BOUNDED TOEPLITZ PRODUCTS
ON WEIGHTED BERGMAN SPACES

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Abstract. We consider the question for which square integrable analytic functions f and g on the unit disk the densely defined products $T_f T_{\bar{g}}$ are bounded on the Bergman space. We prove results analogous to those we obtained in the setting of the unweighted Bergman space [17]. We will furthermore completely describe when the Toeplitz product $T_f T_{\bar{g}}$ is invertible or Fredholm and prove results generalizing those we obtained for the unweighted Bergman space in [18].

1. Introduction

The Bergman space A^2_α is the space of analytic functions on $\mathbb{D}$ which are square-integrable with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha\, dA(z)$, where dA denotes normalized Lebesgue area measure on $\mathbb{D}$. The reproducing kernel in A^2_α is given by

$$K^{(\alpha)}_w(z) = \frac{1}{(1-\bar{w}z)^{2+\alpha}},$$

for $z, w \in \mathbb{D}$. If $\langle\cdot,\cdot\rangle_\alpha$ denotes the inner product in $L^2(\mathbb{D}, dA_\alpha)$, then $\langle h, K^{(\alpha)}_w\rangle_\alpha = h(w)$, for every $h \in A^2_\alpha$ and $w \in \mathbb{D}$. The orthogonal projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α is given by

$$(P_\alpha\, g)(w) = \langle g, K^{(\alpha)}_w\rangle_\alpha = \int_{\mathbb{D}} g(z)\frac{1}{(1-\bar{z}w)^{2+\alpha}}\, dA_\alpha(z),$$

for $g \in L^2(\mathbb{D}, dA_\alpha)$ and $w \in \mathbb{D}$. Given $f \in L^\infty(\mathbb{D})$, the Toeplitz operator T_f is defined on A^2_α by $T_f h = P_\alpha(fh)$. We have

$$(T_f h)(w) = \int_{\mathbb{D}} \frac{f(z)h(z)}{(1-\bar{z}w)^{2+\alpha}}\, dA_\alpha(z),$$

for $h \in A^2_\alpha$ and $w \in \mathbb{D}$. Note that the above formula makes sense, and defines a function analytic on $\mathbb{D}$, also if $f \in L^2(\mathbb{D}, dA_\alpha)$. So, if $g \in A^2_\alpha$ we define $T_{\bar{g}}$ by the formula

$$(T_{\bar{g}} h)(w) = \int_{\mathbb{D}} \frac{\overline{g(z)}h(z)}{(1-\bar{z}w)^{2+\alpha}}\, dA_\alpha(z),$$

for $h \in A^2_\alpha$ and $w \in \mathbb{D}$. If also $f \in A^2_\alpha$, then $T_f T_{\bar{g}} h$ is the analytic function $f\, T_{\bar{g}} h$.

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Problem of Boundedness of Toeplitz Products on A^2_α. For which f and g in A^2_α is the operator $T_f T_{\bar g}$ bounded on A^2_α?

We will first give a necessary condition for boundedness of the Toeplitz product $T_f T_{\bar g}$, and then show that this condition is very close to being sufficient.

To formulate a necessary condition, we need to define the (weighted) Berezin transform: for a function $u \in L^1(\mathbb{D}, dA_\alpha)$, the Berezin transform $B_\alpha[u]$ is the function on $\mathbb{D}$ defined by

$$B_\alpha[u](w) = \int_{\mathbb{D}} u(z) \frac{(1-|w|^2)^{2+\alpha}}{|1-\bar w z|^{4+2\alpha}}\, dA_\alpha(z).$$

The following result gives a necessary condition for the Toeplitz product to be bounded.

Theorem 1.1. *Let $-1 < \alpha < \infty$, and let f and g be in A^2_α. If $T_f T_{\bar g}$ is bounded on A^2_α, then*

$$\sup_{w \in \mathbb{D}} B_\alpha[|f|^2](w)\, B_\alpha[|g|^2](w) < \infty.$$

The following result give a sufficient condition for the Toeplitz product to be bounded close to the above necessary condition.

Theorem 1.2. *Let $\varepsilon > 0$, $-1 < \alpha < \infty$, and let f and g be in A^2_α. If*

$$\sup_{w \in \mathbb{D}} B_\alpha[|f|^{2+\varepsilon}](w)\, B_\alpha[|g|^{2+\varepsilon}](w) < \infty,$$

then the Toeplitz product $T_f T_{\bar g}$ is bounded on A^2_α

Note that in the limiting case $\alpha \downarrow -1$ these transforms correspond to

$$\int_0^{2\pi} u(e^{i\theta}) \frac{1-|w|^2}{|1-\bar w e^{i\theta}|^2} \frac{d\theta}{2\pi} = \widehat{u}(w),$$

the Poisson extension of u on $\mathbb{D}$, as the Hardy space H^2 can be regarded as the limiting case of the weighted Bergman spaces A^2_α (see [22]). It is well-known that a Toeplitz operator on H^2 is bounded if and only if its symbol is bounded on the unit circle $\partial\mathbb{D}$. Sarason([10], [11]) found examples of f and g in H^2 such that the product $T_f T_{\bar g}$ is actually a bounded operator on H^2, though neither T_f nor T_g is bounded. Sarason [12] also conjectured that a necessary and sufficient condition for this product to be bounded is

$$\sup_{w \in \mathbb{D}} \widehat{|f|^2}(w)\widehat{|g|^2}(w) < \infty,$$

Treil proved that the above condition is indeed necessary (see [12]). The second author [20] showed that the stronger condition

$$\sup_{w \in \mathbb{D}} \widehat{|f|^{2+\varepsilon}}(w)\widehat{|g|^{2+\varepsilon}}(w) < \infty,$$

for $\varepsilon > 0$, is sufficient for the Toeplitz product $T_f T_{\bar g}$ to be bounded on H^2.

The above results were proved by the authors for the unweighted case ($\alpha = 0$) in [17]. The proof in [17] does not carry over to the weighted setting without some major adjustments. The proof of the unweighted case of Theorem 2.1 made use of the fact that the reciprocal of the Bergman's kernel's norm is a polynomial. This is, however, not the case in the weighted spaces A^2_α. We will show that the reciprocal of the Bergman's kernel's norm is the sum of a polynomial and a power series absolutely convergent on the closure of the unit disk. The proof of the unweighted case of Theorem 2.2 made use of an inner product formula that involved derivatives. This inner product formula is not enough to prove Theorem 2.1, for which we will need inner product formulas involving higher order derivatives.

Cruz-Uribe [3] showed that if f and g are outer functions, a necessary and sufficient condition for $T_f T_{\bar{g}}$ to be bounded and invertible on H^2 is that $(fg)^{-1}$ is bounded and $\sup\{\widehat{|f|^2}(w)\widehat{|g|^2}(w) : w \in \mathbb{D}\} < \infty$. A similar, though different, characterization of bounded invertible Toeplitz products on H^2 with outer symbols was obtained by the second author [20]. Cruz-Uribe's [3] proof relied on a characterization of invertible Toeplitz operators due to Devinatz and Widom, which in turn is closely related to the Helson-Szegö theorem, that characterizes the weights ω such that the conjugation operator (or Hilbert transform) is bounded on $L^2(\partial\mathbb{D}, \omega\, dm)$. See Sarason's book [9] for more on these results. On the other hand, the proof in [20] is based on a distribution function inequality.

Following our proof of Theorems 2.1 and 2.2 we will consider the special case that $g = 1/f$, in which case it will be possible to remove the $\varepsilon > 0$ in the condition of Theorem 3.1, so that the necessary condition is also sufficient; we will prove the following result.

