COMPACT PRODUCTS OF HANKEL OPERATORS

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In this paper we completely characterize compact products of three Hankel operators on the Hardy space. We obtain a necessary and sufficient condition for that $H_{f_1}^* H_{f_2}^* H_{f_3}^*$, $H_{f_2}^* H_{f_3}^* H_{f_1}^*$, and $H_{f_1}^* H_{f_2}^* H_{f_3}^*$ are simultaneously compact.

Introduction

Let $D$ be the open unit disk in the complex plane. Let $L^2$ denote the Lebesgue space of square integrable functions on the unit circle $\partial D$. The Hardy space $H^2$ is the subspace of $L^2$ of analytic functions on $D$.

Let $P$ be the orthogonal projection from $L^2$ onto $H^2$. Let $Q$ denote $I - P$. Given a function $f$ in $L^\infty$, the Hankel operator $H_f$ with symbol $f$ is defined by $H_f h = Q(f h)$, for $h$ in $H^2$.

In this paper we continue the work in [15] to study the product $H_{f_1}^* H_{f_2}^* H_{f_3}^*$ of three Hankel operators. Our goal is to obtain a function theoretic characterization for the compactness of products of three Hankel operators.

Axler, Chang and Sarason [2], and Volberg [14] proved that $H_{f_1}^* H_{f_2}^*$ is compact if and only if

$$(0.1) \quad H^\infty[f_1] \cap H^\infty[f_2] \subset H^\infty + C,$$

where $H^\infty[f_1]$ is the Douglas algebra generated by $H^\infty$ and $f_1$, and $H^\infty + C$ denotes the Douglas algebra generated by $H^\infty$ and the continuous functions on the unit circle. Recently the second author [16] proved that $H_{f_1}^* H_{f_3}^*$ is compact if and only if

$$(0.2) \quad \lim_{|z| \to 1^-} \| (f_1)_- \circ \varphi_z - (f_1)_-(z) \|_2 \| (f_2)_- \circ \varphi_z - (f_2)_-(z) \|_2 = 0,$$

where $f_- = Q(f)$ and $\varphi_z$ is the Möbius transform

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

In [15], we proved that

$$(0.3) \quad H^\infty[f_1] \cap H^\infty[f_2] \cap H^\infty[f_3] \subset H^\infty + C$$

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is a sufficient condition for $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast$ to be compact for every permutation $\sigma$, but we also gave examples that condition (0.2) is even not necessary for only the product $H_{f_1} H_{f_2} H_{f_3}$ to be zero.

In [15] we also showed that (0.2) is equivalent to
\[
\lim_{|z| \to 1} \prod_{i=1}^{3} \left\| (f_i) \circ \varphi_z - (f_i)(z) \right\|_2 = 0.
\]

The main result of this paper is that $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast$ is compact for every permutation $\sigma$ if and only if
\[
\lim_{|z| \to 1} \prod_{i=1}^{3} \left( \| (f_i) \circ \varphi_z - (f_i)(z) \|_2 \right) \left( \sum_{1 \leq i < j \leq 3} \left\| (f_i) \circ \varphi_z - (f_i)(z) \right\|_2 \left\| (f_j) \circ \varphi_z - (f_j)(z) \right\|_2 \right) = 0.
\]

To prove the above result we will use the Hoffman abstract $H^\ast$-theory on the support set [11]. We will also need some notation. A Douglas algebra is, by definition, a closed subalgebra of $L^\infty$ which contains $H^\infty$. Suppose that $f_1$, ..., $f_n$ are functions in $L^\infty$. Let $H^\infty[f_1, f_2, ..., f_n]$ denote the Douglas algebra generated by the functions $f_1$, $f_2$, ..., and $f_n$. The Gelfand space (space of nonzero multiplicative linear functionals) of the Douglas algebra $B$ will be denoted by $M(B)$. If $B$ is a Douglas algebra, then $M(B)$ can be identified with the set of nonzero linear functionals in $M(H^\infty)$ whose representing measures (on $M(L^\infty)$) are multiplicative on $B$, and we identify the function $f$ with its Gelfand transform on $M(B)$. In particular, $M(H^\infty + C) = M(H^\infty) - D$, and a function $f \in H^\infty$ may be thought of as a continuous function on $M(H^\infty)$. A subset of $M(L^\infty)$ is called a support set if it is the (closed) support of the representing measure for a functional in $M(H^\infty + C)$. When $f_1$, $f_2$ and $f_3$ are in $\overline{H^\infty}$, we will show that $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast$ is compact for every permutation $\sigma$ if and only if for each support set $S$, one of the following conditions hold.

1. $f_1|_S$ is constant.
2. $f_2|_S$ is constant.
3. $f_3|_S$ is constant.
4. $(f_i - f_i(m))(f_j - f_j(m))|_S = 0$ for $1 \leq i < j \leq 3$.

Using his work [1], Sheldon Axler shows us examples that for some support set $S$, there are two functions $f$ and $g$ in $L^\infty$ such that $fg = 0$ on $S$ but neither $f$ nor $g$ is constant on $S$. We thank Sheldon Axler for his examples. So in general condition (4) is not equivalent to one of conditions (1)-(3). Thus the condition that either $f_1$ or $f_2$ or $f_3$ is constant on each support set, is not a necessary condition for $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast$ to be compact for every permutation $\sigma$. So

\[
H^\infty[f_1] \cap H^\infty[f_2] \cap H^\infty[f_3] \subset H^\infty + C
\]

is not a necessary condition for $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast$ to be compact for every permutation $\sigma$.

On the other hand, in [15] we proved that $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast = 0$ for every permutation $\sigma$ if and only if either $H_{f_1}$ or $H_{f_2}$ or $H_{f_3}$ is zero. More recently Gu [8] proved that $H_{f_{\sigma(1)}}^\ast H_{f_{\sigma(2)}}^\ast H_{f_{\sigma(3)}}^\ast$ is of finite rank for every permutation $\sigma$ if and only if either $H_{f_1}$ or $H_{f_2}$ or $H_{f_3}$ is of finite rank. In the special case that $f_1$, $f_2$ and $f_3$ are conjugates of bounded analytic functions, we will show that
$H^{\sigma}_{f_{\sigma_1}} H^{\infty}_{f_{\sigma_2}} H^\infty_{f_{\sigma_3}}$ is compact for every permutation $\sigma$ if and only if
\[
\cap_{i=1}^3 H^\infty_{\{f_i\}} \bigcap_{\lambda_i \in \mathbb{C}} H^\infty \left[ \cup_{1 \leq i < j \leq 3} \{(f_i - \lambda_i)(f_j - \lambda_j)\}, (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3) \right] \subset H^\infty + C.
\]

In Section 1 we give an abstract criterion for the compactness of products of three Hankel operators. We also obtain a function theoretic criterion in Section 2. In Section 3 we obtain a criterion in terms of local properties of their symbols. In the last section we derive a criterion in terms of Douglas algebras for some special cases, which are similar to condition (0.1).

