})h}$ is compact. So the Hartman theorem gives that $[F - \tilde{F}]h|_S$ is in $H^\infty|_S$. On the other hand, the compactness of $[H_f H_g - H_h]T_{F-\tilde{F}}$ gives that $H_f H_{(F-\tilde{F})g}$ is compact. By the Axler-Chang-Sarason-Volberg theorem [2], [37], we have that either $(F - \tilde{F})g|_S$ or $f^*|_S$ is in $H^\infty|_S$, to obtain that either $(F - \tilde{F})g|_S$ or $\big((F - \tilde{F})f\big)^*|_S$ is in $H^\infty|_S$. This completes the proof.

The above two lemmas suggest Conditions (2a) and (3a) in Theorem 1. On the other hand, for each thin Blaschke product B and each m in the zero set $Z_{H^\infty+C}(B)$, Hedenmalm [24] showed that

$$B \circ \phi_m(\lambda) = \xi\lambda$$

for a unimodular constant ξ. Those examples in Section 3 suggest Conditions (2b) and (3b).

5. Necessary part

In this section we will prove the necessary part of Theorem 1. By the definition of the Hankel operator, clearly,

$$H_f^* = H_{f^*},$$

where $f^*(w) = \overline{f(\overline{w})}$.

The following lemma follows from a simple computation and will be used later. For a function f in L^2, let $f_+ = Pf$ and $f_- = (I-P)f$. Then $H_f = H_{f_-}$.

Lemma 10. *Suppose that f is in L^∞. For each $z \in D$,*

$$\|H_f^* k_{\bar z}\|_2 = \|H_f k_z\|_2.$$

Proof. For each f in L^∞, $f = f_+ + f_-$. Then $f^* = f_+^* + f_-^*$. Simply compute to verify that for each $z \in D$,

$$\begin{aligned}
\|H_f^* k_{\bar z}\|^2 &= \|H_{f_-}^* k_{\bar z}\|^2 = \|UH_{f_-^*} k_{\bar z}\|^2 \\
&= \|(I-P)f_-^* k_{\bar z}\|^2 = \big\|\big(f_-^* - f_-^*(\bar z)\big)k_{\bar z}\big\|^2 \\
&= \big\|\big(f_- - f_-(z)\big)k_z\big\|^2; \\
\|H_f k_z\|^2 &= \|H_{f_-} k_z\|^2 = \|UH_{f_-} k_z\|^2 \\
&= \|(I-P)f_- k_z\|^2 = \big\|\big(f_- - f_-(z)\big)k_z\big\|^2.
\end{aligned}$$

The last equality follows from

$$P(f_- k_z) = f_-(z) k_z.$$

Combining the above two equalities gives

$$\|H_f^* k_{\bar z}\| = \|H_f^* k_{\bar z}\|,$$

to complete the proof.

Lemma 11. *Suppose that $H_f H_g - H_h$ is compact. Let S be a support set. If either $f^*|_S$ or $g|_S$ is in $H^\infty|_S$, then $h|_S$ is in $H^\infty|_S$.*

Proof. Suppose that S is the support set for a point $m \in M(H^\infty + C)$. If either $f^*|_S$ or $g|_S$ is in $H^\infty|_S$, by [18], Lemma 2.5, we have that either

$$\lim_{z \to m} \|H_g k_z\|_2 = 0,$$

or

$$\lim_{z \to m} \|H_{f^*} k_z\|_2 = 0.$$

From the proof of Lemma 10, we see that

$$\|H_g k_z\|_2 = \big\|\big(g_- - g_-(z)\big)k_z\big\|_2,$$

and

$$\|H_{f^*} k_z\|_2 = \big\|\big(f_-^* - f_-^*(z)\big)k_z\big\|_2.$$

An easy calculation gives

$$\|H_f H_g k_z\|_2 = \left\|P\left(\overline{(f_-^* - f_-^*(z))}\right)(g_- - g_-(z))k_z\right\|_2.$$

Since both f_- and g_- are in BMO, we obtain that

$$\lim_{z\to m} \|H_f H_g k_z\|_2 = 0.$$

On the other hand, by the compactness of $H_f H_g - H_h$, we have

$$\lim_{z\to m} \|[H_f H_g - H_h]k_z\|_2 = 0,$$

getting

$$\lim_{z\to m} \|H_h k_z\|_2 = 0.$$

Thus $h|_S$ is in $H^\infty|_S$.

Lemma 12. *Suppose that m is a point in $M(H^\infty + C)$ and S is the support set for m. If $H_f H_g - H_h$ is compact and neither $f^*|_S$ nor $g|_S$ is in $H^\infty|_S$, then there is a point $\tilde{m}$ in the Gleason part $P(m)$ such that $[F - \tilde{m}(F)]^* f^*|_S$, $[F - \tilde{m}(F)]g|_S$ and $[F - \tilde{m}(F)]h|_S$ are in $H^\infty|_S$ for each $F \in H^\infty$. Moreover, the mapping*

$$m \to \tilde{m}$$

is constant on $P(y)$ for each nontrivial point y.

Proof. Let m be in $M(H^\infty + C)$ and S the support set for m. For each F in H^∞, we have

$$T_{\tilde{F}} H_h = H_{Fh} = H_h T_F.$$

By the compactness of $H_f H_g - H_h$, we see that

$$T_{\tilde{F}} H_f H_g - H_f H_g T_F = H_{Ff} H_g - H_f H_{Fg}$$

is also compact. Thus the main result in [23] implies

$$\lim_{|z|\to 1} \|T_{\phi_z}^* [H_{Ff} H_g - H_f H_{Fg}] T_{\phi_z} - [H_{Ff} H_g - H_f H_{Fg}]\| = 0.$$

An easy calculation gives

$$\begin{aligned} T_{\phi_z}^* [H_{Ff} H_g] T_{\phi_z} &= H_{Ff} T_{\phi_{\bar{z}}} T_{\overline{\phi_{\bar{z}}}} H_g \\ &= H_{Ff} H_g + H_{Ff}[T_{\phi_{\bar{z}}} T_{\overline{\phi_{\bar{z}}}} - 1] H_g \\ &= H_{Ff} H_g - H_{Ff}[k_{\bar{z}} \otimes k_{\bar{z}}] H_g \\ &= H_{Ff} H_g - [H_{Ff} k_{\bar{z}}] \otimes [H_g^* k_{\bar{z}}], \end{aligned}$$

to obtain

$$T_{\phi_z}^*[H_{Ff}H_g - H_f H_{Fg}]T_{\phi_z} - [H_{Ff}H_g - H_f H_{Fg}]$$
$$= [H_f k_{\bar z}] \otimes [H_{Fg}^* k_{\bar z}] - [H_{Ff} k_{\bar z}] \otimes [H_g^* k_{\bar z}].$$

Thus

$$\lim_{|z|\to 1} \|[H_f k_{\bar z}] \otimes [H_{Fg}^* k_{\bar z}] - [H_{Ff} k_{\bar z}] \otimes [H_g^* k_{\bar z}]\| = 0. \tag{6}$$

Since neither $f^*|_S$ nor $g|_S$ is in $H^\infty|_S$, by [18], Lemma 2.5,

$$\underline{\lim}_{z\to m} \|H_{f^*} k_z\|_2 > 0$$

and

$$\underline{\lim}_{z\to m} \|H_g k_z\|_2 > 0.$$

Letting

$$\lambda_z(F) = \frac{\langle H_{Ff} k_{\bar z}, H_g^* k_{\bar z}\rangle}{\|H_g^* k_{\bar z}\|_2^2},$$

we have

$$|\lambda_z(F)| \leqq \|F\|_\infty \frac{\|H_f k_{\bar z}\|_2}{\|H_g^* k_{\bar z}\|_2},$$

to obtain that $\lambda_z(F) \to \tilde m(F)$ for some finite number $\tilde m(F)$ and

$$|\tilde m(F)| \leqq C\|F\|_\infty \tag{7}$$

for some positive constant C.

First we show that $\tilde m$ is in $M(H^\infty + C)$. Apply the operator

$$[H_f k_{\bar z}] \otimes [H_{Fg}^* k_{\bar z}] - [H_{Ff} k_{\bar z}] \otimes [H_g^* k_{\bar z}]$$

to the function $H_g^* k_{\bar z}$, solve for $H_{Ff} k_{\bar z}$ and then use (6) to obtain

$$\lim_{z\to m} \|H_{Ff} k_{\bar z} - \tilde m(F) H_f k_{\bar z}\|_2 = 0.$$

Substituting the above limit in (6) gives

$$\lim_{z\to m} \|H_{Fg}^* k_{\bar z} - \overline{\tilde m(F)} H_g^* k_{\bar z}\|_2 = 0.$$

By Lemma 10, the first limit gives that $\big(F - \tilde m(F)\big)^* f^*|_S$ is also in $H^\infty|_S$, and the second limit gives that $\big(F - \tilde m(F)\big) g|_S$ is in $H^\infty|_S$. Noting that $H_f H_g - H_h$ is compact, we have

$$[H_f H_g - H_h] T_{F-\tilde{m}(F)} = H_f H_{g(F-\tilde{m}(F))} - H_{h(F-\tilde{m}(F))}$$

is also compact, getting that

$$\varliminf_{z\to m} H_{h(F-\tilde{m}(F))} k_z = 0.$$

So $h(F-\tilde{m}(F))|_S$ is in $H^\infty|_S$.

Second we show that $\tilde{m}$ is a bounded linear multiplicative functional on H^∞. Noting that for each F, and G in H^∞,

$$(F-\tilde{m}(F))(G-\tilde{m}(G)) = FG - \tilde{m}(F)(G-\tilde{m}(G)) - \tilde{m}(G)(F-\tilde{m}(F)) - \tilde{m}(F)\tilde{m}(G)$$

we have

$$(FG - \tilde{m}(F)\tilde{m}(G))g|_S \in H^\infty|_S.$$

On the other hand, we also have

$$(FG - \tilde{m}(FG))g|_S \in H^\infty|_S,$$

to obtain that $\tilde{m}(FG) - \tilde{m}(F)\tilde{m}(G) = 0$. Similarly we see that $\tilde{m}$ is linear on H^∞. By (7), we obtain that $\tilde{m}$ is in $M(H^\infty)$.

Third we show that $\tilde{m}$ is in the Gleason part $P(m)$. If this is false, then $\rho(m,\tilde{m}) = 1$. Thus there is a sequence $\{b_k\}$ of functions in the unit ball of H^∞ such that $b_k(m) = 0$ and $b_k(\tilde{m}) \to 1$. Since the unit ball of H^∞ is weakly $*$ compact, we assume that b_k weakly $*$ converges to b in H^∞. Clearly, $b(m) = 0$ and $\|b\| \leqq 1$.

On the other hand, $[f(b_k - \tilde{m}(b_k))]|_{S_m}$ is in $H^\infty|_{S_m}$. Thus for each $H \in H_0^2(m)$,

$$\int_{S_m} f(b_k - \tilde{m}(b_k)) H \, d\mu_m = 0,$$

and $\int_{S_m} f(b-1)H \, d\mu_m$ is a cluster point of

$$\left\{ \int_{S_m} f(b_k - \tilde{m}(b_k)) H \, d\mu_m \right\},$$

and so we have

$$\int_{S_m} f(b-1)H \, d\mu_m = 0$$

to get that $f(b-1)|_{S_m}$ is in $H^\infty|_{S_m}$. From the proof of [23], Lemma 1, we see that $(b-1)$ is an outer function in $H^2(m)$, getting that $f|_{S_m}$ is in $H^\infty|_{S_m}$. This is a contradiction.