Theorem 1.3. *If $f \in A^2_\alpha$ satisfies the condition*

$$\sup_{w\in\mathbb{D}} B_\alpha[|f|^2](w) B_\alpha[|f|^{-2}](w) < \infty,$$

then the Toeplitz product $T_f T_{\overline{1/f}}$ is bounded on A^2_α.

We will give applications of this result to describe invertible and Fredholm products $T_f T_{\bar{g}}$, for $f, g \in A^2_\alpha$. The results extend those we obtained for the unweighted case in [18]. As in [18], we extend the basic techniques of the real-variable theory of weighted norm inequalities [2], [4], [5], [8] and [13] to the weighted Bergman spaces. We make use of dyadic rectangles on the unit disk and dyadic maximal operators. We will show that every dyadic rectangle that has positive distance to the unit circle is always contained in the pseudohyperbolic disk with the same center as the dyadic rectangle and a fixed radius independent of the dyadic rectangle. This observation simplifies the arguments even for the unweighted case.

2. Necessary Condition for Boundedness

Suppose f and g are in $L^2(\mathbb{D}, dA_\alpha)$. Consider the operator $f \otimes g$ on A^2_α defined by

$$(f \otimes g)h = \langle h, g\rangle_\alpha\, f,$$

for $h \in A^2_\alpha$. It is easily proved that $f \otimes g$ is bounded on A^2_α with norm equal to $\|f \otimes g\| = \|f\|_\alpha \,\|g\|_\alpha$, where $\|h\|_\alpha$ denotes the norm $\left(\int_{\mathbb{D}} |h|^2\, dA_\alpha\right)^{1/2}$ in A^2_α.

We will obtain an expression for the operator $f \otimes g$ in terms of the operators involving the Toeplitz product $T_f T_{\bar{g}}$, where $f, g \in A^2_\alpha$. This is most easily accomplished by using the Berezin transform, which has been useful in the study of operators on the Bergman space [1] and the Hardy space [15]: writing $k^{(\alpha)}_w$ for the normalized reproducing kernels in A^2_α, we define the Berezin transform of a bounded linear operator S on A^2_α to be the function $B_\alpha[S]$ defined on $\mathbb{D}$ by

$$B_\alpha[S](w) = \langle S k^{(\alpha)}_w, k^{(\alpha)}_w\rangle_\alpha,$$

for $w \in \mathbb{D}$. The boundedness of operator S implies that the function $B_\alpha[S]$ is bounded on $\mathbb{D}$. The Berezin transform is injective, for $B_\alpha[S](w) = 0$, for all $w \in \mathbb{D}$, implies that $S = 0$, the zero operator on A^2_α (see [14] for a proof). Using the reproducing property of $K^{(\alpha)}_w$ we have

$$\|K^{(\alpha)}_w\|^2_\alpha = \langle K^{(\alpha)}_w, K^{(\alpha)}_w \rangle_\alpha = K^{(\alpha)}_w(w) = \frac{1}{(1-|w|^2)^{2+\alpha}},$$

thus

$$k^{(\alpha)}_w(z) = \frac{(1-|w|^2)^{(2+\alpha)/2}}{(1-\bar{w}z)^{2+\alpha}}, \tag{2.1}$$

for $z, w \in \mathbb{D}$. It follows from (2.1) that

$$B_\alpha[S](w) = (1-|w|^2)^{2+\alpha} \langle S K^{(\alpha)}_w, K^{(\alpha)}_w \rangle_\alpha,$$

for $w \in \mathbb{D}$. It is easily seen that $T_{\bar{g}} K^{(\alpha)}_w = \overline{g(w)} K^{(\alpha)}_w$. Thus $\langle T_f T_{\bar{g}} K^{(\alpha)}_w, K^{(\alpha)}_w \rangle_\alpha = \langle T_{\bar{g}} K^{(\alpha)}_w, T_{\bar{f}} K^{(\alpha)}_w \rangle_\alpha = \langle \overline{g(w)} K^{(\alpha)}_w, \overline{f(w)} K^{(\alpha)}_w \rangle_\alpha = f(w)\overline{g(w)} \langle K^{(\alpha)}_w, K^{(\alpha)}_w \rangle_\alpha$, and we see that

$$B_\alpha[T_f T_{\bar{g}}](w) = f(w)\overline{g(w)}.$$

We also have

$$\begin{aligned} B_\alpha[f \otimes g](w) &= (1-|w|^2)^{2+\alpha} \langle (f \otimes g) K^{(\alpha)}_w, K^{(\alpha)}_w \rangle_\alpha \\ &= (1-|w|^2)^{2+\alpha} \langle \langle K^{(\alpha)}_w, g \rangle_\alpha \, f, K^{(\alpha)}_w \rangle_\alpha \\ &= (1-|w|^2)^{2+\alpha} \langle K^{(\alpha)}_w, g \rangle_\alpha \langle f, K^{(\alpha)}_w \rangle_\alpha \\ &= (1-|w|^2)^{2+\alpha} f(w)\overline{g(w)}. \end{aligned}$$

We will use the last formulas to obtain an expression foroperator $f \otimes g$ in terms of the operators involving the Toeplitz product $T_f T_{\bar{g}}$, where $f, g \in A^2_\alpha$. We need the following lemma, which may be of independent interest. For a real number β, let $[\beta]$ denote the integer part of β and $\{\beta\} = \beta - [\beta] \geqslant 0$.

Lemma 2.2. *Suppose that α is a real number in $(-1, \infty)$. The function $(1-t)^{2+\alpha}$ has the power series expansion*

$$\begin{aligned} (1-t)^{2+\alpha} = & \sum_{j=0}^{2+[\alpha]} (-1)^j \frac{\Gamma(3+\alpha)}{j!\,\Gamma(3+\alpha-j)} t^j \\ & + (-1)^{1+[\alpha]} \frac{\Gamma(3+\alpha)\sin(\pi\{\alpha\})}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!} t^{3+n+[\alpha]}. \end{aligned}$$

Proof. We will show that

$$\begin{aligned} (1-t)^{-\beta+k} = & \sum_{j=0}^{k-1} (-1)^j \frac{\Gamma(-\beta+k+1)}{\Gamma(-\beta+k+1-j)} \frac{t^j}{j!} \\ & + (-1)^k \frac{\Gamma(-\beta+k+1)}{\Gamma(\beta)\Gamma(-\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{(n+k)!} t^{n+k}, \end{aligned}$$

for $0 < \beta < 1$ and every positive integer k. Interpreting the first sum as 0 when $k = 0$, this formula is the usual binomial expansion for $(1-t)^{-\beta}$. Assuming the above formula to hold, integration with respect to t yields

$$\begin{aligned}\frac{1-(1-t)^{-\beta+k+1}}{-\beta+k+1} &= \sum_{j=0}^{k-1}(-1)^j \frac{\Gamma(-\beta+k+1)}{\Gamma(-\beta+k+1-j)} \frac{t^{j+1}}{(j+1)!} \\ &\quad + (-1)^k \frac{\Gamma(-\beta+k+1)}{\Gamma(\beta)\Gamma(-\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{(n+k+1)!} t^{n+k+1} \\ &= -\sum_{j=1}^{k}(-1)^j \frac{\Gamma(-\beta+k+1)}{\Gamma(-\beta+k+2-j)} \frac{t^j}{j!} \\ &\quad + (-1)^k \frac{\Gamma(-\beta+k+1)}{\Gamma(\beta)\Gamma(-\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{(n+k+1)!} t^{n+k+1}\end{aligned}$$