1 An Abstract Criterion

For $f$ in $L^\infty$, the Toeplitz operator with symbol $f$ is the operator $T_f$ on $H^2$ defined by $T_f h = P(fh)$ for $h \in H^2$. The dual-Toeplitz operator $S_f$ with symbol $f$ is defined by $S_f p = Q(fp)$, for $p \in [H^2]^1$. We have the following useful observation:

\begin{equation}
H^*_f H_g = T_f g - T_f T_g,
\end{equation}

and moreover if $f$ is in $H^\infty$, we have

\begin{equation}
S_f H_g = H_g T_f.
\end{equation}

In [16], the second author proved that if $K$ is a compact operator on the Hardy space, then

\[
\lim_{|z| \to 1} \|K - T^*_{\varphi_z} KT_{\varphi_z}\| = 0,
\]

and moreover the converse is true if $K = H^*_f H_g$ for $f, g \in L^\infty$. Later in [7], introducing the generalized area integral and the distribution function inequality Gorkin and the second author proved that the converse is also true if $K$ is the commutator $[T_f, T_g]$ of two Toeplitz operators $T_f$ and $T_g$. In [10], Gu and the second author proved that the converse is true if $K$ is a finite sum of products of two Hankel operators. Recently [9] Gu even proved that the converse is true if $K$ is a finite sum of finite products of Toeplitz operators with zero symbol by combining some ideas in [10] with his ideas that a finite sum of finite products of Toeplitz operators can be written a finite sum of products of two Hankel operators densely defined on $H^2$. In this section we will use his results to obtain the following abstract criterion for the compactness of products of three Hankel operators.

**Theorem 1.3** Let $f_1$, $f_2$, and $f_3$ be in $L^\infty$. $H^*_f H^*_f H^*_f$ is compact if and only if

\begin{equation}
\lim_{|z| \to 1} \|T^*_{\varphi_z} H^*_f H^*_f H^*_f - H^*_f H^*_f H^*_f \| = 0.
\end{equation}

**Proof:** Let $f_1$, $f_2$, and $f_3$ be in $L^\infty$. Suppose that $H^*_f H^*_f H^*_f$ is compact. Lemma 4.4 in [15] gives that (1.4) holds.

Conversely assuming (1.4) we need to prove that $H^*_f H^*_f H^*_f$ is compact. Let $K = H^*_f H^*_f H^*_f$. Note that

\[
KK^* = H^*_f H^*_f H^*_f H^*_f H^*_f.
\]
Let
\[ T = H_{f_1}H_{f_2}H_{f_3}^*H_{f_3}. \]
It is sufficient to show that $T$ is compact. Because that (1.1) gives
\[
T = [T_{f_1}T_{f_2}T_{f_3}T_{f_3}^*] - T_{f_1}T_{f_2}[T_{f_3}T_{f_3}^* - T_{f_3}],
\]
we have
\[
T = T_{f_1}T_{f_2}T_{f_3}^* - T_{f_1}T_{f_2}T_{f_3}^*T_{f_3} + T_{f_1}T_{f_2}T_{f_3}^*T_{f_3}^* + T_{f_1}T_{f_2}T_{f_3}^*T_{f_3}^* - T_{f_3}.
\]
Thus $T$ is a finite sum of finite products of Toeplitz operators with zero symbol. Theorem 7 in [9] implies that $T$ is compact if
\[
\lim_{|z| \to 1} \|T - T_{\varphi_z}^*TT_{\varphi_z}\| = 0.
\]
So we need to verify condition (1.5). Note
\[
T - T_{\varphi_z}^*TT_{\varphi_z} = KH_{f_3} - T_{\varphi_z}^*KH_{f_3}T_{\varphi_z} = KH_{f_3} - T_{\varphi_z}^*KS_{\varphi_z}H_{f_3} = (K - T_{\varphi_z}^*KS_{\varphi_z})H_{f_3}.
\]
The second equality above follows from (1.2). Since $H_{f_3}$ is a bounded operator, we conclude that condition (1.4) implies (1.5). This completes the proof.

2 Function Theoretic Version

In this section we will obtain function theoretic criterion for the compactness of products of three Hankel operators.

Define an operator $V$ on $L^2$ by
\[
Vf(w) = \overline{f(w)}
\]
for $f \in L^2$. It is easy to check that $V$ is anti-unitary. The operator $V$ enjoys the following property:
\[
(2.1) \quad VH_fV = H_f^*.
\]

For a fixed point $z$ in $D$, let $k_z$ denote the normalized reproducing kernel at $z$, and let $\varphi_z(w)$ denote the Möbius transform
\[
\varphi_z(w) = \frac{z - w}{1 - \overline{z}w}.
\]

Let $U_z$ be the unitary operator on $L^2$ defined by
\[
U_z h(w) = h \circ \varphi_z(w)k_z(w).
\]

For two vectors $x$ and $y$ in $L^2$, we use $x \otimes y$ to denote the operator of rank one given by
\[
(x \otimes y)(f) = (f, y)x
\]
for $f \in L^2$.

We need the following simple but useful lemma in [15].
Lemma 2.2 Let $f_1$, $f_2$, and $f_3$ be in $L^{\infty}$, and $z \in D$. Then
\[
T_{\varphi_z}^*H_{f_3}^*H_{f_2}H_{f_1}^*S_{\varphi_z} - H_{f_3}^*H_{f_2}H_{f_1}^*
= -(H_{f_3}^*H_{f_2}H_{f_1}^*V k_z) \otimes (V k_z) - (V H_{f_3} k_z) \otimes (VT_{\varphi_z}T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) +
(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) \otimes (S_{\varphi_z}^*H_{f_3} k_z).
\]

Before we state and prove another lemma, we introduce more notation. For $z \in D$ we use $F_z$ to denote the following finite rank operator
\[
F_z = (V H_{f_1} k_z) \otimes (VT_{\varphi_z}T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - (T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) \otimes (S_{\varphi_z}^*H_{f_3} k_z).
\]

Lemma 2.3 Let $f_1$, $f_2$, and $f_3$ be in $L^{\infty}$. If $f_1$ is not analytic on $D$, then
\[
\|F_z\|^2 \leq \|[(VT_{\varphi_z}T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \overline{\lambda_z}(S_{\varphi_z}^*H_{f_3} k_z)]\|^2 \|V H_{f_1} k_z\|^2 +
\|[(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \lambda_z(V H_{f_1} k_z)]\|^2 \|S_{\varphi_z}^*H_{f_3} k_z\|^2 \leq 2\|F_z\|^2,
\]
where
\[
\lambda_z = \frac{< T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z, V H_{f_1} k_z >}{\|V H_{f_1} k_z\|^2}.
\]

Proof: For each $z \in D$, let
\[
\lambda_z = \frac{< V H_{f_1} k_z, T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z >}{\|V H_{f_1} k_z\|^2}.
\]
Observe that
\[
[(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \lambda_z(V H_{f_1} k_z)] \perp (V H_{f_1} k_z).
\]
Writing
\[
F_z = ((V H_{f_1} k_z) \otimes [(VT_{\varphi_z}T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \overline{\lambda_z}(S_{\varphi_z}^*H_{f_3} k_z)] -
[(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \lambda_z(V H_{f_1} k_z)] \otimes (S_{\varphi_z}^*H_{f_3} k_z),
\]
we obtain that
\[
F_z^* F_z = \|(VT_{\varphi_z}T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \overline{\lambda_z}(S_{\varphi_z}^*H_{f_3} k_z)\|^2 \|V H_{f_1} k_z\|^2 +
\|(S_{\varphi_z}^*H_{f_3} k_z)\|^2 \|(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \lambda_z(V H_{f_1} k_z)\| \|(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \lambda_z(V H_{f_1} k_z)\| \leq \lambda_z(V H_{f_1} k_z). \]
Thus $F_z^* F_z$ is in the trace class and
\[
\text{trace}(F_z^* F_z) = \|[(VT_{\varphi_z}T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \overline{\lambda_z}(S_{\varphi_z}^*H_{f_3} k_z)]\|^2 \|V H_{f_1} k_z\|^2 +
\|[(T_{\varphi_z}^*H_{f_3}^*H_{f_2} k_z) - \lambda_z(V H_{f_1} k_z)]\|^2 \|S_{\varphi_z}^*H_{f_3} k_z\|^2 \leq \lambda_z(V H_{f_1} k_z).
\]
Since $F_z^* F_z$ is an operator of rank at most 2, we have
\[
\frac{1}{2} \text{trace}(F_z F_z^*) \leq \|F_z\|^2 \leq \text{trace}(F_z^* F_z).
\]
to get
\[ \|F_z\|^2 \leq \| (V T_{\varphi_z} T_{\varphi_z}^* H_{f_3}^* H_{f_2} k_z) - \lambda_z (S_{\varphi_z}^* H_{f_2} k_z) \|^2 + \| (T_{\varphi_z}^* H_{f_3}^* H_{f_2} k_z) - \lambda_z (V H_{f_1} k_z) \|^2 \| (S_{\varphi_z}^* H_{f_2} k_z) \|^2 \]
\[ \leq 2 \|F_z\|^2. \]