Finally, we show that for each nontrivial point y, the mapping $m \to \tilde{m}$ is constant on $P(y)$. If this is false, there are two distinct points $\tilde{m}_1$ and $\tilde{m}_2$ in $P(y)$ such that $g[F - \tilde{m}_i(F)]|_{S_{m_i}}$ is in $H^\infty|_{S_{m_i}}$ for $i = 1, 2$ and each F in H^∞. Since m_1, m_2, and y are in the same Gleason part, they have the same support set. Thus we have that $g[F - \tilde{m}_i(F)]|_{S_y}$ is in $H^\infty|_{S_y}$ for $i = 1, 2$. Noting m_1 does not equal m_2, we see that for some function b in H^∞,

$$\tilde{m}_1(b) \neq \tilde{m}_2(b),$$

getting that

$$g[\tilde{m}_1(b) - \tilde{m}_2(b)]|_{S_y} = g\big[\big(b - \tilde{m}_2(b)\big) - \big(b - \tilde{m}_1(b)\big)\big]|_{S_y}$$

is in $H^\infty|_{S_y}$, so $g|_{S_y}$ is in $H^\infty|_{S_y}$. This is a contradiction, to complete the proof.

Lemma 13. *If $H_f H_g - H_h$ is compact, then for each trivial point m, either $f^*|_{S_m}$ or $g|_{S_m}$ is in $H^\infty|_{S_m}$.*

Proof. Assuming that neither $f^*|_{S_m}$ nor $g|_{S_m}$ is in $H^\infty|_{S_m}$, we will derive a contradiction.

First we show that for each nontrivial point y with $S_y \subset S_m$, either $f^*|_{S_y}$ or $g|_{S_y}$ is in $H^\infty|_{S_y}$.

Suppose that y is a nontrivial point with $S_y \subset S_m$. Thus for some interpolating Blaschke product b_y, $b_y(\tilde{y}) = 0$, but $m(b_y) \neq 0$. Here $\tilde{y}$ is a point in $P(y)$ as in Lemma 12. Noting that $m = \tilde{m}$, by Lemma 12, we have

$$f^*\big[\big(b_y - m(b_y)\big)\big]^*|_{S_m} \in H^\infty|_{S_m}.$$

Now we consider two cases.

In the first case that $|m(b_y)| = 1$, by a lemma [23], we have that $f^*|_{S_m}$ is in $H^\infty|_{S_m}$.

In the second case that $|m(b_y)| < 1$, letting $\lambda = m(b_y)$ and using the function $\phi_\lambda(z) = \dfrac{z - \lambda}{1 - \bar{\lambda} z}$, we have

$$f^*\big(\phi_\lambda(b_y)\big)^*|_{S_m} \in H^\infty|_{S_m},$$

to obtain

$$f^*\big(\phi_\lambda(b_y)\big)^*|_{S_y} \in H^\infty|_{S_y}.$$

We claim that either $f^*|_{S_y}$ or $g|_{S_y}$ is in $H^\infty|_{S_y}$. If this is false, by Lemma 12 we have that $f^*\big[\phi_\lambda(b_y) - \tilde{y}\big(\phi_\lambda(b_y)\big)\big]^*|_{S_y} \in H^\infty|_{S_y}$, to obtain

$$\big[f \tilde{y}\big(\phi_\lambda(b_y)\big)\big]^*|_{S_y} \in H^\infty|_{S_y}.$$

Thus either $f^*|_{S_y}$ is in $H^\infty|_{S_y}$ or

$$0 = \tilde{y}\big(\phi_\lambda(b_y)\big) = \phi_\lambda\big(\tilde{y}(b_y)\big).$$

But the above equation gives

$$\lambda = \tilde{y}(b_y) = 0$$

which contradicts to $\lambda \neq 0$. So either $f^*|_{S_y}$ or $g|_{S_y}$ is in $H^\infty|_{S_y}$, for each nontrivial point y with $S_y \subset S_m$.

Finally, we derive a contradiction. By [17], Corollary 3.2, there is a net $\{y_\alpha\}$ of nontrivial points with $S_{y_\alpha} \subset S_m$ such that

$$y_\alpha \to m.$$

Since either $f^*|_{S_{y_\alpha}}$ or $g|_{S_{y_\alpha}}$ is in $H^\infty|_{S_{y_\alpha}}$, we may assume that $f^*|_{S_{y_\alpha}}$ is in $H^\infty|_{S_{y_\alpha}}$ for each α. For each $H \in H^\infty$,

$$\begin{aligned} 0 = \int f^*\big(H - H(y_\alpha)\big)\,d\mu_{y_\alpha} &= \int f^* H\,d\mu_{y_\alpha} - H(y_\alpha)f^*(y_\alpha) \\ &\to \int f^* H\,d\mu_m - H(m)f^*(m) \\ &= \int f^*\big(H - H(m)\big)\,d\mu_m. \end{aligned}$$

Thus $f^*|_{S_m}$ is in $H^\infty|_{S_m}$. This is a contradiction to complete the proof.

The following lemma is a consequence of the extension of Beurling's invariant subspace theorem ([27], Theorem 20, page 137).

Lemma 14. *If m is a nontrivial point, then there is an inner function Z in $H^\infty(m)$ with $Z(m) = 0$ and*

$$H_0^2(m) = ZH^2(m).$$

Proof. Since m is a nontrivial point, there is a point $\tilde{m}$ in $P(m)$ distinct from m. Thus we can find a function $f \in H^\infty$ such that $m(f) = 0$, but $\tilde{m}(f) \neq 0$. Note that $H_0^2(m)$ is a closed subspace of $H^2(m) = H^2(\tilde{m})$ which is invariant under multiplication by H^∞. Since f is in $H_0^2(m)$ and

$$\tilde{m}(f) = \int_{S_{\tilde{m}}} f\,d\mu_{\tilde{m}} \neq 0,$$

the function 1 is not orthogonal to $H_0^2(m)$ in $L^2(d\mu_{\tilde{m}})$. By Beurling's invariant subspace theorem, we deduce that

$$H_0^2(m) = ZH^2(d\mu_{\tilde{m}}) = ZH^2(d\mu_m)$$

for some inner function Z in $H^2(d\mu_{\tilde{m}}) = H^2(d\mu_m)$, to complete the proof.

Lemma 15. *Suppose that m is a nontrivial Gleason part and B is a Blaschke product. If $[\bar{B} - c_0\bar{Z}]|_{S_m}$ is in $H^\infty|_{S_m}$, for some nonzero constant c_0 and inner function Z in $H_0^2(m)$ satisfying*

$$Z \circ \phi_m(\lambda) = \eta\lambda$$

for some unimodular constant η, then $B|_{S_m} = \bar{c}_0 Z|_{S_m}$.

Proof. Let m be a nontrivial Gleason part. Then the support set S_m is also nontrivial. Since every real-valued function in $H^\infty|_{S_m}$ is constant and $[\bar{B} - c_0\bar{Z}]|_{S_m}$ is in $H^\infty|_{S_m}$, we have

$$B - \bar{c}_0 Z = c_1$$

on S_m for some constant c_1 to obtain

$$B \circ \phi_m(\lambda) = \bar{c}_0 Z \circ \phi_m(\lambda) + c_1 = \bar{c}_0 \eta\lambda + c_1, \tag{8}$$

for $\lambda \in D$.

On the other hand, since $|B| = 1$ on S_m, we have that $|\bar{c}_0 Z + c_1| = 1$ and $|Z| = 1$ on S_m. Noting that Z is not constant on S_m and S_m is nontrivial, we see that the intersection of two circles $|\bar{c}_0\lambda + c_1| = 1$ and $|\lambda| = 1$ contains at least two points, to obtain that the open unit disk $|\lambda| < 1$ contains an open arc of the circle $|\bar{c}_0\lambda + c_1| = 1$ or $c_1 = 0$.

If the open unit disk $|\lambda| < 1$ contains an open arc of the circle $|\bar{c}_0\lambda + c_1| = 1$, by (8) we have that $|B \circ \phi_m(\lambda)| = 1$ for some $\lambda \in D$. But $B \circ \phi_m(\lambda)$ is analytic on the unit disk and $|B \circ \phi_m(\lambda)| \leqq 1$. Thus $B \circ \phi_m(\lambda)$ is constant. This contradicts that c_0 is not zero.

If $c_1 = 0$, then $B = \bar{c}_0 Z$ on S_m. The proof is completed.

Lemma 16. *Suppose that B is a Blaschke product associated with* $\{z_n\}$ *in D. If m is a nontrivial point so that*

$$B \circ \phi_m(\lambda) = \eta\lambda$$

for some unimodular constant η, *then m is in the closure of* $\{z_n\}$.

Proof. Suppose that m is not in the closure of $\{z_n\}$. For $\delta > 0$, set

$$K_\delta(B) = \bigcap_{n=1}^{\infty} \{z : \rho(z, z_n) > \delta\}.$$

According to the Hoffman theorem [26], factor $B = B_1 B_2$ on $K_\delta(B)$ with

$$B_1(m) = B_2(m) = 0,$$

to obtain that

$$\eta\lambda = B_1 \circ \phi_m(\lambda) B_2 \circ \phi_m(z).$$

But

$$B_1(m) = B_2(m) = 0.$$

We conclude that

$$B_1 \circ \phi_m(\lambda) B_2 \circ \phi_m(\lambda) = \lambda^2 h(\lambda)$$

for some analytic function h on D getting

$$\eta = \lambda h(\lambda),$$

for $\lambda \in D$, which is a contradiction.

Lemma 17. *Suppose $1 - |z|$ is sufficiently small. For $z \in D$, if*

$$\frac{|1-z|}{1-|z|^2} \leqq C_1,$$

for some positive constant C_1, then there are positive constants $C_2 > 0$ and $0 < C_3 < 1$ such that

$$\frac{|\theta|}{1-r^2} \leqq C_2,$$

and

$$\rho(x,z) < C_3,$$

where $z = re^{i\theta}$, and $z = x + iy$.

Proof. To write $z = re^{i\theta}$ and $z = x + iy$ in polar coordinate and the Cartesian coordinate, respectively, we have that $x = r\cos\theta$ and $y = r\sin\theta$. Simply compute to verify that

$$\frac{|1-z|}{1-|z|^2} = \frac{\sqrt{(1-r\cos\theta)^2 + (r\sin\theta)^2}}{1-r^2} \tag{9}$$
$$= \sqrt{\frac{1}{(1+r)^2} + \left[\frac{2r\sin\theta/2}{1-r^2}\right]^2},$$

and

$$\rho(x,z) = \left|\frac{x-z}{1-xz}\right| = \left|\frac{iy}{1-x^2-ixy}\right| \tag{10}$$
$$= \frac{|r\sin\theta|}{\sqrt{r^2(2-r^2)\sin^2\theta + 1}}$$
$$\leqq \frac{|r\sin\theta|}{\sqrt{r^2(2-r^2)\sin^2\theta + (1-r^2)^2}}$$
$$= \frac{\frac{|r\sin\theta|}{1-r^2}}{\sqrt{(2-r^2)\left[\frac{|r\sin\theta|}{1-r^2}\right]^2 + 1}}.$$

Let C_1 be the positive constant such that

$$\frac{|1-z|}{1-|z|^2} \leqq C_1.$$

As $1-|z|^2$ is small, (9) gives that $|\theta|$ is small. Thus there is a positive constant C_2, depending only on C_1, such that

$$\frac{|r\sin\theta|}{1-r^2} \leqq C_2.$$

By (10), we see that for some positive constant C_3, depending only on C_2,

$$\rho(x,z) < C_3 < 1,$$

to complete the proof.

The following lemma suggests conditions (2c) and (3c) in Theorem 1.