which implies

$$\begin{aligned}1-(1-t)^{-\beta+k+1} &= -\sum_{j=1}^{k}(-1)^j \frac{(-\beta+k+1)\Gamma(-\beta+k+1)}{\Gamma(-\beta+k+2-j)} \frac{t^j}{j!} \\ &\quad + (-1)^k \frac{(-\beta+k+1)\Gamma(-\beta+k+1)}{\Gamma(\beta)\Gamma(-\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{(n+k+1)!} t^{n+k+1} \\ &= -\sum_{j=1}^{k}(-1)^j \frac{\Gamma(-\beta+k+2)}{\Gamma(-\beta+k+2-j)} \frac{t^j}{j!} \\ &\quad + (-1)^k \frac{\Gamma(-\beta+k+2)}{\Gamma(\beta)\Gamma(-\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{(n+k+1)!} t^{n+k+1},\end{aligned}$$

and thus

$$\begin{aligned}(1-t)^{-\beta+k+1} &= 1+\sum_{j=1}^{k}(-1)^j \frac{\Gamma(-\beta+k+2)}{\Gamma(-\beta+k+2-j)} \frac{t^j}{j!} \\ &\quad + (-1)^{k+1} \frac{\Gamma(-\beta+k+2)}{\Gamma(\beta)\Gamma(-\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{(n+k+1)!} t^{n+k+1}.\end{aligned}$$

This proves the induction step. Assuming α to be a non-integer, the lemma follows by taking $\beta = 1-\{\alpha\}$ and $k=[\alpha]+3$. Then $0<\beta<1$ and $-\beta+k = 2+\{\alpha\}+[\alpha] = 2+\alpha$. Using

$$\Gamma(\beta)\Gamma(-\beta+1) = \Gamma(1-\{\alpha\})\Gamma(\{\alpha\}) = \frac{\pi}{\sin(\pi\{\alpha\})}$$

the stated identity follows. □

Applying the above lemma to $t=|w|^2 = w\bar{w}$ we have

$$\begin{aligned}(1-|w|^2)^{2+\alpha} &= \sum_{j=0}^{2+[\alpha]} (-1)^j \frac{\Gamma(3+\alpha)}{j!\,\Gamma(3+\alpha-j)} w^j \bar{w}^j \\ &+ (-1)^{1+[\alpha]} \frac{\Gamma(3+\alpha)\sin(\pi\{\alpha\})}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!} w^{3+n+[\alpha]} \bar{w}^{3+n+[\alpha]}.\end{aligned}$$

Multiply by $f(w)\overline{g(w)}$ to obtain

$$B_\alpha[f\otimes g](w) = \sum_{j=0}^{2+[\alpha]} (-1)^j \frac{\Gamma(3+\alpha)}{j!\,\Gamma(3+\alpha-j)} w^j f(w)\overline{w^j\, g(w)}$$
$$+ (-1)^{1+[\alpha]} \frac{\Gamma(3+\alpha)\sin(\pi\{\alpha\})}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!}\, w^{3+n+[\alpha]} f(w)\overline{w^{3+n+[\alpha]} g(w)}.$$

Using that for analytic functions h and k the Toeplitz product $T_hT_{\bar k}$ has Berezin transform $B_\alpha[T_hT_{\bar k}](w) = h(w)\overline{k(w)}$, the above formula and the unicity of the Berezin transform imply the following operator identity

$$\begin{aligned} f\otimes g &= \sum_{j=0}^{2+[\alpha]} (-1)^j \frac{\Gamma(3+\alpha)}{j!\,\Gamma(3+\alpha-j)}\, T_{z^j f} T_{\overline{z^j g}} \\ &\quad + (-1)^{1+[\alpha]} \frac{\Gamma(3+\alpha)\sin(\pi\{\alpha\})}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!}\, T_{z^{3+n+[\alpha]} f} T_{\overline{z^{3+n+[\alpha]} g}} \\ &= \sum_{j=0}^{2+[\alpha]} (-1)^j \frac{\Gamma(3+\alpha)}{j!\,\Gamma(3+\alpha-j)}\, T_z^j T_f T_{\bar g} T_{\bar z}^j \\ &\quad + (-1)^{1+[\alpha]} \frac{\Gamma(3+\alpha)\sin(\pi\{\alpha\})}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!}\, T_z^{3+n+[\alpha]} T_f T_{\bar g} T_{\bar z}^{3+n+[\alpha]}. \end{aligned}$$

This operator identity in turn implies

$$\begin{aligned} \|f\otimes g\| \leqslant \sum_{j=0}^{2+[\alpha]} &\frac{\Gamma(3+\alpha)}{j!\,\Gamma(3+\alpha-j)}\, \|T_fT_{\bar g}\| \\ &+ \frac{\Gamma(3+\alpha)\sin(\pi\{\alpha\})}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!}\, \|T_fT_{\bar g}\|. \end{aligned}$$

Using Stirling's formula it is easy to verify that

$$\frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!} \sim \frac{1}{n^{3+\alpha}},$$

so the positive series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1-\{\alpha\})}{(3+n+[\alpha])!}$$

converges. Hence there exists a finite positive number C_α such that

$$\|f\|_\alpha\, \|g\|_\alpha = \|f\otimes g\| \leqslant C_\alpha \|T_fT_{\bar g}\|.$$

For $w\in\mathbb{D}$ the function φ_w has real Jacobian equal to

$$|\varphi_w'(z)|^2 = \frac{(1-|w|^2)^2}{|1-\bar w z|^4}.$$

Using the identity

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar w z|^2} \tag{2.3}$$

it is readily verified that

$$(1-|z|^2)^\alpha\, |k_w^{(\alpha)}(z)|^2 = |\varphi_w'(z)|^2\, (1-|\varphi_w(z)|^2)^\alpha,$$

which implies the change-of-variable formula

$$\int_{\mathbb{D}} h(\varphi_w(z))\,|k_w^{(\alpha)}(z)|^2\,dA_\alpha(z) = \int_{\mathbb{D}} h(u)\ dA_\alpha(u), \tag{2.4}$$

for every $h \in L^1(\mathbb{D})$. It follows from (2.4) that the mapping $U_w^{(\alpha)}h = (h\circ\varphi_w)k_w^{(\alpha)}$ is an isometry on A_α^2:

$$\|U_w^{(\alpha)}h\|_\alpha^2 = \int_{\mathbb{D}} |h(\varphi_w(z))|^2|k_w^{(\alpha)}(z)|^2\,dA_\alpha(z) = \int_{\mathbb{D}} |h(u)|^2\,dA_\alpha(u) = \|h\|_\alpha^2,$$

for all $h \in A_\alpha^2$. Using the identity

$$1 - \overline{\varphi_w(z)}w = \frac{1-|w|^2}{1-\bar{z}w},$$

we have

$$k_w^{(\alpha)}(\varphi_w(z)) = \frac{(1-|w|^2)^{(2+\alpha)/2}}{(1-\overline{\varphi_w(z)}w)^{2+\alpha}} = \frac{(1-\bar{z}w)^{2+\alpha}}{(1-|w|^2)^{(2+\alpha)/2}} = \frac{1}{k_w^{(\alpha)}(z)}.$$