This completes the proof. \( \blacksquare \)

From now on for \( f \in L^2 \) we use \( f_+ \) and \( f_- \) to denote \( P(f) \) and \( Q(f) \) respectively. Because that \( f = f_+ + f_- \), it is easy to see that
\[ H_f k_z = (f_+ - f_- (z)) k_z. \]

**Lemma 2.4** Let \( f, g \) and \( h \) be in \( L^\infty \). Let \( m \) be in \( M(H^\infty + C) \). If
\[ \lim_{z \to m} \| H_f k_z \|_2 = 0 \]
then
\[ \lim_{z \to m} \| H_f^* H_g k_z \|_2 = 0, \]
and
\[ \lim_{z \to m} \| H_f^* H_g H_h^* V k_z \|_2 = 0. \]

**Proof:** Since \( f \) and \( g \) are in \( L^\infty \) and the Hardy projection \( P \) is a bounded operator from \( L^\infty \) to \( BMO \), \( f_+ \), \( f_- \), \( g_+ \) and \( g_- \) are in \( BMO \). Thus they are in \( L^p \) for all \( 1 \leq p < \infty \). Using the unitary operator \( U_z \) we have
\[ H_f k_z = U_z (f_- \circ \varphi_z - f_- (z)) \]
and
\[ H_f^* H_g k_z = P U_z [(f_+ \circ \varphi_z - f_+ (z))(g_- \circ \varphi_z - g_- (z))]. \]

The Cauchy-Schwarz inequality gives that
\[ \| H_f^* H_g k_z \|_2 \leq \| [(f_- \circ \varphi_z - f_- (z))(g_- \circ \varphi_z - g_- (z))] \|_2 \]
\[ \leq \| f_- \circ \varphi_z - f_- (z) \|_4 \| (g_- \circ \varphi_z - g_- (z)) \|_4. \]

If
\[ \lim_{z \to m} \| H_f k_z \|_2 = 0 \]
then
\[ \lim_{z \to m} \| f_- \circ \varphi_z - f_- (z) \|_2 = 0. \]

By the Schwarz inequality we have
\[ \| f_- \circ \varphi_z - f_- (z) \|_4 \leq \| f_- \circ \varphi_z - f_- (z) \|_2^{1/4} \| f_- \circ \varphi_z - f_- (z) \|_6^{3/4} \]
\[ \leq M \| f_- \circ \varphi_z - f_- (z) \|_B^{1/4} \| f_- \|_{BMO}^{3/4}. \]
for some positive constant $M$, where the last inequality follows from the John-Nirenberg theorem. Thus
\[
\lim_{z \to m} \| f^- \circ \varphi - f^-(z) \|_4 = 0.
\]
So we obtain
\[
\lim_{z \to m} \| H_{f}^* H_{g} k_z \|_2 = 0.
\]
Similarly we can prove that
\[
\lim_{z \to m} \| H_{f}^* H_{g} H_{f}^* V k_z \|_2 = 0.
\]
This completes the proof. 

In the following theorem we obtain a function theoretic criterion for the compactness of the product of three Hankel operators.

**Theorem 2.5** Let $f_1$, $f_2$, and $f_3$ be in $L^\infty$. $H_{f_1}^* H_{f_2} H_{f_3}^*$ is compact if and only if for each $m \in M(H^\infty + C)$ one of the following conditions must hold.

(a) \[
\lim_{z \to m} \| (f_1)^- \circ \varphi - (f_1)^-(z) \|_2 = 0.
\]
(b) \[
\lim_{z \to m} \| (f_3)^- \circ \varphi - (f_3)^-(z) \|_2 = 0.
\]
(c) There are constants $\lambda_i(m)$ such that
\[
\lim_{z \to m} \| Q[(f_1)^- \circ \varphi - (f_1)^-(z) - \lambda_1(m)][(f_2)^- \circ \varphi - (f_2)^-(z) - \lambda_2(m)] \|_2 = 0,
\]
\[
\lim_{z \to m} \| Q[(f_3)^- \circ \varphi - (f_3)^-(z) - \lambda_3(m)][(f_2)^- \circ \varphi - (f_2)^-(z) - \lambda_2(m)] \|_2 = 0,
\]
and
\[
\lim_{z \to m} \| Q[(f_1)^- \circ \varphi - (f_1)^-(z) - \lambda_1(m)][(f_2)^- \circ \varphi - (f_2)^-(z) - \lambda_2(m)] \|_2 = 0.
\]

**Proof:** Suppose that $H_{f_1}^* H_{f_2} H_{f_3}^*$ is compact. Let $m$ be a point in $M(H^\infty + C)$. By the Carleson Corona Theorem [4] there is a net \{z_\alpha\} in $D$ converging to $m$. Thus \(z_\alpha \to 1^-\), Theorem 1.3 gives
\[
\lim_{z_\alpha \to m} \| T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2} H_{f_3}^* S_{\varphi_{z_\alpha}} - H_{f_1}^* H_{f_2} H_{f_3}^* \| = 0.
\]
By Lemma 2.2 we have
\[
T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2} H_{f_3}^* S_{\varphi_{z_\alpha}} - H_{f_1}^* H_{f_2} H_{f_3}^*
\]
\[
= -(H_{f_1}^* H_{f_2} H_{f_3}^* V k_{z_\alpha}) \otimes (V k_{z_\alpha}) - (V H_{f_1} k_{z_\alpha}) \otimes (V T_{\varphi_{z_\alpha}}^* H_{f_2} H_{f_3}^* k_{z_\alpha}) + (T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3}^* k_{z_\alpha}).
\]
The compactness of $H_{f_1}^* H_{f_2} H_{f_3}^*$ implies that
\[
\lim_{z_\alpha \to m} \| H_{f_1}^* H_{f_2} H_{f_3}^* V k_{z_\alpha} \|_2 = 0.
\]
Thus we have
\[
\lim_{z_\alpha \to m} \| (V H_{f_1} k_{z_\alpha}) \otimes (V T_{\varphi_{z_\alpha}}^* H_{f_2} H_{f_3}^* k_{z_\alpha}) - (T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3}^* k_{z_\alpha}) \| = 0.
\]
Let

$$\lambda_{ZA} = \frac{< T_{\varphi z}^* H_f^* H_f z_z A, VH_f H_k z_z >}{||VH_f H_k z_z||^2}.$$ 

By Lemma 2.3 we have that

$$\lim_{z_a \to m} \{||(VT_{\varphi z} T_{\varphi z}^* H_f^* H_f k_z A) - \lambda_{ZA}(S_{\varphi z A}^* H_f k_z A)||^2 + \lambda_{ZA}^2 ||VH_f k_z A||^2 \} = 0.$$ 

Note that

$$|\lambda_{ZA}| \leq \frac{||T_{\varphi z}^* H_f^* H_f k_z A||^2}{||VH_f k_z A||^2}.$$ 

If (a) fails, by Lemma 2.7 [7] we see that $$\lambda_{ZA}$$ is bounded and (2.6) gives

$$||(VT_{\varphi z} T_{\varphi z}^* H_f^* H_f k_z A) - \lambda_{ZA}(S_{\varphi z A}^* H_f k_z A)||^2 \to 0.$$ 