Lemma 18. *Suppose that $H_f H_g - H_h$ is compact and $\tilde{m}$ is in the closure of the sequence $\{z_n\}$ with the following property*:

$$\rho\big(\Re(z_n), z_n\big) < c$$

for some positive constant $c < 1$. If for the support set $S_{\tilde{m}}$, there are constants c_f, c_g, and c_h and an inner function Z in $H^\infty(m)$ with $Z(\tilde{m}) = 0$, such that $[f - c_f\overline{Z}]^*|_{S_{\tilde{m}}}$, $[g - c_g\overline{Z}]|_{S_{\tilde{m}}}$, and $[h - c_h\overline{Z}]|_{S_{\tilde{m}}}$ are in $H^\infty|_{S_{\tilde{m}}}$, then one of the following holds:

(1) $c_h = 0$ *and either* $c_f = 0$ *or* $c_g = 0$.

(2) $Z \circ \phi_{\tilde{m}}(\lambda) = \xi\lambda$ *for* $\lambda \in D$, *where ξ is a unimodular constant ξ, and*

$$\rho\big(\Re(z_\alpha), z_\alpha\big) \to 0$$

whenever $z_\alpha \to \tilde{m}$.

Proof. Because Z is defined only on the support set $S_{\tilde{m}}$, we can use functions in H^∞ to approximate Z. To simplify the proof, we may assume that Z is in H^∞.

Suppose that $\tilde{m}$ is in the closure of $\{z_n\}$ and $\rho\big(\Re(z_n), z_n\big) < c < 1$. Choose m in the closure of $\{\Re(z_n)\}$ so that $\tilde{m} = \phi_m(z_0)$ for some z_0 in cD. Then m and $\tilde{m}$ are in the same Gleason part and so $S_m = S_{\tilde{m}}$.

Since $H_f H_g - H_h$ is compact,

$$\lim_{z\to\partial D} \|[H_f H_g - H_h]k_z\|_2 = 0.$$

Noting that for the support set $S_{\tilde{m}}$, there are constants c_f, c_g, and c_h and an inner function Z in H^∞ such that $[f - c_f\bar{Z}]^*|_{S_{\tilde{m}}}, [g - c_g\bar{Z}]|_{S_{\tilde{m}}}$, and $[h - c_h\bar{Z}]|_{S_{\tilde{m}}}$ are in $H^\infty|_{S_{\tilde{m}}}$, by [18], Lemma 2.5, we have that for each $y \in P(\tilde{m})$,

$$\lim_{z\to y} \|H_{(g-c_g\bar{Z})}k_z\|_2 = 0,$$

$$\lim_{z\to y} \|H_{(f-c_f\bar{Z})^*}k_z\|_2 = 0,$$

and

$$\lim_{z\to y} \|H_{(h-c_h\bar{Z})}k_z\|_2 = 0,$$

getting that

$$\lim_{z\to y} \|[\bar{c}_f c_g H_{\tilde{Z}} H_{\bar{Z}} - c_h H_{\bar{Z}}]k_z\|_2^2 = 0. \tag{11}$$

For each $z \in D$, evaluate the Hankel operator $H_{\bar{Z}}$ on the normalized reproducing kernel k_z to verify that

$$H_{\tilde{Z}} H_{\bar{Z}} k_z = \big(1 - \bar{Z}(z)Z\big)k_z$$

and

$$H_{\bar{Z}} k_z = [Z^* - \bar{Z}(z)]\bar{w}\tilde{k}_z.$$

Since Z is an inner function in H^∞, we have

$$\big\|\big(1 - \bar{Z}(z)Z\big)k_z\big\|_2^2 = 1 - |Z(z)|^2,$$

and

$$\|[Z^* - \bar{Z}(z)]\bar{w}\tilde{k}_z\|_2^2 = 1 - |Z(z)|^2.$$

By the fact that $1 - |Z(\tilde{m})|^2 = 1$, we have

$$|c_f c_g| = |c_h|,$$

to obtain that there is a unimodular constant η such that $\bar{c}_f c_g \eta = c_h$.

If $c_h = 0$, then either c_f or c_g must be zero. In this case, Condition (1) holds.

If $c_h \neq 0$, use (11) to obtain

$$\lim_{z\to y} \|[\eta H_{\tilde{Z}} H_{\bar{Z}} - H_{\bar{Z}}]k_z\|_2^2 = 0.$$

Thus

$$\lim_{z\to y} \big\|\eta\big(1 - \bar{Z}(z)Z\big)k_z - [Z^* - \bar{Z}(z)]\bar{w}\tilde{k}_z\big\|_2^2 = 0,$$

so we have

$$\lim_{z\to y}[\|\eta k_z + \bar{Z}(z)\bar{w}\tilde{k}_z\|_2^2 - \|\eta\bar{Z}(z)Zk_z - Z^*\bar{w}\tilde{k}_z\|_2^2] = 0.$$

Easy calculations give

$$\lim_{z\to y}\big[\|\eta k_z + \bar{Z}(z)\bar{w}\tilde{k}_z\|_2^2 - \big(1 + |\bar{Z}(z)|^2\big)\big] = 0,$$

and

$$\lim_{z\to y}[\|\eta\bar{Z}(z)Zk_z - Z^*\bar{w}\tilde{k}_z\|_2^2 - [1 + |\bar{Z}(z)|^2 - 2\Re\{\langle\eta\bar{Z}(z)Zk_z, Z^*\bar{w}\tilde{k}_z\rangle\}] = 0,$$

to obtain

$$\lim_{z\to y}\Re\{\langle\eta\bar{Z}(z)Zk_z, Z^*\bar{w}\tilde{k}_z\rangle\} = 0.$$

Now we consider two cases. In the first case that z is a real number, we have

$$\begin{aligned}&\big\|\eta\big(1 - \bar{Z}(z)Z\big)k_z - [Z^* - \bar{Z}(z)]\bar{w}\tilde{k}_z\big\|_2^2\\ &\quad = 2\big[\big(1 - |Z(z)|^2\big) - \big(\Re\{\eta\overline{(Z^*)'(z)}(1 - |z|^2)\} + \Re\{\langle\eta\bar{Z}(z)Zk_z, Z^*\bar{w}\tilde{k}_z\rangle\}\big)\big].\end{aligned}$$

Let $z = \phi_{\Re(z_k)}(\lambda)$ for the fixed real number λ in the unit disk D. Then $\bar{z} = z$ and $z \to \phi_m(\lambda)$, and so

$$\begin{aligned}(Z^*\circ\phi_z)'(0) &\to \overline{(Z\circ\phi_{\phi_m(\lambda)})'}(0),\\ Z(z) = Z\big(\phi_{x_k}(\lambda)\big) &\to Z\big(\phi_m(\lambda)\big).\end{aligned}$$

By

$$\big(1 - |Z(z)|^2\big) + \Re\{\eta[\overline{(Z^*\circ\phi_z)'(0)}]\} = \big(1 - |Z(z)|^2\big) - \Re\{\eta[\overline{(Z^*)'(z)}(1 - |z|^2)]\},$$

we have

$$1 - \big|Z\big(\phi_m(\lambda)\big)\big|^2 = \Re\{\eta\overline{(Z\circ\phi_{\phi_m(\lambda)})'(0)}\},$$

to obtain

$$1 - |F(\lambda)|^2 = \Re\{F'(\lambda)\}(1 - |\lambda|^2),$$

for each real number λ with $|\lambda| < c$. Here $F = -\eta Z\circ\phi_m$. In other words,

$$\frac{1}{1 - |\lambda|^2} = \Re\left\{\frac{F'(\lambda)}{1 - |F(\lambda)|^2}\right\}. \tag{12}$$

Since F is an analytic function and

$$|F(z)| \leqq 1$$

for $z \in D$, the Schwarz Lemma ([15], Lemma 1.2) states that

$$\frac{|F'(z)|}{1-|F(z)|^2} \leqq \frac{1}{1-|z|^2},$$

and the above equality holds at some $z \in D$ if and only if $F(z)$ is a Möbius transformation. By (12), we see that

$$\frac{|F'(\lambda)|}{1-|F(\lambda)|^2} = \frac{1}{1-|\lambda|^2}$$

for real numbers λ with $|\lambda| < c$, to conclude that F is a Möbius transformation. That is,

$$Z \circ \phi_m(\lambda) = \xi \phi_{z_1}(\lambda),$$

for some unimodular constant ξ and a point $z_1 \in D$. Since $\tilde{m} = \phi_m(z_0)$ and $Z(\tilde{m}) = 0$ we have

$$0 = Z \circ \phi_m(z_0) = \xi \phi_{z_1}(z_0),$$

to obtain that $z_1 = z_0$.

Now we show that z_0 is a real number. If this is false, for complex numbers z with $\bar{z} \neq z$, we have

$$\begin{aligned}\|\eta(1-\bar{Z}(z)Z)k_z - [Z^* - \bar{Z}(z)]\overline{w}\tilde{k}_z\|_2^2 \\ = 2\left[(1-|Z(z)|^2) - \left(\Re\left\{\eta\frac{(1-|z|^2}{\bar{z}-z}(Z(\bar{z}) - Z(z))\right\} + \Re\{\langle \eta\bar{Z}(z)Zk_z, Z^*\overline{w}\tilde{k}_z\rangle\}\right)\right].\end{aligned}$$

Let $z = \phi_{\Re z_n}(z_0)$ in the above equality and take the limit as $\Re z_n \to m$ to obtain

$$1 - |Z \circ \phi_m(z_0)|^2 = \Re\left\{\frac{1-|z_0|^2}{\bar{z}_0 - z_0}[Z \circ \phi_m(\bar{z}_0) - Z \circ \phi_m(z_0)]\right\}.$$

Thus

$$1 = \Re\left\{\frac{1-|z_0|^2}{\bar{z}_0 - z_0}\phi_{z_0}(\bar{z}_0)\right\},$$

and so

$$1 = \Re\left\{\frac{1-|z_0|^2}{1-\bar{z}_0^2}\right\}$$

to force that z_0 is real.

Next we show that $z_0 = 0$. To do this, let $x_n = \Re(z_n)$. Noting that $z_n = \phi_{x_n}(\lambda_n)$ and $z_n \to \tilde{m} = \phi_m(z_0)$, we have

$$\lambda_n = \frac{x_n - z_n}{1 - x_n z_n} = \frac{-iy_n}{1 - x_n^2 - ix_n y_n}$$
$$= \frac{x_n\left(\dfrac{y_n}{1-x_n^2}\right)^2 - i\dfrac{y_n}{1-x_n^2}}{1 + x_n^2\left(\dfrac{y_n}{1-x_n^2}\right)^2} \to z_0,$$

getting

$$\Im(\lambda_n) = -\frac{\dfrac{y_n}{1-x_n^2}}{1 + x_n^2\left(\dfrac{y_n}{1-x_n^2}\right)^2} \to 0.$$

Since $\left|\dfrac{y_n}{1-x_n^2}\right| < M$ for some constant M and $x_n \to 1$, we have

$$\frac{y_n}{1-x_n^2} \to 0,$$

to conclude that $\lambda_n \to 0$ and so $z_0 = 0$. This gives that

$$Z \circ \phi_m(\lambda) = \xi\lambda.$$

The above proof also works for any net $w_\alpha \to m_1$ with

$$\sup_\alpha \rho\big(\Re(w_\alpha), w_\alpha\big) < 1.$$

If $w_\alpha \to m_1$, by Lemmas 8 and 9, we may assume that

$$\sup_\alpha \rho\big(\Re(w_\alpha), w_\alpha\big) < 1,$$

since $f^*|_{S_{m_1}}$ is not in $H^\infty|_{S_{m_1}}$.

Now we are ready to prove the main result in this section, which is the necessary part of Theorem 1.