Since $\varphi_w\circ\varphi_w = id$, we see that

$$(U_w^{(\alpha)}(U_w^{(\alpha)}h))(z) = (U_w^{(\alpha)}h)(\varphi_w(z))k_w^{(\alpha)}(z) = h(z)k_w^{(\alpha)}(\varphi_w(z))k_w^{(\alpha)}(z) = h(z),$$

for all $z \in \mathbb{D}$ and $h \in A_\alpha^2$. Thus $(U_w^{(\alpha)})^{-1} = U_w^{(\alpha)}$, and hence $U_w^{(\alpha)}$ is unitary on A_α^2. Furthermore,

$$T_{f\circ\varphi_w}U_w^{(\alpha)} = U_w^{(\alpha)}T_f. \tag{2.5}$$

Proof. For $h \in H^\infty$ and $g \in A_\alpha^2$ we have

$$\begin{aligned}
\langle U_w^{(\alpha)}T_f h, U_w^{(\alpha)}g\rangle_\alpha &= \langle T_f h, g\rangle_\alpha = \langle fh, g\rangle_\alpha \\
&= \int_{\mathbb{D}} f(u)h(u)\overline{g(u)}\,dA_\alpha(z) \\
&= \int_{\mathbb{D}} f(\varphi_w(z))h(\varphi_w(z))\overline{g(\varphi_w(z))}|k_w^{(\alpha)}(z)|^2\,dA_\alpha(z) \\
&= \int_{\mathbb{D}} f(\varphi_w(z))h(\varphi_w(z))k_w^{(\alpha)}(z)\overline{g(\varphi_w(z))k_w^{(\alpha)}(z)}\,dA_\alpha(z) \\
&= \langle fU_w^{(\alpha)}h, U_w^{(\alpha)}g\rangle_\alpha = \langle T_{f\circ\varphi_w}U_w^{(\alpha)}h, U_w^{(\alpha)}g\rangle_\alpha,
\end{aligned}$$

establishing (2.5). □

It follows from (2.5), applied to f and $\bar{g}$, that

$$\begin{aligned}
T_{f\circ\varphi_w}T_{\bar{g}\circ\varphi_w} &= (T_{f\circ\varphi_w}U_w^{(\alpha)})U_w^{(\alpha)}(T_{\bar{g}\circ\varphi_w}U_w^{(\alpha)})U_w^{(\alpha)} \\
&= (U_w^{(\alpha)}T_f)U_w^{(\alpha)}(U_w^{(\alpha)}T_{\bar{g}})U_w^{(\alpha)} = U_w^{(\alpha)}(T_fT_{\bar{g}})U_w^{(\alpha)},
\end{aligned}$$

thus

$$\|f\circ\varphi_w\|_\alpha\|g\circ\varphi_w\|_\alpha \leqslant C_\alpha\|T_{f\circ\varphi_w}T_{\bar{g}\circ\varphi_w}\| = C_\alpha\|T_fT_{\bar{g}}\|,$$

hence

$$B_\alpha[|f|^2](w)\,B_\alpha[|g|^2](w) \leqslant C_\alpha^2\,\|T_fT_{\bar{g}}\|^2,$$

for all $w \in \mathbb{D}$. So, for $f, g \in A_\alpha^2$, a necessary condition for the Toeplitz product $T_fT_{\bar{g}}$ to be bounded on A_α^2 is

$$\sup_{w\in\mathbb{D}} B_\alpha[|f|^2](w)\,B_\alpha[|g|^2](w) < \infty. \tag{2.6}$$

This completes the proof of Theorem 1.1.

3. Sufficient Condition for Boundedness

Theorem 1.2 states that a condition slightly stronger than the necessary condition (2.6) is sufficient, namely the condition that for $f, g \in A^2_\alpha$

$$\sup_{w \in \mathbb{D}} B_\alpha[|f|^{2+\varepsilon}](w)\, B_\alpha[|g|^{2+\varepsilon}](w) < \infty, \tag{3.1}$$

for $\varepsilon > 0$.

Estimates. We establish some estimates the the n-th order derivatives of images of Toeplitz operators.

Lemma 3.2. *Let $-1 < \alpha < \infty$ and let n be a non-negative integer. For $f \in A^2_\alpha$ and $h \in H^\infty(\mathbb{D})$ we have*

$$|(T_f^* h)^{(n)}(w)| \leqslant 2^n \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \frac{1}{(1-|w|^2)^{n+1+\alpha/2}} B_\alpha[|f|^2](w)^{1/2} \|h\|_\alpha,$$

for all $w \in \mathbb{D}$.

Proof. Differentiating the formula

$$\left(T_f^* h\right)(w) = (\alpha+1) \int_{\mathbb{D}} \frac{\overline{f(z)} h(z)}{(1-w\bar{z})^{2+\alpha}} (1-|z|^2)^\alpha \, dA(z)$$

n times yields

$$\left(T_f^* h\right)^{(n)}(w) = \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \int_{\mathbb{D}} \frac{\overline{z^n f(z)} h(z)}{(1-w\bar{z})^{2+n+\alpha}} (1-|z|^2)^\alpha \, dA(z). \tag{3.3}$$

It follows that

$$\begin{aligned}
\left|\left(T_f^* h\right)^{(n)}(w)\right| &\leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \int_{\mathbb{D}} \frac{|f(z)|\,|h(z)|}{|1-w\bar{z}|^{2+n+\alpha}} (1-|z|^2)^\alpha \, dA(z) \\
&\leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \left(\int_{\mathbb{D}} \frac{|f(z)|^2}{|1-w\bar{z}|^{4+2n+2\alpha}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{D}} |h(z)|^2 (1-|z|^2)^\alpha \, dA(z) \right)^{1/2} \\
&\leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \frac{1}{(1-|w|)^n} \left(\int_{\mathbb{D}} \frac{|f(z)|^2}{|1-w\bar{z}|^{4+2\alpha}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{D}} |h(z)|^2 (1-|z|^2)^\alpha \, dA(z) \right)^{1/2} \\
&= \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \frac{1}{(1-|w|)^n} \left(\frac{B_\alpha[|f|^2](w)}{(1-|w|^2)^{2+\alpha}} \right)^{1/2} \|h\|_\alpha \\
&= \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \frac{2^n}{(1-|w|^2)^{n+1+\alpha/2}} B_\alpha[|f|^2](w)^{1/2} \|h\|_\alpha,
\end{aligned}$$

as desired. □

Lemma 3.4. *Let $-1 < \alpha < \infty$, let $\varepsilon > 0$, and let n be an integer at least as large as $(2+\alpha)/(2+\varepsilon)$. There exists a constant C, only depending on α and n, such that for $f \in A^2_\alpha$ and $h \in H^\infty(\mathbb{D})$ we have*

$$|(T_f^* h)^{(n)}(w)| \leqslant \frac{C}{(1-|w|^2)^n} B_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)} \left(\frac{|h(z)|^\delta}{|1-\bar{z}w|^{2+\alpha}} \, dA_\alpha(z) \right)^{1/\delta},$$

for all $w \in \mathbb{D}$, where $\delta = (2+\varepsilon)/(1+\varepsilon)$.