If (b) fails by Lemma 2.7 [7] again and (2.6) we have

$$||(T_{\varphi z}^* H_f^* H_f k_z A) - \lambda_{ZA}(VH_f H_k z_z)||^2 \to 0.$$ 

Without loss of generality we may assume that $$\lambda_{ZA}$$ converges to some complex number $$\lambda_2(m)$$. Then

$$||(VT_{\varphi z} T_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(S_{\varphi z A}^* H_f k_z A)||^2 \to 0,$$

and

$$||(T_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(VH_f H_k z_z)||^2 \to 0.$$ 

Since $$S_{\varphi z} S_{\varphi z}^* = S_{\varphi z} S_{\varphi z}^* = S_1$$, applying $$S_{\varphi z}$$ to $$[(VT_{\varphi z} T_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(S_{\varphi z A}^* H_f k_z A)]$$ gives that

$$||(VT_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(H_f k_z A)||^2 \to 0.$$ 

Note

$$H_f k_z = [(f_2)_-(f_2)_-(z)] k_z,$$

$$H_f^* H_f k_z = P\{[(f_2)_-(f_3)_-(z)][(f_2)_-(f_2)_-(z)]k_z\},$$

and

$$H_f^* H_f k_z = P\{[(f_3)_-(f_3)_-(z)][(f_2)_-(f_2)_-(z)]k_z\}.$$ 

Using the properties of $$V$$ and the fact that $$\varphi z V k_z = k_z$$ we have

$$(VT_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(H_f k_z A) = Q\{[(f_3)_-(f_3)_-(z)] [(f_2)_-(f_2)_-(z) - \lambda_2(m)] k_z\}.$$ 

Since $$[(f_2)_-(f_2)_-(z) - \lambda_2(m)]$$ is in $$H^\infty$$, for any constant $$\lambda_3(m)$$ the above equality gives that

$$(VT_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(H_f k_z A) = Q\{[(f_3)_-(f_3)_-(z) - \lambda_3(m)] [(f_2)_-(f_2)_-(z) - \lambda_2(m)] k_z\}.$$ 

Similarly we obtain that for any constant $$\lambda_1(m),$$

$$(VT_{\varphi z}^* H_f^* H_f k_z A) - \lambda_2(m)(H_f k_z A) = Q\{[(f_1)_-(f_1)_-(z) - \lambda_1(m)] [(f_2)_-(f_2)_-(z) - \lambda_2(m)] k_z\}.$$
So the first two limits hold in (c). On the other hand

\[ H_{f_1}^* H_{f_2}^* H_{f_3} k_z = U_z Q \{(f_1) - \circ \varphi_z - (f_1)(z) - \lambda_1(m)) (f_2) - \circ \varphi_z - (f_2)(z) - \lambda_2(m)\} \times ((f_3) - \circ \varphi_z - (f_3)(z) - \lambda_3(m)) - H_{f_1}^* Q \{(f_3) - \circ \varphi_z - (f_3)(z) - \lambda_3(m)\} \} \).

Because that \( H_{f_1}^* H_{f_2}^* H_{f_3} \) is compact, the left hand side of the above equality converges to zero and the second term in the right hand side converges to zero also. We conclude that the first term in the right hand must converge to zero. This means that the third limit holds in (c).

To prove the other direction, suppose that for each \( m \in M(\mathcal{H}^\infty + C) \), one of (a), (b) and (c) holds. By Theorem 1.3 we need only prove that (1.4) holds. If this is not true, choose a net \( z_\alpha \in D \) converging to some \( m \in M(\mathcal{H}^\infty + C) \) and

\[ \lim_{z_\alpha \to m} \|T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2}^* H_{f_3}^* S_{\varphi_{z_\alpha}} - H_{f_1}^* H_{f_2}^* H_{f_3}^* \| > 0. \]

By Lemma 2.2, we have

\[ T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2}^* H_{f_3}^* S_{\varphi_{z_\alpha}} - H_{f_1}^* H_{f_2}^* H_{f_3}^* = - \left( H_{f_1}^* H_{f_2}^* H_{f_3}^* V k_{z_\alpha} \right) \otimes (V k_{z_\alpha}) - (V H_{f_1} k_{z_\alpha}) \otimes (VT_{\varphi_{z_\alpha}} T_{\varphi_{z_\alpha}}^* H_{f_3}^* H_{f_3}^* k_{z_\alpha}) + (T_{\varphi_{z_\alpha}}^* H_{f_1} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3} k_{z_\alpha}). \]

If either (a) or (b) holds, by Lemma 2.4, the norm of each term in the right hand side of the above equality converges to zero. If (c) holds, the third condition in (c) implies that the first term in the right hand side of the above equality converges to zero. Thus

\[ \lim_{z_\alpha \to m} \| \left( V H_{f_1} k_{z_\alpha} \right) \otimes (VT_{\varphi_{z_\alpha}} T_{\varphi_{z_\alpha}}^* H_{f_3}^* H_{f_3}^* k_{z_\alpha}) - (T_{\varphi_{z_\alpha}}^* H_{f_1} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3} k_{z_\alpha}) \| = \lim_{z_\alpha \to m} \|T_{\varphi_{z_\alpha}}^* H_{f_1}^* H_{f_2}^* H_{f_3}^* S_{\varphi_{z_\alpha}} - H_{f_1}^* H_{f_2}^* H_{f_3}^* \| > 0. \]

On the other hand, we also have that

\[ (V H_{f_1} k_{z_\alpha}) \otimes (VT_{\varphi_{z_\alpha}} T_{\varphi_{z_\alpha}}^* H_{f_3}^* H_{f_3}^* k_{z_\alpha}) - (T_{\varphi_{z_\alpha}}^* H_{f_1} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3} k_{z_\alpha}) = (V H_{f_1} k_{z_\alpha}) \otimes (VT_{\varphi_{z_\alpha}} T_{\varphi_{z_\alpha}}^* H_{f_3}^* H_{f_3}^* k_{z_\alpha}) - \lambda_2(m) S_{\varphi_{z_\alpha}}^* H_{f_3} k_{z_\alpha} + (T_{\varphi_{z_\alpha}}^* H_{f_1} k_{z_\alpha} - \lambda_2(m) V H_{f_1} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3} k_{z_\alpha}). \]

Hence the first and second conditions in (c) give that

\[ \| \left( V H_{f_1} k_{z_\alpha} \right) \otimes (VT_{\varphi_{z_\alpha}} T_{\varphi_{z_\alpha}}^* H_{f_3}^* H_{f_3}^* k_{z_\alpha}) - (T_{\varphi_{z_\alpha}}^* H_{f_1} k_{z_\alpha}) \otimes (S_{\varphi_{z_\alpha}}^* H_{f_3} k_{z_\alpha}) \| \to 0. \]

The above limit leads to a contradiction of the previous limit. This completes the proof.■

Recall some notation and facts about abstract \( H^p \)-theory on a support set. Let \( m \) be in \( M(\mathcal{H}^\infty + C) \) and let \( d\mu_m \) denote the unique representing measure for \( m \) with support \( S \). That is,

1. \( m(f g) = \int_S f g d\mu_m = \int_S f d\mu_m \int_S g d\mu_m \) for all \( f, g \in \mathcal{H}^\infty \).
2. If \( h \) is an a.e. nonnegative function in \( L^1(d\mu_m) \) such that \( \int_S fhd\mu_m = \int_S f d\mu_m \) for all \( f \in \mathcal{H}^\infty \), then \( h = 1 \) a.e. \( d\mu_m \).
Define \( H^p(m) \) be the closure of \( H^\infty \) in \( L^p(d\mu_m) \). Let \( H^\infty_m = \{ f \in H^\infty : m(f) = 0 \} \) and \( H^2_0(m) = \{ f \in H^2(m) : \int_S f d\mu_m = 0 \} \). Hoffman ([11], page 289) proved that

(H1) \( H^\infty + H^\infty_m \) is dense in \( L^2(d\mu_m) \).