Theorem 19. *Suppose that B_1 and B_2 are Blaschke products.*

If $H_{\bar{B}_1}H_{\bar{B}_2} - H_h$ is compact for some h in L^∞, then for each support set $S(= S_m)$, one of the following holds:

(1) *$h|_S$ is in $H^\infty|_S$ and either $\bar{B}_1|_S$ or $\bar{B}_2|_S$ is in $H^\infty|_S$.*

(2) *m is a thin part in the fibre $M_1(H^\infty)$ with the following properties*:

(2a) *m is in the closure of a sequence* $\{z_n\}$ *in* D *satisfying*

$$\left|\frac{1 - z_n}{1 - |z_n|^2}\right| < M$$

for n. Here M is a positive contradiction.

(2b) $B_1|_{S_m} = cB_2|_{S_m}$, $[h - \bar{c}\bar{Z}]|_{S_m}$ *is in* $H^\infty|_{S_m}$, *and* $B_2 \circ \phi_m(\lambda) = \xi\lambda$ *for some unimodular constants c and* ξ.

(2c) *If m is in the closure of some sequence* $\{w_n\} \subset D$, *then*

$$\rho(\Re(w_\alpha), w_\alpha) \to 0$$

whenever the subnet $\{w_\alpha\}$ *converges to m.*

(3) *m is a thin part in the fibre* $M_{-1}(H^\infty)$ *with the following properties*:

(3a) *m is in the closure of a sequence* $\{z_n\}$ *in* D *satisfying*

$$\left|\frac{1 + z_n}{1 - |z_n|^2}\right| < M$$

for n. Here M is a positive contradiction.

(3b) $B_1|_{S_m} = cB_2|_{S_m}$, $[h - \bar{c}\bar{Z}]|_{S_m}$ *is in* $H^\infty|_{S_m}$, *and* $B_2 \circ \phi_m(\lambda) = \xi\lambda$ *for some unimodular constants c and* ξ.

(3c) *If m is in the closure of some sequence* $\{w_n\} \subset D$, *then*

$$\rho(\Re(w_\alpha), w_\alpha) \to 0$$

whenever the subnet $\{w_\alpha\}$ *converges to m.*

Proof. First we introduce some notation to simplify this proof. Use G_1 to denote the set $\{m \in M(H^\infty + C) : 1 - |B_1(m)|^2 = 0 \text{ or } 1 - |B_2(m)|^2 = 0\}$ and G_2 to denote the set $\{m \in M(H^\infty + C) : 1 - |B_1(m)|^2 > 0 \text{ and } 1 - |B_2(m)|^2 > 0\}$. By a lemma in [18], $G_1 = \{m \in M(H^\infty + C), B_1|_{S_m} \text{ or } B_2|_{S_m} \text{ is constant}\}$.

Suppose that m is a point in $M(H^\infty + C)$. We consider two cases.

In the first case that m is a trivial point, by Lemma 13, thus m is in G_1. So condition (1) holds for the support set S_m, and G_2 does not contain any trivial points.

In the second case that m is a nontrivial point such that condition (1) does not hold, we show that condition (2) or (3) holds. Clearly, m must be in G_2. By Lemma 7, m is in either $M_1(H^\infty)$ or $M_{-1}(H^\infty)$. We consider only the case that m is in $M_1(H^\infty)$. In the case that m is in $M_{-1}(H^\infty)$, the argument below also works.

Assume that for some positive constant γ such that $1 - |B_1(m)|^2 > \gamma$ and $1 - |B_2(m)|^2 > \gamma$. Let $N(m)$ denote the set $\{m_1 \in M(H^\infty) : 1 - |B_1(m_1)|^2 > \gamma/2$ and $1 - |B_2(m_1)|^2 > \gamma/2\}$. Thus $N(m)$ is an open neighborhood of m and the Carleson corona theorem [8] gives that the intersection of $N(m)$ and the unit disk D is dense in $N(m)$. Let $\{z_n\}$ be the intersection of the zeros of B_1 and $N(m) \cap D$. Set $D_k = \left\{ z \in D : \dfrac{|1 - z|}{(1 - |z|^2)} \geqq 2^k \right\}$, and $\mathscr{D}_k$ denotes the closure of D_k in the maximal ideal space of H^∞.

We claim that the intersection of the closure of $\{z_n\}$ in $M(H^\infty)$ and $\bigcap_k \mathscr{D}_k$ is empty. If this is not true, let m_2 be a point in the intersection. Then there is a sequence $\{w_k\}$ with $w_k \in D_k$ such that $\{w_k\}$ captures m. By Lemma 8, there is a function u in QC such that $u = -\tilde{u}$ and $u(w_k) \to 1$. By the compactness of $H_{(u-\tilde{u})\bar{B}_1}$, we see that $(u - \tilde{u})\bar{B}_1|_{S_{m_2}}$ is in $H^\infty|_{m_2}$, getting that $\bar{B}_1|_{S_{m_2}}$ is in $H^\infty|_{S_{m_2}}$, since $|u(m_2)| = 1$. This implies that

$$1 - |B_1(m)|^2 = 0.$$

On the other hand, m_2 is in the closure of $\{z_n\}$ in $M(H^\infty)$ such that $\left(1 - |B_1(z_n)|^2\right) \geqq \frac{\gamma}{2}$, to obtain that

$$1 - |B_1(m_2)|^2 = \lim_{z_n \to m_2} \left(1 - |B_1(z_n)|^2\right) \geqq \frac{\gamma}{2},$$

which is a contradiction.

From now on we assume that $\{z_n\}$ are contained in some D/D_{k_0}. That is,

$$\frac{|1 - z_n|}{1 - |z_n|^2} \leqq 2^{k_0},$$

for every n.

Write $z_n = x_n + iy_n$ where x_n and y_n are real numbers. If we write $z_n = r_n e^{i\theta_n}$, then $x_n = r_n \cos\theta_n$ and $y_n = r_n \sin\theta_n$. By Lemma 17, we have

$$\left|\frac{\theta_n}{1 - r_n^2}\right| \leqq C_2,$$

and

$$\rho(x_n, z_n) = \left|\frac{x_n - z_n}{1 - \bar{z}_n x_n}\right| < C_3 < 1$$

for some positive constants C_2, and C_3 to obtain that the closure of $\{x_n\}$ in the maximal ideal space of H^∞ intersects with the Gleason part $P(m)$.

Let Φ be the Blaschke product associated with $\{z_n\}$. Next we show that m is in the zero set of Φ and Φ is locally thin at each point in $Z_{H^\infty + C}(\Phi)$.

Let m_3 be either m or a point in $Z_{H^\infty+C}(\Phi)$ and S the support set for m_3. Clearly, m_3 is in G_2. That means that neither $\bar{B}_1|_S$ nor $\bar{B}_2|_S$ is in $H^\infty|_S$. By Lemma 12, we have that for some point $\tilde{m}_3$ in the Gleason part $P(m_3)$, and for each $F \in H^\infty$, $[F - \tilde{m}_3(F)]^*\bar{B}_1|_S$, $\big(F - \tilde{m}_3(F)\big)\bar{B}_2|_S$ and $\big(F - \tilde{m}_3(F)\big)h|_S$ are in $H^\infty|_S$. Noting that $\tilde{m}_3$ is in $P(m_3)$, we have that $S_{\tilde{m}_3} = S_{m_3} = S$, getting that by Lemma 14, for some inner function Z in $H^\infty(\tilde{m}_3)$, $Z(\tilde{m}_3) = 0$ and

$$H_0^2(\tilde{m}_3) = ZH^2(\tilde{m}_3).$$

Choose a sequence $\{f_n\} \subset H^\infty$ such that

$$\|f_n - Z\|_{H^2(\tilde{m}_3)} \to 0,$$

to obtain that $f_n(\tilde{m}_3) \to Z(\tilde{m}_3) = 0$. Thus

$$\|\bar{B}_1[f_n - f_n(\tilde{m}_3)] - \bar{B}_1 Z\|_{H^2(\tilde{m}_3)} \to 0,$$

and so $\bar{B}_1 Z|_{S_{\tilde{m}_3}}$ is in $H^\infty|_{S_{\tilde{m}_3}}$, because $\bar{B}_1[f_n - f_n(\tilde{m}_3)]|_{S_{\tilde{m}_3}}$ is in $H^\infty|_{S_{\tilde{m}_3}}$. This implies that for some constant c_1 and function $L_1 \in H^\infty(\tilde{m}_3)$,

$$\bar{B}_1 Z = c_1 + ZL_1$$

on $S_{\tilde{m}_3}$. Therefore we have that $[\bar{B}_1 - c_1\bar{Z}]|_{S_{\tilde{m}_3}}$ is in $H^\infty|_{S_{\tilde{m}_3}}$. Similarly we have that for some constants c_2 and c_h, $[\bar{B}_2 - c_2\bar{Z}]|_{S_{\tilde{m}_3}}$ and $[h - c_h\bar{Z}]|_{S_{\tilde{m}_3}}$ are in $H^\infty|_{S_{\tilde{m}_3}}$. Since $H^\infty(\tilde{m}_3)|_{S_{\tilde{m}_3}}$ does not contain any nonconstant real valued functions, we have $[B_1 - \bar{c}_1 Z]|_{S_{\tilde{m}_3}}$ is constant. Thus we assume Z is in H^∞. Lemma 18 gives

$$Z \circ \phi_{\tilde{m}_3}(\lambda) = \eta\lambda,$$

and Lemma 15 gives that $B_1|_{S_{\tilde{m}_3}} = \bar{c}_1 Z|_{S_{\tilde{m}_3}}$. So we have

$$B_1 \circ \phi_{\tilde{m}_3}(\lambda) = c_1 Z \circ \phi_{\tilde{m}_3}(\lambda) = \bar{c}_1\eta\lambda,$$

to obtain that $\tilde{m}_3 = m_3$ and

$$B_1 \circ \phi_{m_3}(\lambda) = \bar{c}_1\eta\lambda,$$

because $B_1(m_3) = 0$, $Z(m_3) = 0$ and Z has only one zero in $P(m_3)$. By Lemma 16, m_3 is in the closure of $\{z_n\}$. Factor $B_1 = \Phi\Psi$ and

$$\Phi \circ \phi_{m_3}(\lambda) = \lambda\Phi_1(\lambda),$$

for some Blaschke product Ψ and function Φ_1 in H^∞ with

$$|\Phi_1(\lambda)| \leqq 1.$$

Thus we have

$$\bar{c}_1\eta\lambda = B_1 \circ \phi_{m_3}(\lambda) = \Phi \circ \phi_{m_3}(\lambda)\Psi \circ \phi_{m_3}(\lambda) = \lambda\Phi_1(\lambda)\Psi \circ \phi_{m_3}(\lambda),$$

getting that $\Phi_1(\lambda)$ is a constant. Hence

$$\Phi \circ \phi_{m_3}(\lambda) = c\lambda$$

for some unimodular constant c. By [16], Theorem 3.2, Φ is locally thin at m_3. By Lemma 18, we see that $\rho(\Re(z_\alpha), z_\alpha) \to 0$ whenever $z_\alpha \to m$, and $\sup_\alpha \rho(\Re(z_\alpha), z_\alpha) < 1$, to obtain that Φ is a thin Blaschke product and m is a thin part.

Using the same procedure as above, we obtain that

$$B_2|_{S_m} = cB_1|_{S_m} = c_2 Z|_{S_m}$$

and $[h - c_h\overline{B_1}]|_{S_m}$ is in $H^\infty|_{S_m}$ for some unimodular constant c_2, to complete the proof.

6. Sufficient part

In this section, we will present the proof of the sufficient part of Theorem 1.