Proof. Using formula (3.3) and Hölder's inequality we have

$$\begin{aligned}
&\left| \left(T_f^* h\right)^{(n)}(w) \right| \\
&\quad \leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \int_{\mathbb{D}} \frac{|f(z)|\,|h(z)|}{|1-w\bar{z}|^{2+n+\alpha}} (1-|z|^2)^\alpha \, dA(z) \\
&\quad \leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \left(\int_{\mathbb{D}} \frac{|f(z)|^{2+\varepsilon}}{|1-w\bar{z}|^{2+\alpha+n(2+\varepsilon)}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/(2+\varepsilon)} \\
&\qquad\quad \times \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-w\bar{z}|^{2+\alpha}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/\delta} \\
&\quad = \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \left(\int_{\mathbb{D}} \frac{|f(z)|^{2+\varepsilon}}{|1-w\bar{z}|^{4+2\alpha+n(2+\varepsilon)-(2+\alpha)}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/(2+\varepsilon)} \\
&\qquad\quad \times \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-w\bar{z}|^{2+\alpha}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/\delta} \\
&\quad \leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \left(\int_{\mathbb{D}} \frac{|f(z)|^{2+\varepsilon}}{|1-w\bar{z}|^{4+2\alpha}(1-|w|)^{n(2+\varepsilon)-(2+\alpha)}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/(2+\varepsilon)} \\
&\qquad\quad \times \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-w\bar{z}|^{2+\alpha}} (1-|z|^2)^\alpha \, dA(z) \right)^{1/\delta} \\
&\quad \leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+1)} \frac{1}{(1-|w|)^{n-(2+\alpha)/(2+\varepsilon)}} \left(\frac{1}{\alpha+1} \frac{B_\alpha[|f|^{2+\varepsilon}](w)}{(1-|w|^2)^{2+\alpha}} \right)^{1/(2+\varepsilon)} \\
&\qquad\quad \times \left(\frac{1}{\alpha+1} \int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-w\bar{z}|^{2+\alpha}} \, dA_\alpha(z) \right)^{1/\delta} \\
&\quad = \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \frac{(1+|w|)^{n-(2+\alpha)/(2+\varepsilon)}}{(1-|w|^2)^n} \left(B_\alpha[|f|^{2+\varepsilon}](w) \right)^{1/(2+\varepsilon)} \\
&\qquad\quad \times \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-w\bar{z}|^{2+\alpha}} \, dA_\alpha(z) \right)^{1/\delta} \\
&\quad \leqslant \frac{\Gamma(\alpha+2+n)}{\Gamma(\alpha+2)} \frac{2^{n-(2+\alpha)/(2+\varepsilon)}}{(1-|w|^2)^n} B_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)} \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-w\bar{z}|^{2+\alpha}} \, dA_\alpha(z) \right)^{1/\delta},
\end{aligned}$$

which gives the desired estimate. □

Inner Product Formula in A^2_α. In this subsection we will establish a formula for the inner product in A^2_α needed to prove our sufficiency condition for boundedness of Toeplitz products.

If f and g satisfy the sufficiency condition (3.1), and h and k are polynomials, Lemma 3.2 shows that analytic functions $F = T_f^* h$ and $G = T_g^* k$ satisfy

$$(1 - |z|^2)^{2k+2+\alpha} |u^{(k)}(z) \overline{v^{(k)}(z)}| \leqslant C_{\alpha,k} \|h\|_\alpha \, \|k\|_\alpha,$$

while Lemma 3.4, combined by the L^p-boundedness of the Bergman projection on A^2_α will be used to show that

$$\int_{\mathbb{D}} (1 - |z|^2)^{2n+\alpha} |u^{(n)}(z) \overline{v^{(n)}(z)}| \, dA(z) \leqslant C_{\alpha,k} \|h\|_\alpha \, \|k\|_\alpha,$$

provided $n \geqslant (2 + \alpha)/(2 + \varepsilon)$ (details will follow). So we need to rewrite the inner product in such a way that the above estimates can be used. Write

$$\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f \bar{g} \, dA_\alpha = (\alpha + 1) \int_{\mathbb{D}} f(z) \overline{g(z)} (1 - |z|^2)^\alpha \, dA(z).$$

Note that

$$\langle z^n, z^n \rangle_\alpha = \frac{n! \Gamma(\alpha + 2)}{\Gamma(n + \alpha + 2)}.$$

A calculation shows that

$$\langle f, g \rangle_\alpha = \langle f, g \rangle_{\alpha+2} + \frac{\langle f', g' \rangle_{\alpha+2}}{(\alpha + 2)(\alpha + 3)} + \frac{\langle f', g' \rangle_{\alpha+3}}{(\alpha + 3)(\alpha + 4)}, \tag{3.5}$$

for all $f, g \in A^2_\alpha$.

We iterate formula (3.5) to obtain an inner product formula useful in estabilishing the sufficiency condition sufficiency condition (3.1) for boundedness of Toeplitz products on the weighted Bergman space A^2_α.

Lemma 3.6. *Let $-1 < \alpha < \infty$. There exist constants $b_{n,1}, \ldots, b_{n,2n+1}$ such that*

$$\begin{aligned} \langle f, g \rangle_\alpha = \langle f, g \rangle_{\alpha+2} &+ \sum_{j=1}^{2} \sum_{k=1}^{n-1} b_{n,2k+j-2} \langle f^{(k)}, g^{(k)} \rangle_{\alpha+2k+j+1} \\ &+ \sum_{j=1}^{3} b_{n,2n+j-2} \langle f^{(n)}, g^{(n)} \rangle_{\alpha+2n+j-1}, \end{aligned} \tag{3.7}$$

for all $f, g \in A^2_\alpha$.

Proof. The inductive step is to use (3.5) on

$$\begin{aligned} \langle f^{(n)}, g^{(n)} \rangle_{\alpha+2n+j-1} = \langle f^{(n)}, g^{(n)} \rangle_{\alpha+2n+j+1} &+ \frac{\langle f^{(n+1)}, g^{(n+1)} \rangle_{\alpha+2n+j+1}}{(\alpha + 2n + j + 1)(\alpha + 2n + j + 2)} \\ &+ \frac{\langle f^{(n+1)}, g^{(n+1)} \rangle_{\alpha+2n+j+2}}{(\alpha + 2n + j + 2)(\alpha + 2n + j + 3)}, \end{aligned}$$

for $j = 1, 2$. The following definitions establish the induction step, and can be used to determine these inner product formulas recursively.

$$
\begin{aligned}
b_{n+1,2n+3} &= \frac{b_{n,2n}}{(\alpha+2n+4)(\alpha+2n+5)}, \\
b_{n+1,2n+2} &= \frac{b_{n,2n}+b_{n,2n-1}}{(\alpha+2n+3)(\alpha+2n+4)}, \\
b_{n+1,2n+1} &= \frac{b_{n,2n-1}}{(\alpha+2n+2)(\alpha+2n+3)}, \\
b_{n+1,k} &= b_{n,k}, \text{ for } 1 \leqslant k \leqslant 2n.
\end{aligned}
$$

This proves the result. □

Proof Sufficiency Condition

The inner product formula (3.7) and the estimates discussed will establish that for analytic functions f and g satisfying condition (3.1) the Toeplitz operator $T_f T_{\bar{g}}$ is bounded on A^2_α.

Let f and g be analytic functions satisfying the condition (3.1), and let h and k be polynomials. Put $F = T^*_f h$ and $G = T^*_g k$, and choose a positive integer n such that $n \geqslant (2+\alpha)/(2+\varepsilon)$. By Lemma 3.2, there are finite constants $C_{\alpha,k}$ (depending on the constant in condition (3.1)) such that

$$
(1-|z|^2)^{2k+2+\alpha}|F^{(k)}(z)\overline{G^{(k)}(z)}| \leqslant C_{\alpha,k}\|h\|_\alpha \, \|k\|_\alpha,
$$

for all $z \in \mathbb{D}$. This implies that

$$
|\langle F^{(k)}, G^{(k)}\rangle_{\alpha+2k+j+1}| \leqslant C_{\alpha,k}\|h\|_\alpha \, \|k\|_\alpha,
$$

for $k = 1, \dots, n-1$ and $j = 1, 2$.