(H2) \( L^2(d\mu_m) = H^2(m) \oplus H^2_0(m) \).

For each \( f \in L^\infty \) we can think that \( f \) is also in \( L^2(d\mu_m) \). By (H2) we can write \( f = f^+_m + f^-_m \) where \( f^+_m \) is in \( H^2(m) \) and \( f^-_m \) is in \( H^2_0(m) \).

**Lemma 2.7** Let \( f \) and \( g \) be in \( L^\infty \). Let \( m \) be in \( M(H^\infty + C) \). For some constants \( \lambda_1 \) and \( \lambda_2 \), if

\[
\lim_{z \to m} \| Q([f \circ \varphi_z - f_-(z) - \lambda_1](g \circ \varphi_z - g_-(z) - \lambda_2)]_2) = 0
\]

then \( [f^+_m - \lambda_1][g^+_m - \lambda_2] \in H^2(m) \).

**Proof:** Let \( m \) be in \( M(H^\infty + C) \). Without lost of generality we may assume that \( ||f||_{\infty} \leq 1 \) and \( ||g||_{\infty} \leq 1 \). First we show that for each \( h \in H^\infty_m \) with \( ||h||_{\infty} \leq 1 \),

\[
\lim_{z \to m} \int (f \circ \varphi_z - f_-(z) - \lambda_1)(g \circ \varphi_z - g_-(z) - \lambda_2) h \circ \varphi_z d\theta = \int [f^+_m - \lambda_1][g^+_m - \lambda_2] h d\mu_m.
\]

For given \( \epsilon > 0 \), by (H1), there are functions \( f^+_m \in H^\infty \) and \( f^-_m \in H^\infty_m \) such that

\[
\int_{S_m} |f - f^+_m - f^-_m|^2 d\mu_m < \epsilon.
\]

Taking the projection of \( f - f^+_m - f^-_m \) into \( H^2_0(m) \) gives that

\[
\int_{S_m} |f^+_m - f^-_m|^2 d\mu_m < \epsilon.
\]

Moreover for some neighborhood \( U_m \) of \( m \) we have

\[
\int |f - f^+_m - f^-_m|^2 d\mu_z < 2\epsilon,
\]

for \( z \in U_m \cap D \). Note that \( d\mu_z = |k_z|^2 d\theta \). Making the change of variable \( w = \varphi_z(\lambda) \) and then taking the projection of \( f \circ \varphi_z - f^+_m \circ \varphi_z - f^-_m \circ \varphi_z \) give

\[
\int ||[f \circ \varphi_z - f_-(z) - (f^+_m \circ \varphi_z - f^-_m(z))]^2 d\theta < 2\epsilon
\]

for all \( z \in U_m \cap D \).

Similarly there are two functions \( g^+_m \in H^\infty \) and \( g^-_m \in H^\infty_m \) such that

\[
\int_{S_m} |g^+_m - g^-_m|^2 d\mu_m < \epsilon.
\]

For some neighborhood \( V_m \) of \( m \) we have

\[
\int |g^-_m - g^+_m|^2 d\mu_m < \epsilon.
\]

for all \( z \in V_m \cap D \).
Combining (2.9) with (2.11) gives that for \(z \in U_m \cap V_m \cap D\)
\[
| \int_D (f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2) h \circ \varphi z d\theta
- \int [(f_\circ z - f_\circ (z)) - \lambda_1][g_\circ z - g_\circ (z) - \lambda_2] h \circ \varphi z d\theta| < 4\epsilon.
\]
Similarly combining (2.8) with (2.10) yields that
\[
| \int ((f_\circ z - f_\circ (m) - \lambda_1)(g_\circ z - g_\circ (m) - \lambda_2) h d\mu_m - \int (f_\circ m - \lambda_1)(g_\circ m - \lambda_2) h d\mu_m| < 2\epsilon.
\]
On the other hand, we also have that for some neighborhood \(W_m\) of \(m\),
\[
| \int (f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2) h \circ \varphi z d\theta - \int [f_\circ z - \lambda_1][g_\circ z - \lambda_2] h d\mu_m| < \epsilon.
\]
for all \(z \in W_m \cap D\). Noting that \(f_\circ (m) = g_\circ (m) = 0\), we have
\[
| \int (f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2) h \circ \varphi z d\theta - \int [f_\circ m - \lambda_1][g_\circ m - \lambda_2] h d\mu_m| < 7\epsilon.
\]
for \(z \in U_m \cap V_m \cap W_m \cap D\). This implies that
\[
\lim_{z \to m} \int (f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2) h \circ \varphi z d\theta = \int [f_\circ m - \lambda_1][g_\circ m - \lambda_2] h d\mu_m.
\]
If
\[
\lim_{z \to m} ||Q[(f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2)]||_2 = 0
\]
then
\[
\lim_{z \to m} | \int (f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2) h \circ \varphi z d\theta|
\leq \lim_{z \to m} ||Q[(f_\circ z - f_\circ (z) - \lambda_1)(g_\circ z - g_\circ (z) - \lambda_2)]||_2.
\]
Hence
\[
\int [f_\circ m - \lambda_1][g_\circ m - \lambda_2] h d\mu_m = 0
\]
for all \(h \in H^\infty (m)\). By Theorem 4.2.12 in ([3], page 226) we have that
\[
[f_\circ m - \lambda_1][g_\circ m - \lambda_2] \in H^1 (m).
\]
This completes the proof.

Using the same method as in the proof of the above lemma we can prove the following lemma.

**Lemma 2.12** Let \(f\) and \(g\) be in \(L^\infty\). For each \(m \in M(H^\infty + C)\) and two constants \(\lambda_1\) and \(\lambda_2\), \([f_\circ m - \lambda_1][g_\circ m - \lambda_2]\) equals zero a.e. \(d\mu_m\) if and only if
\[
\lim_{z \to m} ||Q[f_\circ z - f_\circ (z) - \lambda_1][g_\circ z - g_\circ (z) - \lambda_2]||_1 = 0.
\]

Remark: Because that both \(f_\circ\) and \(g_\circ\) are in BMO, we see that
\[
\lim_{z \to m} ||Q[f_\circ z - f_\circ (z) - \lambda_1][g_\circ z - g_\circ (z) - \lambda_2]||_1 = 0
\]
is equivalent to
\[ \lim_{z \to m} \|[f_+ \circ \varphi_z - f_- (z) - \lambda_1] [g_+ \circ \varphi_z - g_- (z) - \lambda_2]\|_2 = 0. \]

Now we turn to the characterization for the compactness of all products of three Hankel operators.

**Theorem 2.13** Let \( f_1, f_2, \) and \( f_3 \) be in \( L^\infty \). \( H^*_\sigma(f_1) H^*_\sigma(f_2) \) \( H^*_\sigma(f_3) \) is compact for every permutation \( \sigma \) if and only if for each \( m \in M(H^\infty + C) \), one of the following holds.

(a) \( \lim_{z \to m} \|[f_1]_+ \circ \varphi_z - (f_1)_-(z)\|_2 = 0. \)

(b) \( \lim_{z \to m} \|[f_2]_+ \circ \varphi_z - (f_2)_-(z)\|_2 = 0. \)

(c) \( \lim_{z \to m} \|[f_3]_+ \circ \varphi_z - (f_3)_-(z)\|_2 = 0. \)

(d) \( \lim_{z \to m} \|[f_i]_+ \circ \varphi_z - (f_i)_-(z)\|[f_j]_+ \circ \varphi_z - (f_j)_-(z)\|_2 = 0. \) for all \( 1 \leq i < j \leq 3. \)

**Proof:** Suppose that \( H^*_\sigma(f_1) H^*_\sigma(f_2) \) \( H^*_\sigma(f_3) \) is compact for every permutation \( \sigma \). We shall prove that one of conditions (a), (b), (c) and (d) holds. So it suffices to show that (d) holds if none of conditions (a), (b), and (c) holds.