Suppose that $\{x_n\}$ is a thin Blaschke sequence in D. As in Section 3, define

$$\delta_n = \left| \prod_{m \neq n} \frac{x_m - x_n}{1 - x_m x_n} \right|.$$

By the Sundberg-Wolff interpolation theorem [36], there is a function σ in QA such that

$$\sigma(x_n) = \frac{\prod_{m \neq n} \frac{x_m - x_n}{1 - x_m x_n}}{\delta_n}.$$

For each integer $k > 0$, we write $\tau_k(t)$ for the kth Rademacher function defined on $[0, 1]$ by

$$\tau_k(t) = \operatorname{sign} \sin 2^k \pi t.$$

Clearly, $\{\tau_k\}$ is orthonormal in $L^2[0, 1]$ ([29], [14]). The following theorem is inspired by [37], Lemma 7.

Theorem 20. *Suppose that $\{x_n\}$ is a thin sequence on the real axis and B is a thin Blaschke product associated with $\{x_n\}$. Let $B^{(n)}$ be the Blaschke product associated with the subsequence $\{x_k\}_{k \geqq n}$. If for each factorization $B^{(n)} = B_1 B_2$, $\|H_{\bar{B}_1} H_{\bar{B}_2}\| < \varepsilon$, then for each $\phi = \sum_{k=n}^{\infty} c_k k_{x_k} \in [BH^2]^\perp$,*

$$\|[H_{\bar{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]\phi\|_2^2 \leqq \frac{8\varepsilon}{1 - \varepsilon} \|\phi\|_2^2 + \sum_{k=n}^{\infty} |c_k|^2 \|[H_{\bar{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] k_{x_k}\|_2^2.$$

Proof. Suppose that $\phi = \sum_{k=n}^{\infty} c_k k_{x_k} \in [BH^2]^\perp$, for some sequence $\{c_k\}$ in l^2. For each $t \in [0, 1]$, define

$$L(t)\phi = \sum_{k=n}^{\infty} c_k \tau_k(t)[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] k_{x_k}$$

$$= \sum_{k=n}^{\infty} c_k \tau_k(t)[k_{x_k} - B\overline{w}\overline{k}_{x_k}],$$

where $\{\tau_k(t)\}$ are Rademacher functions. The last equality follows from that as we did in Section 3,

$$[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] k_{x_k} = k_{x_k} - B\overline{w}\overline{k}_{x_k}.$$

For each fixed t in $[0, 1]$, let $\sigma_+ = \{k \geqq n : \tau_k(t) = 1\}$ and $\sigma_- = \{k \geqq n : \tau_k(t) = -1\}$. Let B_+ be the Blaschke product associated with $\{x_k\}_{k \in \sigma_+}$ and B_- the Blaschke product associated with $\{x_k\}_{k \in \sigma_-}$. Thus $B_+ B_-$ is the Blaschke product associated with $\{x_k\}_{k \geqq n}$, and so

$$\|H_{\tilde{B}_+} H_{\bar{B}_-}\| < \varepsilon.$$

Define

$$X_+ = \sum_{k \in \sigma_+} c_k k_{x_k},$$

$$X_- = \sum_{k \in \sigma_-} c_k k_{x_k},$$

$$Y_+ = \sum_{k \in \sigma_+} c_k B\bar{z}\bar{\sigma}(x_k)\overline{k}_{x_k},$$

$$Y_- = \sum_{k \in \sigma_-} c_k B\bar{z}\bar{\sigma}(x_k)\overline{k}_{x_k}.$$

Then

$$L(t)\phi = X_+ + Y_- - (X_- + Y_+)$$

and

$$[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]\phi = X_+ - Y_- + X_- - Y_+.$$

Let P_+ be the projection onto the space spanned by $\{k_{x_k}\}_{k \in \sigma_+}$ and P_- the projection onto the space spanned by $\{k_{x_k}\}_{k \in \sigma_-}$. Since

$$\langle B\bar{z}\overline{k}_{x_k}, k_{x_l} \rangle = 0,$$

for $k \neq l$ and

$$\langle B\bar{z}\overline{k}_{x_k}, k_{x_k} \rangle = (1 - x_k^2) B'(x_k),$$

we have that $X_+ \perp Y_-$ and $X_- \perp Y_+$. An easy calculation gives

$$\|[H_{\tilde{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{\sigma}}]\phi\|_2^2 = \|X_+ - Y_-\|_2^2 + \|X_- - Y_+\|_2^2 + 2\Re\{\langle X_+ - Y_-, X_- - Y_+\rangle\}$$
$$= \|X_+\|_2^2 + \|Y_-\|_2^2 + \|X_-\|_2^2 + \|Y_+\|_2^2$$
$$+ 2\Re\{\langle X_+, X_-\rangle - \langle X_+, Y_+\rangle - \langle Y_-, X_-\rangle + \langle Y_-, Y_+\rangle\},$$

and

$$\|L(t)\phi\|_2^2 = \|X_+ + Y_- - (X_- + Y_+)\|_2^2$$
$$= \|X_+ + Y_-\|_2^2 + \|(X_- + Y_+)\|_2^2 - 2\Re\{\langle X_+ + Y_-, X_- + Y_+\rangle\}$$
$$= \|X_+\|_2^2 + \|Y_-\|_2^2 + \|X_-\|_2^2 + \|Y_+\|_2^2$$
$$- 2\Re\{\langle X_+, X_-\rangle + \langle X_+, Y_+\rangle + \langle Y_-, X_-\rangle + \langle Y_-, Y_+\rangle\}.$$

Thus

$$\|[H_{\tilde{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{\sigma}}]\phi\|_2^2 - \|L(t)\phi\|_2^2 = 4\Re\{\langle X_+, X_-\rangle + \langle Y_-, Y_+\rangle\}$$
$$= 4\Re\{\langle P_+X_+, P_-X_-\rangle + \langle P_-T_{\bar{\sigma}}X_-, P_+T_{\bar{\sigma}}X_+\rangle\},$$

where the last equality comes from that $P_+X_+ = X_+$, $P_-X_- = X_-$, and

$$\langle Y_-, Y_+\rangle = \left\langle \sum_{k\in\sigma_-} c_k B\overline{\sigma(x_k)}\,\overline{zk_{x_k}}, \sum_{k\in\sigma_+} c_k B\overline{\sigma(x_k)}\,\overline{zk_{x_k}}\right\rangle$$
$$= \left\langle \sum_{k\in\sigma_-} c_k\overline{\sigma(x_k)}k_{x_k}, \sum_{k\in\sigma_+} c_k\overline{\sigma(x_k)}k_{x_k}\right\rangle$$
$$= \langle T_{\bar{\sigma}}X_-, T_{\bar{\sigma}}X_+\rangle$$
$$= \langle P_-T_{\bar{\sigma}}X_-, P_+T_{\bar{\sigma}}X_+\rangle$$

because $P_+T_{\bar{\sigma}}X_+ = T_{\bar{\sigma}}X_+$ and $P_-T_{\bar{\sigma}}X_- = T_{\bar{\sigma}}X_-$. So

$$\|[H_{\tilde{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{\sigma}}]\phi\|_2^2 \leqq \|L(t)\phi\|_2^2 + 4[|\langle P_+X_+, P_-X_-\rangle| + |\langle P_+Y_-, P_-Y_+\rangle|]$$
$$\leqq \|L(t)\phi\|_2^2 + 4[\|P_+P_-\|\,\|X_+\|_2\|X_-\|_2 + \|P_+P_-\|\,\|T_{\bar{\sigma}}X_+\|_2\|T_{\bar{\sigma}}X_-\|_2]$$
$$\leqq \|L(t)\phi\|_2^2 + \frac{4\|P_+P_-\|(1 + \|\sigma\|_\infty)}{(1 - \|P_+P_-\|)^2}\|\phi\|_2^2$$

where the last equality follows from

$$\|\phi\|_2^2 = \|X_+ + X_-\|_2^2$$
$$= \|X_+\|_2^2 + \|X_-\|_2^2 + 2\Re\{\langle X_+, X_-\rangle\}$$
$$\geqq \|X_+\|_2^2 + \|X_-\|_2^2 - 2\|P_+P_-\|\,\|X_+\|_2\|X_-\|_2$$
$$\geqq \|X_+\|_2^2 + \|X_-\|_2^2 - \|P_+P_-\|[\|X_+\|_2^2 + \|X_-\|_2^2],$$

and

$$Y_+ = T_{\tilde\sigma} H_{\bar B} X_+, \quad Y_- = T_{\tilde\sigma} H_{\bar B} X_-.$$

Noting

$$P_+ P_- = H_{\tilde B_+} H_{\bar B_+} H_{\tilde B_-} H_{\bar B_-},$$

we have

$$\|[H_{\tilde B} H_{\bar B} - H_{\bar B} T_{\bar\sigma}]\phi\|_2^2 \leqq \|L(t)\phi\|_2^2 + \frac{8\varepsilon}{(1-\varepsilon)^2}\|\phi\|_2^2.$$

Take the integral the both sides of the above inequality, to obtain

$$\|[H_{\tilde B} H_{\bar B} - H_{\bar B} T_{\bar\sigma}]\phi\|_2^2 \leqq \int_0^1 \|L(t)\phi\|_2^2 \, dt + \frac{8\varepsilon}{(1-\varepsilon)^2}\|\phi\|_2^2.$$

Since $\{\tau_k(t)\}$ is orthonormal, we have

$$\int_0^1 \|L(t)\phi\|_2^2 \, dt = \sum_{k=n}^{\infty} |c_k|^2 \|[H_{\tilde B} H_{\bar B} - H_{\bar B} T_{\bar\sigma}] k_{x_k}\|_2^2,$$

to complete the proof.

The following theorem is motivated by examples in Section 3.

Theorem 21. *Suppose that $\{x_n\}$ is a thin sequence on the real axis and B is a thin Blaschke product associated with $\{x_n\}$. Then $H_{\tilde B} H_{\bar B} - H_{\bar B \bar\sigma}$ is a compact operator.*

Proof. First we will show that

$$H_{\tilde B} H_{\bar B} - H_{\bar B} T_{\bar\sigma}$$

is compact on the kernel of $T_{\bar B}$. It is well known [37] that the kernel of $T_{\bar B}$ is spanned by $\{k_{x_n}\}$ and $\{k_{x_n}\}$ is a $\mathscr{U} + K_\infty$ basis. That is, for some unitary operator V and compact operator K from l^2 to the kernel of $T_{\bar B}$,

$$k_{x_n} = (V + K)e_n,$$

where $\{e_n\}$ is the standard orthogonal basis of l^2. Let P_n be the projection from the kernel of $T_{\bar B}$ onto the space spanned by $\{k_{x_k}\}_{k=1}^n$. Clearly, P_n is a compact operator. Let B_n be the Blaschke product associated with the sequence $\{x_k\}_{k>n}$. Then B_n is a thin Blaschke product. By [37], Lemma 6, for any factorization $B_n = B_{n1} B_{n2}$,

$$\inf_{z \in D} \max\{|B_{n1}(z)|, |B_{n2}(z)|\} > \left[\frac{\delta_n}{1 + \sqrt{1 - \delta_n^2}}\right]^2.$$

By the main lemma from [2] ([37], Lemma 5) or a distribution function inequality [40],

$$\|H_{\tilde{B}_{n1}} H_{\bar{B}_{n2}}\| \leqq C_1 \left(\frac{2\sqrt{1-\delta_n^2}}{1+\sqrt{1-\delta_n^2}} \right)^{1/4}$$

for some positive constant C_1, independent of n. As $\delta_n \to 1$, for any $\varepsilon > 0$, choose N so large that for $n > N$,

$$C_1 \left(\frac{2\sqrt{1-\delta_n^2}}{1+\sqrt{1-\delta_n^2}} \right)^{1/4} < \varepsilon.$$

By Theorem 20, for each $\phi = \sum_{k=1}^{\infty} c_k k_{x_k} \in [BH^2]^{\perp}$,

$$\begin{aligned}
\|[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}][I - P_n]\phi\|_2^2 &= \left\| [H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] \Big(\sum_{k>n} c_k k_{x_k} \Big) \right\| \\
&\leqq \frac{8\varepsilon}{1-\varepsilon} \|\phi\|_2^2 + \sum_{k>n}^{\infty} |c_k|^2 \|[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] k_{x_k}\|_2^2 \\
&\leqq \frac{8\varepsilon}{1-\varepsilon} \|\phi\|_2^2 + \max_{k>n} \|[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] k_{x_k}\|_2^2 \sum_{k>n}^{\infty} |c_k|^2.
\end{aligned}$$

On the other hand, for each $k > n$, simply compute to verify that

$$\|[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] k_{x_k}\|_2^2 = 2(1 - \delta_k) \to 0.$$

Thus we have that $[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]$ is compact on the kernel of $T_{\bar{B}}$.