Using Lemma 3.4,

$$
\begin{aligned}
&(1-|w|^2)^{2n}|(T^*_f h)^{(n)}(w)|\,|(T^*_g k)^{(n)}(w)| \\
&\quad \leqslant CB_\alpha[|f|^{2+\varepsilon}](w)^{1/(2+\varepsilon)} B_\alpha[|g|^{2+\varepsilon}](w)^{1/(2+\varepsilon)} \\
&\qquad \times \left(\int_{\mathbb{D}} \frac{|h(z)|^\delta}{|1-\bar{z}w|^{2+\alpha}}\, dA_\alpha(z)\right)^{1/\delta} \left(\int_{\mathbb{D}} \frac{|k(z)|^\delta}{|1-\bar{z}w|^{2+\alpha}}\, dA_\alpha(z)\right)^{1/\delta} \\
&\quad \leqslant CM \left(Q_\alpha|h|^\delta(w)\right)^{1/\delta} \left(Q_\alpha|k|^\delta(w)\right)^{1/\delta},
\end{aligned}
$$

where Q_α denotes the integral operator defined by

$$
Q_\alpha u(w) = \int_{\mathbb{D}} \frac{|u(z)|}{|1-\bar{z}w|^{2+\alpha}}\, dA_\alpha(z).
$$

Using the inequality of Cauchy-Schwarz,

$$
\begin{aligned}
&\int_{\mathbb{D}} (1-|w|^2)^{2n}|(T^*_f h)^{(n)}(w)|\,|(T^*_g k)^{(n)}(w)|\, dA_\alpha(w) \\
&\quad \leqslant CM \left(\int_{\mathbb{D}} \left(Q_\alpha|h|^\delta(w)\right)^{2/\delta}\, dA_\alpha(w)\right)^{1/2} \left(\int_{\mathbb{D}} \left(Q_\alpha|k|^\delta(w)\right)^{2/\delta}\, dA_\alpha(w)\right)^{1/2}.
\end{aligned}
$$

Since $p = 2/\delta > 1$, the L^p-boundedness of operator Q_α on A^2_α (which can be proved similarly to Theorem 4.2.3 and Remark 4.2.5 in [21] considering the test function

$(1-|z|^2)^{-(\alpha+1)/(pq)})$, shows that

$$\int_{\mathbb{D}} \left(Q_\alpha|v|^\delta(w)\right)^{2/\delta}\,dA_\alpha(w) \leqslant C' \int_{\mathbb{D}} \left(|v|^\delta(w)\right)^{2/\delta}\,dA_\alpha(w) = \|v\|_\alpha^2,$$

thus

$$\int_{\mathbb{D}} (1-|z|^2)^{2n+\alpha}|u^{(n)}(z)\overline{v^{(n)}(z)}|\,dA(z) \leqslant C_{\alpha,n}\|h\|_\alpha\,\|k\|_\alpha.$$

This implies

$$|\langle F^{(k)}, G^{(k)}\rangle_{\alpha+2n+j-1}| \leqslant C_{\alpha,n}\|h\|_\alpha\,\|k\|_\alpha,$$

for $j = 1, 2, 3$. Also, by Lemma 3.2,

$$|\langle F, G\rangle_{\alpha+2}| \leqslant C_{\alpha,0}\|h\|_\alpha\,\|k\|_\alpha.$$

With the help of inner product formula (3.7) it follows that

$$|\langle F, G\rangle_\alpha| \leqslant \left(\sum_{j=1}^{2n+1} |b_{n,j}| \max_{0\leqslant k\leqslant n} C_{\alpha,k}\right) \|h\|_\alpha\,\|k\|_\alpha,$$

proving that the Toeplitz product $T_f T_{\bar{g}}$ is bounded on A^2_α. □

4. A Reversed Hölder Inequality

In this section we will prove a reverse Hölder inequality for f in A^2_α satisfying the following invariant weight condition:

$$\sup_{w\in\mathbb{D}} B_\alpha[|f|^2](w) B_\alpha[|f|^{-2}](w) < \infty. \qquad (M_2)$$

We will prove that the above condition implies that

$$\sup_{w\in\mathbb{D}} B_\alpha[|f|^{2+\varepsilon}](w) B_\alpha[|f|^{-(2+\varepsilon)}](w) < \infty. \qquad (M_{2+\varepsilon})$$

for sufficiently small $\varepsilon > 0$. By Hölder's inequality,

$$\left(\int_{\mathbb{D}} |f|^2\,dA_\alpha\right)^{1/2} \leqslant \left(\int_{\mathbb{D}} |f|^{2+\varepsilon}\,dA_\alpha\right)^{1/(2+\varepsilon)}.$$

Applying this to the function $f\circ\varphi_w$ it follows that

$$B_\alpha[|f|^2](w) \leqslant B_\alpha[|f|^{2+\varepsilon}](w)^{2/(2+\varepsilon)},$$

and thus

$$B_\alpha[|f|^2](w) B_\alpha[|f|^{-2}](w) \leqslant \left(B_\alpha[|f|^{2+\varepsilon}](w) B_\alpha[|f|^{-(2+\varepsilon)}](w)\right)^{2/(2+\varepsilon)},$$

so condition $(M_{2+\varepsilon})$ implies (M_2). Thus, the above implication will follow once we prove a reversed Hölder inequality:

Theorem 4.1. *Suppose that $f \in A^2_\alpha$ satisfies condition (M_2) with constant*

$$M = \sup_{w\in\mathbb{D}} B_\alpha[|f|^2](w) B_\alpha[|f|^{-2}](w) < \infty.$$

There exist constants $\varepsilon_M > 0$ and $C_M > 0$ such that

$$B_\alpha[|f|^{2+\varepsilon}](w) \leqslant C_M \left(B_\alpha[|f|^2](w)\right)^{(2+\varepsilon)/2},$$

for every $w \in \mathbb{D}$ and $0 < \varepsilon < \varepsilon_M$.

As in [18], our proof will make use of dyadic rectangles and the dyadic maximal function. We first discuss the dyadic rectangles and prove some elementary properties related to these rectangles.

Dyadic rectangles. Any set of the form

$$Q_{n,m,k} = \{re^{i\theta} : (m-1)2^{-n} \leqslant r < m2^{-n} \text{ and } (k-1)2^{-n+1}\pi \leqslant \theta < k2^{-n+1}\pi\},$$

where n, m and k are positive integers such that $m \leqslant 2^n$ and $k \leqslant 2^n$ is called a dyadic rectangle. The center of the above dyadic rectangle $Q = Q_{n,m,k}$ is the point $z_Q = (m - \frac{1}{2})2^{-n}e^{i\vartheta}$, with $\vartheta = (k - \frac{1}{2})2^{1-n}\pi$. If $d(Q)$ denotes the distance between Q and $\partial\mathbb{D}$, and $\ell(Q)$ denotes the length of the square in the radial direction ($\ell(Q_{n,m,k}) = 2^{-n}$), then

$$1 - |z_Q| = d(Q) + \tfrac{1}{2}\ell(Q). \tag{4.2}$$

The following figure shows these quantities for a dyadic rectangle not adjacent to the unit circle $\partial\mathbb{D}$.