To do this, let \( m \) be in \( M(H^\infty + C) \). By Theorem 2.5 the compactness of \( H^*_\sigma(f_1) H^*_\sigma(f_2) \) \( H^*_\sigma(f_3) \) for every permutation \( \sigma \) implies that there are three constants \( \lambda_i(m) \) such that
\[
\lim_{z \to m} \|[Q:\((f_i)_+ \circ \varphi_z - (f_i)_-(z) - \lambda_i(m))((f_j)_+ \circ \varphi_z - (f_j)_-(z) - \lambda_j(m))]\|_2 = 0,
\]
for \( i \neq j \). By Lemma 2.7 we have that
\[
((f_i)_+^m - \lambda_i(m))((f_j)_+^m - \lambda_j(m)) \in H^1(m), \quad \text{and} \quad ((f_j)_+^m - \lambda_j(m))((f_i)_+^m - \lambda_i(m)) \in H^1(m).
\]

Let \( G \) be the real part of
\[
((f_i)_+^m - \lambda_i(m))((f_j)_+^m - \lambda_j(m)) - \int_{S_m} ((f_i)_+^m - \lambda_i(m))((f_j)_+^m - \lambda_j(m))d\mu_m.
\]
It is easy to check that for each \( h \in H^\infty \)
\[
\int_{S_m} Ghd\mu_m = 0.
\]

Let
\[
\tilde{H} = \{ f \in L^1(d\mu_m) : \int_{S_m} fhd\mu_m = 0 \ \text{for all} \ h \in H^\infty \}.
\]
We see that \( tG \in \tilde{H} \) for all real number \( t \). By Lemma 4.2.9 ([3], page 224) we have
\[
\int_{S_m} \ln |1 - tG|d\mu_m \geq 0.
\]

By Lemma 4.2.8 ([3], page 223) the above inequality gives that \( G = 0 \ a.e. \ d\mu_m \). This implies that \( ((f_i)_+^m - \lambda_i(m))((f_j)_+^m - \lambda_j(m)) \) is a constant \( e_{ij}(m) \). On the other hand we also have that
\[
\lim_{z \to m} \|[Q:\((f_i)_+ \circ \varphi_z - (f_i)_-(z) - \lambda_i(m))((f_j)_+ \circ \varphi_z - (f_j)_-(z) - \lambda_j(m))]\times
\]
\[
[(f_k)_+ \circ \varphi_z - (f_k)_-(z) - \lambda_k(m))]\|_2 = 0,
\]
for distinct integers \( i, j, \) and \( k \). This implies that

\[
[(f_i)^m - \lambda_i(m)][(f_j)^m - \lambda_j(m)][(f_k)^m - \lambda_k(m)]
\]

is in \( H^1(m) \). Thus

\[
c_{ij}(m)[(f_k)^m - \lambda_k(m)]
\]

is in \( H^1(m) \). So either \( c_{ij}(m) \) is zero or \( [(f_k)^m - \lambda_k(m)] \in H^2(m) \). The latter case gives that \( f_k|s_m \in H^\infty|s_m \). By Lemma 4 \([16]\) we have

\[
\lim_{z \to m} ||(f_k)^{-} \circ \varphi_z - (f_k)^{-}(z)||_2 = 0.
\]

The first case gives that

\[
[(f_i)^m - \lambda_i(m)][(f_j)^m - \lambda_j(m)] = c_{ij}(m) = 0.
\]

Hence

\[
[(f_i)^m - \lambda_i(m)][(f_j)^m - \lambda_j(m)] = 0.
\]

We have to show that both \( \lambda_i(m) \) and \( \lambda_j(m) \) are zero. Assume that \( \lambda_j(m) \) is not zero. Then by Theorem 22 \([12]\) there are an inner function \( I \) and an outer function \( O \) such that

\[
(f_j)^m - \lambda_j(m) = IO.
\]

Since \( O \) is an outer function, there is a sequence \( \{g_n\} \subset H^\infty \) such that

\[
\int_{s_m} |Og_n - 1|^2 d\mu_m \to 0.
\]

Thus

\[
0 = \int_{s_m} |[(f_i)^m - \lambda_i(m)][(f_j)^m - \lambda_j(m)]g_n|d\mu_m = \int_{s_m} |[(f_i)^m - \lambda_i(m)]IOg_n|d\mu_m
\]

\[
= \int_{s_m} |(f_i)^m - \lambda_i(m)|Og_n|d\mu_m \to \int_{s_m} |[(f_i)^m - \lambda_i(m)]|d\mu_m,
\]

so we obtain that \( [(f_i)^m - \lambda_i(m)] \) is zero and \( f_i|s_m \) is in \( H^\infty|s_m \). By Lemma 4 \([16]\) we obtain

\[
\lim_{z \to m} ||(f_i)^{-} \circ \varphi_z - (f_i)^{-}(z)||_2 = 0.
\]

Similarly we can prove that either \( \lambda_i = 0 \) or

\[
\lim_{z \to m} ||(f_j)^{-} \circ \varphi_z - (f_j)^{-}(z)||_2 = 0.
\]

Summarizing the above arguments we have obtained that either one of conditions (a), (b), and (c) holds or

\[
(f_i)^m(f_j)^m = 0.
\]

By Lemma 2.12 the above equality is equivalent to

\[
\lim_{z \to m} ||[(f_i)^{-} \circ \varphi_z - (f_i)^{-}(z)][(f_j)^{-} \circ \varphi_z - (f_j)^{-}(z)]||_2 = 0.
\]
Conversely suppose that for each $m$, one of (a), (b), (c), and (d) holds. We shall prove that $H^*_{f_{\sigma(1)}} H^*_{f_{\sigma(2)}} H^*_{f_{\sigma(3)}}$ is compact for every permutation. Clearly one of (a), (b) and (c) implies that one of (a), (b) and (c) in Theorem 2.5 holds. Since $Q$ is bounded on $L^2$, we have

$$
\|Q((f_i) - \varphi_z - (f_i)(z))(f_j) - \varphi_z - (f_j)(z))\|_2 \\
\leq \|((f_i) - \varphi_z - (f_i)(z))(f_j) - \varphi_z - (f_j)(z))\|_2,
$$

and

$$
\|Q[((f_k) - \varphi_z - (f_k)(z))(f_i) - \varphi_z - (f_i)(z))(f_j) - \varphi_z - (f_j)(z))\|_2 \\
\leq \|((f_k) - \varphi_z - (f_k)(z))(f_i) - \varphi_z - (f_i)(z))(f_j) - \varphi_z - (f_j)(z))\|_2.
$$

Thus condition (d) implies that (c) in Theorem 2.5. So by Theorem 2.5 we conclude that $H^*_{f_{\sigma(1)}} H^*_{f_{\sigma(2)}} H^*_{f_{\sigma(3)}}$ is compact for every permutation $\sigma$.

Note that one of conditions (a), (b), (c) and (c) holds in the above theorem equivalent to

$$
\lim_{z \to m} \left[ \prod_{i=1}^{3} (f_i) - \varphi_z - (f_i)(z) \right] \left[ \sum_{1 \leq i < j \leq 3} \|((f_i) - \varphi_z - (f_i)(z))(f_j) - \varphi_z - (f_j)(z))\|_2 \right] = 0.
$$

We restate the above theorem in the following function theoretic condition.