To show that $[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]$ is compact, we need only to show that $[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]$ is compact on BH^2 because of $H^2 = BH^2 \oplus \operatorname{Ker} T_{\bar{B}}$. To do this, for each $\phi \in BH^2$, write $\phi = B\psi$ for some $\psi \in H^2$. Define a bounded linear operator V from BH^2 to H^2 by

$$V\phi = \psi.$$

Then

$$\begin{aligned}
[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]\phi &= [H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}] B\psi \\
&= -H_{\bar{B}} T_{\bar{\sigma}} T_B \psi \\
&= -H_{\bar{B}} T_{\bar{\sigma}} T_B V\phi \\
&= -[H_{\bar{B}} T_B T_{\bar{\sigma}} - H_{\bar{B}} H_{\tilde{B}} H_{\bar{\sigma}}] V\phi \\
&= -H_{\bar{B}} H_{\tilde{B}} H_{\bar{\sigma}} V\phi,
\end{aligned}$$

where the third equality follows from (2). Noting that $H_{\bar{\sigma}}$ is compact on H^2, we see that $H_{\bar{B}} H_{\tilde{B}} H_{\bar{\sigma}} V$ is compact, getting that $[H_{\tilde{B}} H_{\bar{B}} - H_{\bar{B}} T_{\bar{\sigma}}]$ is compact.

Next, we show that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}\bar{\sigma}}$ is compact. By (3), we have

$$H_{\bar{B}\bar{\sigma}} = H_{\bar{B}}T_{\bar{\sigma}} + T_B H_{\bar{\sigma}}.$$

By the compactness of $H_{\bar{\sigma}}$, we conclude that $H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}\bar{\sigma}}$ is compact, to complete the proof.

Lemma 22. *Suppose that $\{z_n\}$ is a thin sequence and B_2 is the Blaschke product associated with $\{z_n\}$ and B the Blaschke product associated with $\{x_n\}$. If $\rho(x_n, z_n) \to 0$, then there is a function ψ in QC such that $B_2 = B\psi$.*

Proof. Since $\rho(x_n, z_n) \to 0$, $Z_{H^\infty+C}(B_2) = Z_{H^\infty+C}(B)$. By the result in [3] and [20], $B_2 = B\psi$ for some function $\psi \in H^\infty + C$. In order to show that ψ is in QC, we need to show that for each support set S_m, $\psi|_{S_m}$ is constant.

If m is in $M(H^\infty + C)$ but not in $Z_{H^\infty+C}(B_2)$, by noting that B and B_2 are thin Blaschke products, by [24], Proposition 2.3, we have that $B_2|_{S_m}$ and $B|_{S_m}$ are unimodular constants, getting that $\psi|_{S_m}$ is constant.

If m is in $Z_{H^\infty+C}(B_2)$, by [24], Proposition 2.3 again, we have that $|(B \circ \phi_m)'(0)| = |(B_2 \circ \phi_m)'(0)| = 1$. Since $\{x_n\}$ is a thin sequence, by the Sundberg-Wolff interpolating theorem [36], we have that for two functions h_1 and h_2 in QA,

$$h_1(x_n) = \frac{|(B \circ \phi_{x_n})'(0)|}{(B \circ \phi_{x_n})'(0)}$$

and

$$h_2(x_n) = \frac{|(B_2 \circ \phi_{x_n})'(0)|}{(B_2 \circ \phi_{x_n})'(0)}.$$

Easy calculations give

$$[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}_1}]k_{x_n} = [k_{x_n} - \bar{h}_1(x_n)\bar{\bar{B}}w\overline{k_{x_n}}]$$

and

$$[H_{\bar{B}_2}H_{\bar{B}_2} - H_{\bar{B}_2}T_{\bar{h}_2}]k_{x_n} = \left[\left(1 - \bar{B}_2(x_n)B\right)k_{x_n} - \bar{h}_2(x_n)\left(\bar{\bar{B}}_2 - \overline{B_2(x_n)}\right)w\overline{k_{x_n}}\right].$$

Thus

$$\|[H_{\bar{B}}H_{\bar{B}} - H_{\bar{B}}T_{\bar{h}_1}]k_{x_n}\|_2^2 = 2[1 - |(B \circ \phi_{x_n})'(0)|] \to 0$$

and

$$\|[H_{\bar{B}_2}H_{\bar{B}_2} - H_{\bar{B}_2}T_{\bar{h}_2}]k_{x_n}\|_2^2 = 2[1 - |(B_2 \circ \phi_{x_n})'(0)|] + o(1 - |x_n|^2) \to 0.$$

So

$$\|[H_{\bar{B}}T_{\bar{h}_1} - H_{\bar{B}_2}T_{\bar{h}_2}]k_{x_n}\|_2^2 \to 0.$$

Since h_1 and h_2 are in QA, we have

$$\|H_{\overline{Bh_1}-\overline{B_1h_2}}k_{x_n}\|_2 \to 0,$$

to obtain that for each $m \in Z_{H^\infty+C}(B)(= Z_{H^\infty+C}(B_2))$, $[\overline{Bh_1} - \overline{B_1h_2}]|_{S_m}$ is in $H^\infty|_{S_m}$. Noting that $h_1|_{S_m}$ and $h_2|_{S_m}$ are unimodular constants, we have

$$[\overline{Bh_1} - \overline{B_1h_2}]|_{S_m} = [\overline{Bh_1} - \overline{B\psi h_2}]|_{S_m},$$

getting that $\bar{\psi}|_{S_m}$ is in $H^\infty|_{S_m}$. Hence $\psi|_{S_m}$ is constant. This completes the proof.

Theorem 23. *Suppose that $\{z_n\}$ is a thin sequence and B_2 is the Blaschke product associated with $\{z_n\}$. If $\rho(\Re(z_n), z_n) \to 0$, then $H_{\tilde{B}_2}H_{\bar{B}_2}$ is a compact perturbation of the Hankel operator $H_{\bar{B}_2h}$ for some function $h \in QC$.*

Proof. Let $x_n = \Re(z_n)$. Let B be the Blaschke product associated with $\{x_n\}$. By Lemma 22, we have that $B_2 = B\psi$ for some function ψ in QC, to obtain

$$H_{\tilde{B}_2}H_{\bar{B}_2} = T_{\tilde{\psi}}H_{\tilde{B}}H_{\bar{B}}T_{\bar{\psi}} + K$$

for some compact operator K since ψ is in QC. On the other hand, by Theorem 23, we see that for some compact operator K_1 and function $\sigma \in QA$,

$$H_{\tilde{B}}H_{\bar{B}} = H_{\bar{B}\bar{\sigma}} + K_1,$$

getting that

$$H_{\tilde{B}_2}H_{\bar{B}_2} = T_{\tilde{\psi}}H_{\bar{B}\bar{\sigma}}T_{\bar{\psi}} + K_2.$$

Here K_2 is a compact operator. So we conclude that

$$H_{\tilde{B}_2}H_{\bar{B}_2} = H_{\tilde{\psi}\bar{B}\bar{\sigma}\bar{\psi}} + K_3$$

for some compact operator K_3.

Now we are ready to give the proof of the sufficient part of Theorem 1.

Theorem 24. *Suppose that B_1 and B_2 are two Blaschke products. $H_{\tilde{B}_1}H_{\bar{B}_2}$ is a compact perturbation of a Hankel operator if for each support set S_m, one of the following holds:*

(1) *Either $\bar{B}_1|_{S_m}$ or $\bar{B}_2|_{S_m}$ is in $H^\infty|_{S_m}$.*

(2) *m is a thin part in the fibre $M_1(H^\infty)$ with the following properties:*

(2a) *m is in the closure of a sequence $\{z_n\}$ in D satisfying*

$$\left|\frac{1 - z_n}{1 - |z_n|^2}\right| < M$$

for n. Here M is a positive constant.

(2b) *$B_1|_{S_m} = cB_2|_{S_m}$ and $B_2 \circ \phi_m(\lambda) = \xi\lambda$ for some unimodular constants c and ξ.*

(2c) *If m is in the closure of some sequence $\{w_n\} \subset D$, then*

$$\rho(\Re(w_\alpha), w_\alpha) \to 0$$

whenever the subnet $\{w_\alpha\}$ converges to m.

(3) *m is a thin part in the fibre $M_{-1}(H^\infty)$ with the following properties:*

(3a) *m is in the closure of a sequence $\{z_n\}$ in D satisfying*

$$\left|\frac{1+z_n}{1-|z_n|^2}\right| < M$$

for n. Here M is a positive constant.

(3b) *$B_1|_{S_m} = cB_2|_{S_m}$ and $B_2 \circ \phi_m(\lambda) = \xi\lambda$ for some unimodular constants c and ξ.*

(3c) *If m is in the closure of some sequence $\{w_n\} \subset D$, then*

$$\rho(\Re(w_\alpha), w_\alpha) \to 0$$

whenever the subnet $\{w_\alpha\}$ converges to m.

Proof. Suppose that B_1 and B_2 satisfy the conditions in the theorem. We will show that $H_{\bar{B}_1}H_{\bar{B}_2}$ is a compact perturbation of a Hankel operator.

For a bounded operator T on the Hardy space H^2, $\operatorname{dist}(T, \mathscr{K})$ denotes the distance from T to the ideal $\mathscr{K}$ of compact operators, given by

$$\operatorname{dist}(T, \mathscr{K}) = \inf_{K \in \mathscr{K}} \|T - K\|.$$

We shall show that for each sufficiently small $\varepsilon >$, there is a function $g \in L^\infty$ such that

$$\operatorname{dist}(H_{\bar{B}_1}H_{\bar{B}_2} - H_g, \mathscr{K}) < 100\varepsilon^{1/4}.$$

To do this, set

$$O_\varepsilon^+ = \{z \in D : 1 - |\tilde{B}_1(z)| > \varepsilon\} \cap \{z \in D : |1 - z| < \varepsilon\}.$$

Let $\{z_n\}$ be zeros of B_2 in O_ε^+ and B the Blaschke product associated with $\{z_n\}$. By condition (2b), we see that for each m in the closure of $\{z_n\}$ in $M(H^\infty)$, neither $B_2|_{S_m}$ nor $B_1|_{S_m}$ is constant.

Claim that

$$\left|\frac{1-z_n}{1-|z_n|^2}\right| < M$$

for some positive constant M. If it is false, by Lemma 9, there are a function $F = -\tilde{F}$ in QC and a point m in the closure of $\{z_n\}$ such that $(F - \tilde{F})\bar{B}_2|_{S_m}$ or $(F - \tilde{F})\bar{B}_1|_{S_m}$ is in $H^\infty|_{S_m}$, and $F|_{S_m} = 1$. This implies that $\bar{B}_1|_{S_m}$ or $\bar{B}_2|_{S_m}$ is constant, which is a contradiction.

Factor $B_2 = BB_3$. Claim the following:

(c1) B is thin.

(c2) $B_1 = B\psi$ for some function ψ in $H^\infty + C$.