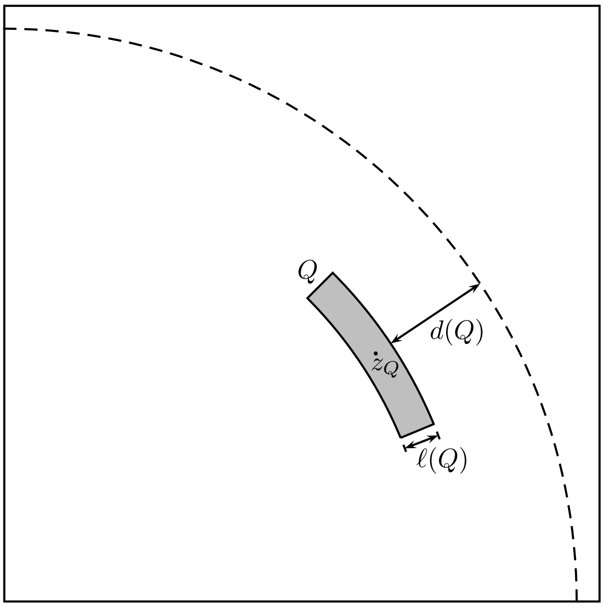

Figure 1: Dyadic rectangle Q with center z_Q

A simple calculation shows that

$$|Q| = 8|z_Q|(1 - |z_Q| - d(Q))^2. \tag{4.3}$$

Write $A_\alpha(E)$ to denote the measure of a measurable set $E \subset \mathbb{D}$ with respect to $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha\, dA(z)$. If Q is a dyadic rectangle, then its weighted area is

$$A_\alpha(Q) = \ell(Q)\Big\{(d(Q)+\ell(Q))^{1+\alpha}\left(1+|z_Q|-\tfrac{1}{2}\ell(Q)\right)^{1+\alpha} \\ - d(Q)^{1+\alpha}\left(1+|z_Q|+\tfrac{1}{2}\ell(Q)\right)^{1+\alpha}\Big\}.$$

The above formula for $A_\alpha(Q)$ can be used to obtain estimates for use in our proofs. However, many different cases need to be considered. As it turns out, dyadic rectangles not in contact with the unit circle can be treated easily without knowing their weighted area. The following formula give the weighted area of a dyadic rectangle that lies adjacent to the unit circle. If Q is a dyadic rectangle in the unit disk other than $\mathbb{D}$ for which $d(Q) = 0$, then

$$A_\alpha(Q) = 2^{3+2\alpha}|z_Q|^{1+\alpha}(1-|z_Q|)^{2+\alpha}. \tag{4.4}$$

Invariant Weight Condition. For $w \in \mathbb{D}$ let $k_w^{(\alpha)}$ denote the normalized reproducing kernel in the weighted Bergman space A^2_α.

Lemma 4.5. *Let $-1 < \alpha < \infty$. There exists a positive number c_α such that*

$$|k_{z_Q}^{(\alpha)}(z)|^2 \geqslant \frac{c_\alpha}{(1-|z_Q|)^{2+\alpha}},$$

for every dyadic square Q in $\mathbb{D}$ and every $z \in Q$.

Proof. If $z = re^{i\theta} \in Q$ and $Q = Q_{n,m,k}$, then $z_Q = 2^{-n}(m-\frac{1}{2})e^{i\vartheta}$, where $\vartheta = 2^{1-n}(k-\frac{1}{2})\pi$, thus

$$|\theta - \vartheta| \leqslant \frac{2\pi}{2^{n+1}} \leqslant 2\pi(1-|z_Q|).$$

Since $r \geqslant |z_Q| - 1/2^{n+1} \geqslant |z_Q| - (1-|z_Q|)$, we have $r|z_Q| \geqslant |z_Q|^2 - |z_Q|(1-|z_Q|)$, thus

$$1 - r|z_Q| \leqslant 1 - |z_Q|^2 + |z_Q|(1-|z_Q|) = (1+2|z_Q|)(1-|z_Q|) \leqslant 3(1-|z_Q|).$$

Hence

$$\begin{aligned} |1-\bar z_Q z|^2 &= 1 + r^2|z_Q|^2 - 2r|z_Q|\cos(\theta-\vartheta) \\ &= (1-r|z_Q|)^2 + 4r|z_Q|\sin^2((\theta-\vartheta)/2) \\ &\leqslant (1-r|z_Q|)^2 + r|z_Q|(\theta-\vartheta)^2 \\ &\leqslant 9(1-|z_Q|)^2 + 4\pi^2 r|z_Q|(1-|z_Q|)^2 \\ &\leqslant 50(1-|z_Q|)^2, \end{aligned}$$

and we obtain

$$|k_{z_Q}^{(\alpha)}(z)|^2 = \frac{(1-|z_Q|^2)^{2+\alpha}}{|1-\bar z_Q z|^{4+2\alpha}} \geqslant \frac{1}{50^{2+\alpha}(1-|z_Q|)^{2+\alpha}}.$$

This proves the inequality with $c_\alpha = 1/50^{2+\alpha}$. □

For $w \in \mathbb{D}$ and $0 < s < 1$ let $D(w,s)$ denote the pseudohyperbolic disk with center w and radius $0 < s < 1$, i.e,

$$D(w,s) = \{z \in \mathbb{C} : |\varphi_w(z)| < s\}.$$

Lemma 4.6. *Suppose that $f \in A^2_\alpha$ satisfies the invariant weight condition (M_2) and let $0 < s < 1$. There is a constant $c_s > 0$ such that*

$$\frac{1}{c_s} \leqslant \frac{|f(z)|}{|f(w)|} \leqslant c_s,$$

whenever $z \in D(w, s)$.

Proof. Fix $w \in \mathbb{D}$. Let u be in $D(0, s)$. Since f is in A^2_α we have $f(u) = \langle f, K_u^{(\alpha)} \rangle_\alpha$. Applying the Cauchy-Schwarz inequality we obtain

$$|f(u)| \leqslant \|f\|_\alpha \|K_u^{(\alpha)}\|_\alpha = \frac{\|f\|_\alpha}{(1 - |u|^2)^{(2+\alpha)/2}} \leqslant \frac{\|f\|_\alpha}{(1 - s^2)^{(2+\alpha)/2}},$$

for each u in $D(0, s)$. Now if $z \in D(w, s)$ then $z = \varphi_w(u)$, for some $u \in D(0, s)$. Replacing f by $f \circ \varphi_w$ in the above inequality gives

$$|f(z)| = |(f \circ \varphi_w)(u)| \leqslant \frac{\|f \circ \varphi_w\|_\alpha}{(1 - s^2)^{(2+\alpha)/2}} = \frac{1}{(1 - s^2)^{(2+\alpha)/2}} B_\alpha[|f|^2](w)^{1/2}.$$

By the Cauchy-Schwarz inequality

$$\frac{1}{|f(w)|} = |(f^{-1} \circ \varphi_w)(0)| \leqslant \|f^{-1} \circ \varphi_w\|_\alpha = B_\alpha[|f^{-1}|^2](w)^{1/2}.$$

Combining these inequalities we have

$$\frac{|f(z)|}{|f(w)|} \leqslant \frac{1}{(1 - s^2)^{(2+\alpha)/2}} B_\alpha[|f|^2](w)^{1/2} B_\alpha[|f|^{-2}](w)^{1/2} \leqslant \frac{M^{1/2}}{(1 - s^2)^{(2+\alpha)/2}},$$

for all $z \in D(w, s)$. Replacing f by its reciprocal f^{-1} gives the other inequality. □

Proposition 4.7. *There exists an $0 < R < 1$ such that*

$$Q \subset D(z_Q, R),$$

for every dyadic rectangle in $\mathbb{D}$ that has positive distance to $\partial\mathbb{D}$.

The following figure illustrates the above proposition.