**Theorem 2.14** Let $f_1$, $f_2$, and $f_3$ be in $L^\infty$. $H^*_{f_{\sigma(1)}} H^*_{f_{\sigma(2)}} H^*_{f_{\sigma(3)}}$ is compact for every permutation $\sigma$ if and only if

$$
\lim_{|z| \to \infty} \left[ \prod_{i=1}^{3} (f_i) - \varphi_z - (f_i)(z) \right] \left[ \sum_{1 \leq i < j \leq 3} \|((f_i) - \varphi_z - (f_i)(z))(f_j) - \varphi_z - (f_j)(z))\|_2 \right] = 0.
$$

### 3 Local Version

Combining Lemma 2.12 with Lemma 2.5 in [7] we can restate Theorem 2.14 in the following local version.

**Theorem 3.1** Let $f_1$, $f_2$, and $f_3$ be in $L^\infty$. $H^*_{f_{\sigma(1)}} H^*_{f_{\sigma(2)}} H^*_{f_{\sigma(3)}}$ is compact for every permutation $\sigma$ if and only if for each support set $S_m$ either one of $f_i|S_m$ for $1 \leq i \leq 3$ is in $H^\infty|S_m$ or $(f_i)^m(f_j)^m = 0$ on $S_m$, for $1 \leq i < j \leq 3$.

In the special case that symbols $f_1$, $f_2$ and $f_3$ are conjugates of some bounded analytic functions we have the following local version of our main result.

**Theorem 3.2** Let $f_1$, $f_2$ and $f_3$ be in $H^\infty$. $H^*_{f_{\sigma(1)}} H^*_{f_{\sigma(2)}} H^*_{f_{\sigma(3)}}$ is compact for every permutation $\sigma$ if and only if for each support set $S$ for $m$, one of the following conditions hold.

1. $f_1|S$ is constant.
2. $f_2|S$ is constant.
3. $f_3|S$ is constant.
(4) \((f_i - f_i(m))(f_j - f_j(m))\) = 0 for \(1 \leq i < j \leq 3\).

**Proof:** Note that \(f_i\) is in \(\overline{H}^\infty\). We have
\[
(f_i)_m = f_i - f_i(m).
\]

This theorem follows from Theorem 3.1.

**Theorem 3.3** Let \(f_1, f_2\) and \(f_3\) be in \(\overline{H}^\infty\). \(H_1^*, H_2^*, H_3^*\) is compact if and only if for each support set \(S\), either \(f_1|_{S}\) or \(f_3|_{S}\) is constant or for some constant \(\lambda_S\), \(f_3(f_2 - \lambda S)|_{S}\), \(f_1(f_2 - \lambda S)|_{S}\), and \(f_1 f_3(f_2 - \lambda S)|_{S}\) are in \(H^\infty|_{S}\).

**Proof:** Now we use the criterion in Theorem 2.5 to prove the result. So it suffices to show that one of three conditions in Theorem 2.5 is equivalent to one condition in the theorem. To do so, let \(S\) be a support set for a point \(m\) in \(M(H^\infty + C)\). Because that \(f_1, f_2\) and \(f_3\) are conjugates of some bounded analytic functions, we have that \((f_i)_- = f_i - f_i(0)\) for \(1 \leq i \leq 3\). Thus
\[
(f_i)_- \circ \varphi_z - (f_i)_-(z) = (f_i) \circ \varphi_z - (f_i)(z),
\]
so that
\[
\|(f_i)_- \circ \varphi_z - (f_i)_-(z)\|_2 = \|(f_i) \circ \varphi_z - (f_i)(z)\|_2 = \|U_z H f_z k_z\|_2 = \|H f_z k_z\|_2.
\]

Hence condition (a) in Theorem 2.5 is equivalent to
\[
\lim_{z \to m} \|H f_z k_z\|_2 = 0.
\]
By Lemma 4 [16] and the fact that \(f_1 \in \overline{H}^\infty\), this holds if and only if \(f_1|_{S}\) is constant. Similarly we can prove that condition (b) in Theorem 2.5 is equivalent to the condition that \(f_3|_{S}\) is constant.

Note that \(f_i\) is continuous on the maximal ideal space of \(H^\infty\). Then we have that
\[
f_i(z) \to f_i(m)
\]
as \(z \to m\) for \(1 \leq i \leq 3\). Therefore
\[
\lim_{z \to m} \|Q[(f_1)_- \circ \varphi_z - (f_1)_-(z)]((f_2)_- \circ \varphi_z - (f_2)_-(z) - \lambda(m))\|_2
\]
\[
= \lim_{z \to m} \|Q[f_1 \circ \varphi_z - f_1(m)](f_2 \circ \varphi_z - f_2(m) - \lambda(m))\|_2
\]
\[
= \lim_{z \to m} \|Q[f_1 \circ \varphi_z(f_2 \circ \varphi_z - f_2(m) - \lambda(m))]\|_2
\]
\[
= \lim_{z \to m} \|U_z Q[f_1(f_2 - f_2(m) - \lambda(m))k_z]\|_2
\]
\[
= \lim_{z \to m} \|H f_z(f_2 - f_2(m) - \lambda(m))k_z\|_2.
\]
Let \(\lambda_S = f_2(m) + \lambda(m)\). By Lemma 4 [16] again, this implies that the first limit in condition (c) in Theorem 2.5 is equivalent to the condition that \([f_1(f_2 - \lambda(S))]|_{S}\) is in \(H^\infty|_{S}\). Similarly we can prove that the second limit in condition (c) in Theorem 2.5 is equivalent to the condition that
\[ [f_3(f_2 - \lambda(S))] S \text{ is in } H^\infty | S \text{ and the third limit in condition (c) in Theorem 2.5 is equivalent to the condition that } [f_1 f_3(f_2 - \lambda(S))] S \text{ is in } H^\infty | S. \] This completes the proof. \[ \blacksquare \]

Let \( \theta_i \) be inner functions. In [15] we proved that \( H^\infty \frac{\sigma_1}{\sigma_2} H^\infty \frac{\sigma_3}{\sigma_2} \) is zero if and only if

\[ \theta_1 \theta_3 |(\theta_2 - \lambda) \]

for some constant \( \lambda \). As a consequence of the above theorem we have the following corollary.

**Corollary 3.4** \( H^\infty \frac{\sigma_1}{\sigma_2} H^\infty \frac{\sigma_3}{\sigma_2} \) is compact if and only if for each support set \( S \), either \( \theta_1 | S \) or \( \theta_3 | S \) is constant or \( \theta_1 \theta_3 |(\theta_2 - \lambda_S) \) on \( S \) for some constant \( \lambda_S \).

### 4 A Criterion in Terms of Douglas Algebras

In this section we will obtain some conditions in terms of Douglas algebras about the compactness of products of Hankel operators.

**Theorem 4.1** Let \( f_1, f_2 \) and \( f_3 \) be in \( \overline{H^\infty} \). \( H^\infty \frac{\sigma_1}{\sigma_2} H^\infty \frac{\sigma_2}{\sigma_2} H^\infty \frac{\sigma_3}{\sigma_2} \) is compact for every permutation \( \sigma \) if and only if

\[
\bigcap_{i=1}^3 H^\infty | f_i \bigcap_\lambda \in \mathbb{C} H^\infty | \bigcup_{1 \leq i < j \leq 3} \{(f_i - \lambda_i)(f_j - \lambda_j)\}, (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3) \subset H^\infty + C.
\]

In order to prove the above theorem it suffices to show the following lemma.

**Lemma 4.2** Let \( f_1, f_2 \) and \( f_3 \) be in \( \overline{H^\infty} \).

\[
\bigcap_{i=1}^3 H^\infty | f_i \bigcap_\lambda \in \mathbb{C} H^\infty | \bigcup_{1 \leq i < j \leq 3} \{(f_i - \lambda_i)(f_j - \lambda_j)\}, (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3) \subset H^\infty + C
\]

if and only if for each support set \( S \) for \( m \), one of the following conditions holds.