(c3) Both B_3 and ψ are unimodular constants on the support set S_m for each $m \in Z_{H^\infty+C}(B)$.

To prove the above claims, for each $m \in Z_{H^\infty+C}(B)$, by condition (2), we have that for some sequence $\{w_n\}$ in the unit disk D satisfying

$$\left|\frac{1 - w_n}{1 - |w_n|^2}\right| < M,$$

m is in the closure of $\{w_n\}$ and

$$B_2 \circ \phi_m(\lambda) = \xi\lambda$$

for some unimodular number ξ. Use that $B_2(m) = B(m)B_3(m) = 0$, to obtain

$$B \circ \phi_m(z) = zh(z)$$

and

$$\begin{aligned} \xi z &= B_2 \circ \phi_m(z) \\ &= B \circ \phi_m(z) B_3 \circ \phi_m(z) \\ &= B_3 \circ \phi_m(z) z h(z), \end{aligned}$$

for $z \in D$, where h is analytic function on D and $|h(z)| \leqq 1$. This gives

$$\xi = h(z) B_3 \circ \phi_m(z).$$

Thus $|B_3 \circ \phi_m(z)|$ reaches its maximal value at some point in the unit disk, and so both $B_3 \circ \phi_m(z)$ and $h(z)$ are unimodular constant. We have that $B_3 \circ \phi_m(z) = \gamma$ and

$$B \circ \phi_m(z) = \xi\bar{\gamma}z,$$

for some unimodular constant γ. Noting

$$B_3(m) = \int_{S_m} B_3 \, d\mu_m$$

and $|B_3| = 1$ on S_m, we obtain that $B_3|_{S_m}$ equals γ, and B is locally thin at m. By [16], Theorem 3.2, B is a thin Blaschke product. By condition (2b), we see that $B_1|_{S_m} = cB_2|_{S_m} = c\gamma B|_{S_m}$. Thus this implies

$$|B_1(m)| \leqq |B(m)|$$

for all $m \in Z_{H^\infty+C}(B)$. If m is in $M(H^\infty + C)$, but not in the Gleason part $P(\tilde{m})$ for some $\tilde{m} \in Z_{H^\infty+C}(B)$, $B|_{S_m}$ is a unimodular constant [24]. Thus

$$|B_1(m)| \leqq 1 = |B(m)|.$$

By a theorem [3] and [20], factor $B_1 = \psi B$ for some function ψ in $H^\infty + C$.

To finish the proof of our claims, we need only show that $\psi|_{S_m}$ is a unimodular constant for each $m \in Z_{H^\infty+C}(B)$. To do this, let $m \in Z_{H^\infty+C}(B)$. As we showed above, $B_2|_{S_m} = c\gamma B|_{S_m}$. Thus $B\psi|_{S_m} = c\gamma B|_{S_m}$, and so $\psi|_{S_m} = c\gamma$.

Replace O_ε^+ by

$$O_\varepsilon^- = \{z \in D : 1 - |B_1(z)| > \varepsilon\} \cap \{z \in D : |1 + z| < \varepsilon\},$$

in the above process to obtain similar factorization of B_1 and B_2. Since the fibre $M_1(H^\infty)$ is disjoint from the fibre $M_{-1}(H^\infty)$, the product of two thin Blaschke products with zeros converging to 1 and -1 respectively is still a thin Blaschke product. For sake of simplicity, use the same notation as above, to obtain that

$$B_1 = \psi B$$

and

$$B_2 = B_3 B,$$

which satisfy:

(a) B is a thin Blaschke product with zeros in $O_\varepsilon^+ \cup O_\varepsilon^-$ converging to either 1 or -1.

(b) Both $B_3|_{S_m}$ and $\psi|_{S_m}$ are unimodular constants for $m \in Z_{H^\infty+C}(B)$.

Now we shall show that:

(c)
$$\lim_{|z|\to 1,\, z \in O_\varepsilon^+ \cup O_\varepsilon^-} |B_3(z)| = 1,$$

and

$$\lim_{|z|\to 1,\, z \in O_\varepsilon^+ \cup O_\varepsilon^-} |\psi(z)| = 1.$$

(d) For each $m \in Z_{H^\infty+C}(B)$, and $m_1 \in P(m)$,

$$(\psi - \tilde{\psi})\bar{B}|_{S_{m_1}} = 0.$$

(e) $T_\psi H_{\bar{B}} - H_{\bar{B}} T_\psi$ is compact.

First we prove (c) by showing that the first limit holds. Similarly we can show that the second limit also holds. If (c) is false, then we assume that for some point m in $M(H^\infty + C)$, m is in the closure of O_ε^+ or O_ε^- in $M(H^\infty + C)$ and $|B_3(m)| < 1$. Thus $B_3|_{S_m}$ is not constant, and so $B_2|_{S_m}$ is not constant. For sake of simplicity, we assume that m is in the closure of O_ε^+. This gives that $B_1|_{S_m}$ is not constant. Thus condition (1) does not hold. By condition (2b) and Lemma 16, we see that m is in the closure of zeros of B_2 in D. Thus m is in $Z_{H^\infty+C}(B)$. By (a), $B_3|_{S_m}$ is a unimodular constant. This contradicts that $|B_3(m)| < 1$.

To show (d), let $m \in Z_{H^\infty+C}(B)$. There is a subnet $\{z_\alpha\}$ of $\{z_n\}$ so that

$$z_\alpha \to m.$$

By condition (2c), we have

$$\rho\big(\Re(z_\alpha), \bar{z}_\alpha\big) \to 0$$

as $z_\alpha \to m$ to obtain

$$\rho(z_\alpha, \bar{z}_\alpha) \to 0$$

as $z_\alpha \to m$. Thus

$$\tilde{\psi}(m) = \lim_{z_\alpha \to m} \psi(\bar{z}_\alpha) = \psi(m),$$

so $\tilde{\psi}|_{S_m} = \psi|_{S_m}$ is a unimodular constant because $\psi|_{S_m}$ is a unimodular constant. Hence

$$(\psi - \tilde{\psi})\bar{B}|_{S_m} = 0.$$

The above equality also holds for each m_1 in the Gleason part $P(m)$ since $S_m = S_{m_1}$.

To prove (e), observe that as we showed above, $\tilde{\psi}|_{S_m} = \psi|_{S_m}$ is a unimodular constant, and

$$(\psi - \tilde{\psi})\bar{B}|_{S_m} = 0,$$

for each $m \in \bigcup_{\tilde{m} \in Z_{H^\infty+C}(B)} P(\tilde{m})$. For each m in $M(H^\infty + C)/\Big[\bigcup_{\tilde{m} \in Z_{H^\infty+C}(B)} P(\tilde{m})\Big]$, $B|_{S_m}$ is a unimodular constant [24]. By the main result [22], $T_\psi H_{\bar{B}} - H_{\bar{B}} T_\psi$ is compact.

Now we are ready to prove

$$\operatorname{dist}(H_{\bar{B}_1} H_{\bar{B}_2} - H_g, \mathscr{K}) < 100\varepsilon^{1/4}.$$

First we consider $H_{\bar{B}_1} H_{\bar{B}_2}$ on $\operatorname{Ker} T_{\bar{B}} = [BH^2]^\perp$. To do this, let $f \in \operatorname{Ker} T_{\bar{B}}$. Noting that by (3),

$$H_{\bar{B}_2} = H_{\bar{B}_3 \bar{B}} = T_{B_3^*} H_{\bar{B}} + H_{\bar{B}_3} T_{\bar{B}},$$

and

$$H_{\bar{B}_1} = H_{\tilde{\psi}\bar{B}} = T_\psi H_{\bar{B}} + H_{\tilde{\psi}} T_{\bar{B}},$$

we obtain that

$$\begin{aligned} H_{\bar{B}_1} H_{\bar{B}_2} f &= [T_\psi H_{\bar{B}} + H_{\tilde{\psi}} T_{\bar{B}}] T_{B_3^*} H_{\bar{B}} f \\ &= [T_\psi H_{\bar{B}} T_{B_3^*} H_{\bar{B}} + H_{\tilde{\psi}} T_{\bar{B}} T_{B_3^*} H_{\bar{B}}] f. \end{aligned}$$

For each $m \in M(H^\infty + C)$, by claims (c1) and (c3), either $B|_{S_m}$ or $B_3|_{S_m}$ is constant. By the Axler-Chang-Sarason-Volberg theorem and (2),

$$T_{\bar{B}} T_{B_3^*} - T_{B_3^*} T_{\bar{B}} = H_{\bar{B}_3} H_{\bar{B}}$$

is compact to obtain that for some compact operator K,

$$\begin{aligned} H_{\bar{B}_1} H_{\bar{B}_2} f &= [T_\psi H_{\bar{B}} T_{B_3^*} H_{\bar{B}} + H_{\tilde{\psi}} T_{B_3^*} T_{\bar{B}} H_{\bar{B}} + K] f \\ &= [T_\psi T_{\bar{B}_3} H_{\bar{B}} H_{\bar{B}} + K] f. \end{aligned}$$

Here the last equality follows from (4):

$$H_{\bar{B}} T_{B_3^*} = T_{\bar{B}_3} H_{\bar{B}},$$

and

$$T_{\bar{B}} H_{\bar{B}} = H_{B\bar{B}} = 0.$$

By Theorem 23 we have that for some compact operator K_1 and $h \in QC$,

$$H_{\bar{B}_1} H_{\bar{B}_2} f = [T_\psi T_{\bar{B}_3} H_{\bar{B}h} + K_1] f.$$

On the other hand, we also have

$$\begin{aligned} T_\psi T_{\bar{B}_3} H_{\bar{B}h} &= T_{\bar{B}_3} T_\psi H_{\bar{B}h} + H_{\tilde{\psi}} H_{\bar{B}_3} H_{\bar{B}h} \\ &= T_{\bar{B}_3} H_{\bar{B}h} T_\psi + H_{\tilde{\psi}} H_{\bar{B}_3} H_{\bar{B}h} + K_0 \end{aligned}$$

for some compact operator K_0. The last equality follows from that $T_\psi H_{\bar{B}} - H_{\bar{B}} T_\psi$ is compact by (e) and $T_\psi T_{\tilde{h}} - T_{\tilde{h}} T_\psi$ is compact because of $h \in QC$, and $H_{\bar{B}h} = T_{\tilde{h}} H_{\bar{B}} + H_h T_{\bar{B}}$. So

$$H_{\bar{B}_1} H_{\bar{B}_2} f = [H_{\psi \bar{B} B_3^* h} + K_2] f + H_{\tilde{\psi}} H_{\bar{B}_3} H_{\bar{B}h} f,$$

for some compact operator K_2.

Use (b), to obtain that for each support set S, either $B|_S$ or $B_3|_S$ is constant. By the Axler-Chang-Sarason-Volberg theorem [2], [37], $H_{\bar{B}_3} H_{\bar{B}}$ is compact, getting that $[H_{\bar{B}_1} H_{\bar{B}_2} - H_{\psi \bar{B} B_3^* h}]|_{\ker T_B^*}$ is compact.

Next we consider $H_{\bar{B}_1}H_{\bar{B}_2} - H_{\psi\bar{B}B_3^*h}$ on BH^2. Noting that h is in QC and ψ is in H^∞, observe that $H_{\psi\bar{B}B_3^*h} - T_{\bar{B}_3}H_{\bar{B}}T_hT_\psi$, $T_hT_B - T_BT_h$ and $T_\psi T_B - T_BT_\psi$ are compact, to obtain that

$$H_{\psi\bar{B}B_3^*h}T_B - T_{\bar{B}_3}H_{\bar{B}}T_BT_hT_\psi$$

is compact. Since $H_{\bar{B}}T_B = H_{\bar{B}B} = 0$, $H_{\psi\bar{B}B_3^*h}T_B$ is compact. Thus we need only estimate $H_{\bar{B}_1}H_{\bar{B}_2}$ on BH^2.

To do this, letting $v \in H^2$, and $f = Bu \in BH^2$, we consider the following inner product:

$$\begin{aligned}\langle H_{\bar{B}_1}H_{\bar{B}_2}f, v\rangle &= \langle H_{\bar{B}_1}H_{\bar{B}_2}Bu, v\rangle = \langle H_{\bar{B}_1}H_{\bar{B}_3}u, v\rangle = \langle H_{\bar{B}_3}u, H_{\bar{B}_1}v\rangle \\ &= \int_{O_\varepsilon^+\cup O_\varepsilon^-} \langle \operatorname{grad}(H_{\bar{B}_3}u)(z), \operatorname{grad}(H_{\bar{B}_1}v)(z)\rangle \log\frac{1}{|z|^2}\,dA(z) \\ &\quad + \int_{D/[O_\varepsilon^+\cup O_\varepsilon^-]} \langle \operatorname{grad}(H_{\bar{B}_3}u)(z), \operatorname{grad}(H_{\bar{B}_1}v)(z)\rangle \log\frac{1}{|z|^2}\,dA(z),\end{aligned}$$

where the last equality follows from the Littlewood-Paley formula [15]. Using the proof of Theorem 7 in [40], we have that for some compact operators K_r for $0 < r < 1$,

$$\begin{aligned}&\left|\int_{O_\varepsilon^+\cup O_\varepsilon^-} \langle \operatorname{grad}(H_{\bar{B}_3}u)(z), \operatorname{grad}(H_{\bar{B}_1}v)(z)\rangle \log\frac{1}{|z|^2}\,dA(z) - \langle K_ru, v\rangle\right| \\ &\qquad \leqq 100 \sup_{|z|>r,\, z\in O_\varepsilon^+\cup O_\varepsilon^-} [1 - |B_3(z)|^2]^{1/4}\|u\|_2\|v\|_2,\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{D/[O_\varepsilon^+\cup O_\varepsilon^-]} \langle \operatorname{grad}(H_{\bar{B}_3}u)(z), \operatorname{grad}(H_{\bar{B}_1}v)(z)\rangle \log\frac{1}{|z|^2}\,dA(z)\right| \\ &\qquad \leqq 100\big(1 - |B_1(z)|^2\big)^{1/4}\|u\|_2\|v\|_2.\end{aligned}$$

Noting

$$|B_3(z)| \to 1$$

as $|z| \to 1$ and $z \in O_\varepsilon^+ \cup O_\varepsilon^-$, and $H_{\psi\bar{B}B_3^*h}T_B$ is compact, we conclude

$$\lim_{r\to 1} \|[[H_{\bar{B}_1}H_{\bar{B}_2} - H_{\psi\bar{B}B_3^*h}] - K_r]|_{BH^2}\| \leqq 100\varepsilon^{1/4}.$$

Sumarizing what we have done above gives that for some compact operators K_n and a sequence $\{f_n\} \subset L^\infty$,

$$\|[H_{\bar{B}_1}H_{\bar{B}_2} - H_{f_n} - K_n\| < 1/n.$$

Let $\|T\|_e$ denote the essential norm of a bounded operator T on H^2, defined by

$$\|T\|_e = \inf_{K \in \mathscr{K}} \|T - K\|.$$

The Calkin algebra $\mathscr{B}(H^2)/\mathscr{K}$ is a closed C^*-algebra under the norm $\|T\|_e$, where $\mathscr{K}$ denotes the ideal of compact operators on H^2. Let $[T]$ denote the element in the Calkin algebra containing T. Let $\mathscr{H}$ denote the space of bounded Hankel operators.

Next we show that for some bounded operator T on H^2,

$$[H_{\bar{B}_1} H_{\bar{B}_2}] = [T].$$

By the above inequalities, use the traingle inequality, to obtain

$$\|H_{f_n} - H_{f_m}\|_e < 1/n + 1/m.$$

Thus $\{[H_{f_n}]\}$ is a Cauchy sequence in the Calkin algebra and so it converges to some $[T]$, getting that

$$[H_{\bar{B}_1} H_{\bar{B}_2}] = [T],$$

and T is in the closure of $\mathscr{H} + \mathscr{K}$.

To finish the proof we need only to show that $T = H_f + K$ for some f in L^∞ and a compact operator K.

By the Axler-Berg-Jewell-Shields theorem [1], for each $f \in L^\infty$, there is a function $g \in H^\infty + C$,

$$\begin{aligned}\operatorname{dist}(H_f, \mathscr{K}) &= \|H_f\|_e = \|f - g\|_\infty \\ &= \|H_f - H_g\| = \operatorname{dist}(H_f, \mathscr{K} \cap \mathscr{H}).\end{aligned}$$

By the same idea in [4] and [33], we have that $\mathscr{H} + \mathscr{K}$ is a closed subspace of $\mathscr{B}(H^2)$, to obtain that for some function $f \in L^\infty$ and compact operator K,

$$T = H_f + K.$$

This implies

$$H_{\bar{B}_1} H_{\bar{B}_2} = H_f + K,$$

to complete the proof.

Acknowledgments. We thank the referee for his useful suggestions. The fourth author would like to thank Xiaoman Chen, Kunyu Guo and Jiaxing Hong for their hospitality while he visited the institute of mathematics at Fudan University and the part of this paper was in progress. The first author is partically supported by 973-project and National Educational Ministry of China. The second author is partially supported by

NNSFC(10171019), Shuguang project in Shanghai, and Young teacher Fund of higher school of National Educational Ministry of China. The third author is partially supported by Grant-in-Aid for Scientific Research (No. 13440043), Ministry of Education, Science and Culture. The fourth author was partially supported by the National Science Foundation.

References

[1] *S. Axler, I. Berg, N. Jewell* and *A. Shields*, Approximation by compact operators and the space $H^\infty + C$, Ann. Math. **109** (1978), 601–612.

[2] *S. Axler, S.-Y. A. Chang, D. Sarason*, Product of Toeplitz operators, Integr. Equ. Oper. Th. **1** (1978), 285–309.

[3] *S. Axler* and *P. Gorkin*, Divisibility in Douglas algebras, Michigan Math. J. **319** (1984), 89–94.

[4] *S. Axler* and *A. Shields*, Algebras generated by analytic and harmonic functions, Indiana Univ. Math. J. **36** (1987), no. 3, 631–638.

[5] *J. Barría*, On Hankel operators not in the Toeplitz algebra, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1507–1511.

[6] *J. Barría* and *P. Halmos*, Asymptotic Toeplitz operators, Trans. Amer. Math. Soc. **273** (1982), 621–630.

[7] *A. Brown* and *P. R. Halmos*, Algebraic Properties of Toeplitz Operators, J. reine angew. Math. **213** (1963/1964), 89–102.

[8] *L. Carleson*, An interpolation problem for bounded analytic functions, Amer. J. Math. **80** (1958), 921–930.

[9] *L. Carleson*, Interpolation by bounded analytic functions and the corona problem, Ann. Math. **76** (1965), 547–559.

[10] *X. Chen* and *F. Chen*, Hankel operators in the set of essential Toeplitz operators, Acta Math. Sinica **6** (1990), 354–363.

[11] *F. Chen* and *X. Chen*, The construction of a special kind Hankel operators and function theory, preprint.

[12] *R. G. Douglas*, Banach algebra techniques in the operator theory, Academic Press, New York and London 1972.

[13] *R. G. Douglas*, Local Toeplitz operators, Proc. London Math. Soc. **36** (1978), 243–272.

[14] *P. Durn*, Theory of H^p spaces, Academic Press, New York-London 1970.

[15] *J. B. Garnett*, Bounded Analytic Functions, Academic Press, New York 1981.

[16] *P. Gorkin, H. Lingenberg* and *R. Mortini*, Homeomorphic disks in the spectrum of H^∞, Indiana Univ. Math. J. **39** (1990), no. 4, 961–983.

[17] *P. Gorkin* and *R. Mortini*, Interpolating Blaschke products and factorization in Douglas algebras, Michigan Math. J. **38** (1991), 147–160.

[18] *P. Gorkin* and *D. Zheng*, Essentially commuting Toeplitz operators, Pacific J. Math. **190** (1999), 87–109.

[19] *C. Guillory* and *D. Sarason*, Division in $H^\infty + C$, Mich. Math. J. **28** (1981), 173–181.

[20] *C. Guillory, K. Izuchi* and *D. Sarason*, Interpolating Blaschke products and division in Douglas algebras, Proc. Roy. Irish Acad. Sect. A **84** (1984), 1–7.

[21] *K. Guo* and *D. Zheng*, Invariant subspaces, quasi-invariant subspaces, and Hankel operators, J. Funct. Anal. **187** (2001), 308–342.

[22] *K. Guo* and *D. Zheng*, Essentially commuting Hankel and Toeplitz operators, J. Funct. Anal. **201** (2003), 121–147.

[23] *K. Guo* and *D. Zheng*, The distribution function inequality for a finite sum of finite products of Toeplitz operators, J. Funct. Anal., to appear.

[24] *H. Hedenmalm*, Thin interpolating sequences and three algebras of bounded functions, Proc. Amer. Math. Soc. **99** (1987), no. 3, 489–495.

[25] *K. Hoffman*, Analytic functions and logmodular Banach algebras, Acta Math. **108** (1962), 271–317.

[26] *K. Hoffman*, Bounded analytic functions and Gleason parts, Ann. Math. **103** (1967), 74–111.

[27] *G. Leibowitz*, Lectures on complex function algebras, Scott, Foresman, Glenview, IL, 1970.

[28] *N. K. Nikolskii*, Treatise on the shift operator, Springer-Verlag, Berlin-Heidelberg-New York 1985.

[29] *J. Partington*, Interpolation, identification, and sampling, Clarendon Press Oxford 1997.

[30] *V. Peller*, Hankel operators and their applications, Springer Monogr. Math., Springer-Verlag, New York 2003.

[31] *S. Power*, Hankel operators on Hilbert space, Bull. London Math. Soc. **12** (1980), 422–442.

[32] *S. Power*, Hankel operators on Hilbert space, Pitman, London 1982.

[33] *W. Rudin*, Spaces of $H^\infty + C$, Ann. Inst. Fourier(Grenoble) **25** (1975), 99–125.
[34] *D. Sarason*, Function theory on the unit circle, Virginia Polytechnic Institute and State University, Blacksburg, VA, 1979.
[35] *D. Sarason*, Toeplitz operators with piecewise quasicontinuous symbols, Indiana Univ. Math. J. **26** (1977), 817–828.
[36] *C. Sundberg* and *T. Wolff*, Interpolating sequences for QA_B, Trans. Amer. Math. Soc. **276** (1983), no. 2, 551–581.
[37] *A. Volberg*, Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang, and D. Sarason, J. Oper. Th. **8** (1982), 209–218.
[38] *T. Wolff*, Two algebras of bounded functions, Duke Math. J. **49** (1982), 321–328.
[39] *T. Yoshino*, The conditions that the product of Hankel operators is also a Hankel operator, Arch. Math. (Basel) **73** (1999), no. 2, 146–153.
[40] *D. Zheng*, The distribution function inequality and products of Toeplitz operators and Hankel operators, J. Funct. Anal. **138** (1996), no. 2, 477–501.

Institute of Mathematics, Fudan University, Shanghai, 200433, The People's Republic of China
e-mail: xchen@fudan.edu.cn
e-mail: kyguo@fudan.edu.cn

Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-21, Japan
e-mail: izuchi@scux.sc.niigata-u.ac.jp

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, USA
e-mail: zheng@math.vanderbilt.edu

Eingegangen 11. November 2003, in revidierter Fassung 7. Januar 2004