1. \( f_1 | S \) is constant.
2. \( f_2 | S \) is constant.
3. \( f_3 | S \) is constant.
4. \( (f_i - f_i(m))(f_j - f_j(m)) | S = 0 \) for \( 1 \leq i < j \leq 3 \).

**Proof:** Without loss of generality we may assume that \( \|f_i\|_\infty \leq 1 \). Let \( A \) denote the Douglas algebra

\[
\bigcap_{i=1}^3 H^\infty | f_i \bigcap_\lambda \in \mathbb{C} H^\infty | \bigcup_{1 \leq i < j \leq 3} \{(f_i - \lambda_i)(f_j - \lambda_j)\}, (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3) \].

By the Sarason Theorem (Lemma 1.3, [7]), we get that \( M(A) \) equals

\[
M(H^\infty | f_1) \cup M(H^\infty | f_2) \cup M(H^\infty | f_3) \cup \overline{\text{closure} \left\{ \bigcup_{\lambda \in \mathbb{C}} M(H^\infty | \bigcup_{1 \leq i < j \leq 3} \{(f_i - \lambda_i)(f_j - \lambda_j)\}, (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)) \right\}}.
\]

Suppose that (4.3) holds. Then \( A \subset H^\infty + C \), and so \( M(H^\infty + C) \subset M(A) \). Let \( m \in M(H^\infty + C) \). Then \( m \) is an element of

\[
M(H^\infty | f_1) \cup M(H^\infty | f_2) \cup M(H^\infty | f_3) \]
\[ \text{closure}\{ \cup_{i,j \in \mathbb{C}} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \}. \]

If \( m \) is in any of the first three sets, Lemma 1.4 [7] gives that either (1) or (2) or (3) holds. Thus, we may assume that \( m \in \text{closure}\{ \cup_{i,j \in \mathbb{C}} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \}. \) Hence there exist constants \( \lambda_i, \alpha \) and points \( m_\alpha \in M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \) such that \( m_\alpha \to m \). Let \( S_\alpha \) be the support set for \( m_\alpha \) and \( S \) the support for \( m \). By Lemma 1.4 [7] \( (f_i - \lambda_i, \alpha)(f_j - \lambda_j, \alpha)|_{S_\alpha} \) and \( (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)|_{S_\alpha} \) are in \( H^\infty|_{S_\alpha} \). Since the support of the measure is an antisymmetric set and \( f_i \) is in \( H^\infty \), \( (f_i - \lambda_i, \alpha)(f_j - \lambda_j, \alpha)|_{S_\alpha} = c_{ij, \alpha} \) and \( (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)|_{S_\alpha} = d_\alpha \) for some constants \( c_{ij, \alpha} \) and \( d_\alpha \).

Since
\[ m \in \text{closure}\{ \cup_{i,j \in \mathbb{C}} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \}, \]
we have
\[ m \in \text{closure}\{ \cup_{\|f_i\| \leq 1, for some i} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \} \cup \]
\[ \cup_{|\lambda_i| \geq 2} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}). \]
If \( m \) is in the first set, note \( |f_i(m)| \leq 1 \) for all \( i \) but \( |\lambda_i| \geq 2 \) for some \( i \). As in the proof of Theorem 3.2 we can show that \( m \in \bigcup_{i=1}^3 M(H^\infty|f_i|) \).

Suppose that \( m \) is in the second set. Let \( \epsilon > 0 \) be given. Note that if we cover the closed disc with radius 2 by finitely many discs \( D_1, \ldots, D_n \) centered at points \( \lambda_i \) of diameter \( \epsilon \), then
\[ m \in \text{closure}\{ \cup_{\lambda_i \in D_i, for all i} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \}. \]
Thus, there exists a disk \( D_i \) with center \( \lambda_i \) of diameter \( \epsilon \) such that
\[ m \in \text{closure}\{ \cup_{\lambda_i \in D_i, for all i} M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}) \}. \]
Let \( (m_\alpha) \) be a net from this set capturing \( m \) in its closure. Note that since
\[ m_\alpha \in \{ M(H^\infty|_{\{i \leq i < j \leq 3\}}\{(f_i - \lambda_i)(f_j - \lambda_j), (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)\}), \}
there exist constants \( c_{ij, \alpha} \) and \( d_\alpha \) such that \( (f_i - \lambda_i)(f_j - \lambda_j)|_{S_\alpha} = c_{ij, \alpha} \) and \( (f_1 - \lambda_1)(f_2 - \lambda_2)(f_3 - \lambda_3)|_{S_\alpha} = d_\alpha \).
Because that \( ||f_i|| \leq 1 \), and \( \lambda_i, \alpha \) are all bounded (independently of \( \alpha \)), there exists a constant \( M \) such that \( |c_{ij, \alpha}| \leq M \) and \( |d_\alpha| \leq M \) for all \( \alpha \). Cover the disc of radius \( M \) by finitely many disks, \( B_1, \ldots, B_n \) of diameter \( \epsilon \). Then
\[ m \in \text{closure}\{ \cup_{i,j} \{ m_\alpha : (f_i - \lambda_i)(f_j - \lambda_j)|_{S_\alpha} = c_{ij, \alpha} \in B_k, d_\alpha \in B_k \} \}
Thus, we may assume that there is a net \( m_\alpha \to m \) with corresponding values \( \lambda_i, \alpha \) and \( c_{ij, \alpha} \) and \( d_\alpha \) satisfying \( \text{diam}\{\lambda_i, \alpha\}, \text{diam}\{c_{ij, \alpha}\} \) and \( \text{diam}\{d_\alpha\} \) are less than \( \epsilon \). Furthermore, we may assume that \( c_{ij, \alpha} \to c_{ij} \), and \( d_\alpha \to d \) for some complex numbers \( c_{ij} \) and \( d \).
Now we claim that $S \subseteq \overline{US_\alpha}$.
Let $S = \overline{US_\alpha}$. By ([6] p. 39),

$$M(H^\infty|_S) = \{ \varphi \in M(H^\infty) \mid \text{supp}\varphi \subseteq S \}.$$

By the definition of $S$,

$$m_\alpha \in M(H^\infty|_S)$$

for each $\alpha$. Since $m_\alpha \to m, m \in M(H^\infty|_S)$, and the claim is established.

Now if $x, y \in US_\alpha$, then there exist $\alpha$, and $\beta$ as above with $x \in S_\alpha$ and $y \in S_\beta$.

Hence

$$|(f_i - \lambda_i)(f_j - \lambda_j)(x) - (f_i - \lambda_i)(f_j - \lambda_j)(y)| \leq |(f_i - \lambda_i)(f_j - \lambda_j)(x) - (f_i - \lambda_i)(f_j - \lambda_j)(x) + |(f_i - \lambda_i)(f_j - \lambda_j)(x) - (f_i - \lambda_i)(f_j - \lambda_j)(y)| + |(f_i - \lambda_i)(f_j - \lambda_j)(y) - (f + ag)(y)| \leq$$

$$|\lambda_i - \lambda_i,\alpha||\|f_j\| + |\lambda_j| + |\lambda_j - \lambda_j,\alpha||\|f_i\| + |\lambda_i| |c_{i,j,\alpha} - c_{i,j,\beta}| +$$

$$|\lambda_i - \lambda_i,\beta||\|f_j\| + |\lambda_j| + |\lambda_j - \lambda_j,\beta||\|f_i\| + |\lambda_i|.$$

So

$$|(f_i - \lambda_i)(f_j - \lambda_j)(x) - (f_i - \lambda_i)(f_j - \lambda_j)(y)| \leq 13\epsilon.$$


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