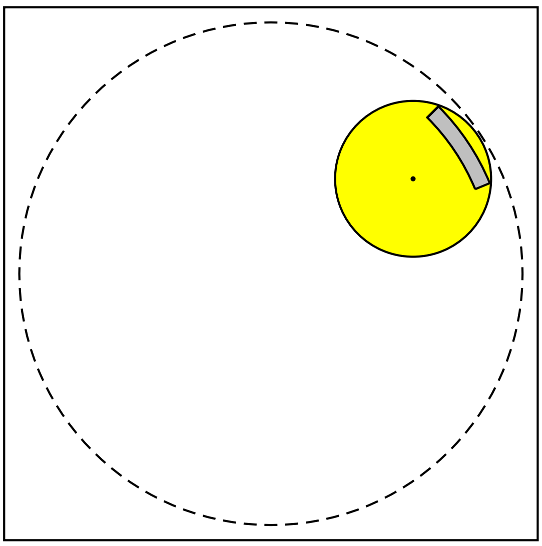

Figure 2: Dyadic rectangle Q included in $D(z_Q, R)$.

Proof. It suffices to consider dyadic rectangles closest to $\partial\mathbb{D}$. Let Q be such a dyadic rectangle with positive distance to $\partial\mathbb{D}$. For $0 < r < 1$ the pseudohyperbolic disk $D(z_Q, r)$ is a euclidean disk in $\mathbb{D}$ whose euclidean center is closer to the origin than z_Q is (the euclidean center of $D(z_Q, r)$ is $(1-r^2)z_Q/(1-r^2|z_Q|^2)$ and the euclidean radius is $(1-|z_Q|^2)r/(1-r^2|z_Q|^2)$; see [6], page 3). Recall that the center z_Q of Q has argument $\vartheta = (2k-1)\pi/2^n$. We need to show that Q's outer corners $(1-2^{-n})e^{i(\vartheta \pm \pi/2^n)}$ belong to $D(z_Q, r)$ for sufficiently large $0 < r < 1$. Using rotation-invariance, it will be enough to estimate the pseudohyperbolic distance d_n between the points $z_n = 1 - \frac{3}{2}2^{-n}$ and $\lambda_n = (1-2^{-n})e^{i\vartheta_n}$, where $\vartheta_n = \pi/2^n$. A calculation shows that

$$|z_n - \lambda_n|^2 = 2^{-2n-2} + 4(1 - \tfrac{3}{2}\,2^{-n})(1-2^{-n})\sin^2(\tfrac{1}{2}\vartheta_n),$$

and

$$|1 - \bar{z}_n\lambda_n|^2 = 25 \times 2^{-2n-2}(1 - \tfrac{3}{5}2^{-n})^2 + 4(1 - \tfrac{3}{2}\,2^{-n})(1-2^{-n})\sin^2(\tfrac{1}{2}\vartheta_n).$$

It follows that

$$d_n^2 = \frac{1 + 4(1-\frac{3}{2}\,2^{-n})(1-2^{-n})\pi^2\left(\sin(\frac{1}{2}\vartheta_n)/(\frac{1}{2}\vartheta_n)\right)^2}{25(1-\frac{3}{5}2^{-n})^2 + 4(1-\frac{3}{2}\,2^{-n})(1-2^{-n})\pi^2\left(\sin(\frac{1}{2}\vartheta_n)/(\frac{1}{2}\vartheta_n)\right)^2} \longrightarrow \frac{1+4\pi^2}{25+4\pi^2},$$

as $n \to \infty$. Consequently, there exists an $0 < R < 1$ such that $d_n < R$, for all positive integers n. Then $Q \subset D(z_Q, R)$, for every dyadic rectangle for which $d(Q) > 0$. □

Lemma 4.8. *If $f \in A^2_\alpha$ satisfies the invariant weight condition (M_2), then there is a constant $C > 0$ such that*

$$\left(\frac{1}{A_\alpha(Q)}\int_Q |f|^2\,dA_\alpha\right)\left(\frac{1}{A_\alpha(Q)}\int_Q |f|^{-2}\,dA_\alpha\right) \leqslant C,$$

for every dyadic rectangle Q.

The following proof of this more general result is actually more elementary than the proof of the corresponding lemma given in [18].

Proof. Suppose $f \in A^2_\alpha$ satisfies the invariant weight condition

$$B_\alpha[|f|^2](w)B_\alpha[|f|^{-2}](w) \leqslant M < \infty,$$

for all $w \in \mathbb{D}$. Let Q be a dyadic square in the unit disk other than $\mathbb{D}$ (if $Q = \mathbb{D}$ the estimate holds, since $\int_{\mathbb{D}} |f|^2\,dA_\alpha = B_\alpha[|f|^2](0)$ and $\int_{\mathbb{D}} |f|^{-2}\,dA_\alpha = B_\alpha[|f|^{-2}](0)$). First assume that $d(Q) > 0$. By Proposition 4.7, $Q \subset D(z_Q, R)$. By Lemma 4.6, there exists a positive constant C such that

$$\frac{1}{C}|f(z_Q)| \leqslant |f(z)| \leqslant C|f(z_Q)|,$$

for all $z \in Q$. Therefore

$$\left(\frac{1}{A_\alpha(Q)}\int_Q |f|^2\,dA_\alpha\right)\left(\frac{1}{A_\alpha(Q)}\int_Q |f|^{-2}\,dA_\alpha\right) \leqslant \left(C^2|f(z_Q)|^2\right)\left(C^2|f(z_Q)|^{-2}\right) = C^4.$$

Next assume that $d(Q) = 0$. Using Lemma 4.5 we have

$$\begin{aligned} B_\alpha[|f|^2](z_Q) &= \int_{\mathbb{D}} |f|^2 |k^{(\alpha)}_{z_Q}|^2 \, dA_\alpha \\ &\geqslant \int_Q |f|^2 |k^{(\alpha)}_{z_Q}|^2 \, dA_\alpha \\ &\geqslant \frac{c_\alpha}{(1-|z_Q|)^{2+\alpha}} \int_Q |f|^2 \, dA_\alpha. \end{aligned}$$

Since $Q \neq \mathbb{D}$ and $d(Q) = 0$ we have $|z_Q| \geqslant 1/2$, and it follows from (4.4) that

$$A_\alpha(Q) \geqslant 2^{2+\alpha}(1-|z_Q|)^{2+\alpha}.$$

Combining the above two inequalities yields

$$B_\alpha[|f|^2](z_Q) \geqslant \frac{2^{2+\alpha}\, c_\alpha}{A_\alpha(Q)} \int_Q |f|^2 \, dA_\alpha.$$

A similar inequality holds for f^{-1}. Thus we have

$$\begin{aligned} &\left(\frac{1}{A_\alpha(Q)} \int_Q |f|^2 \, dA_\alpha\right) \left(\frac{1}{A_\alpha(Q)} \int_Q |f|^{-2} \, dA_\alpha\right) \\ &\qquad \leqslant \left(\frac{B_\alpha[|f|^2](z_Q)}{2^{2+\alpha}\, c_\alpha}\right) \left(\frac{B_\alpha[|f|^{-2}](z_Q)}{2^{2+\alpha}\, c_\alpha}\right) \leqslant \frac{M}{4^{2+\alpha}\, c_\alpha^2}, \end{aligned}$$

as desired. □

Lemma 4.9. *Let $-1 < \alpha < \infty$ and suppose that $f \in A^2_\alpha$ satisfies the invariant weight condition (M_2). For every $w \in \mathbb{D}$ let $d\mu^{(\alpha)}_w = |f \circ \varphi_w|^2 \, dA_\alpha$. If $0 < \gamma < 1$, then there exists a $0 < \delta < 1$ such that*

$$\mu^{(\alpha)}_w(E) \leqslant \delta \mu^{(\alpha)}_w(Q),$$

whenever E a subset of Q with $A_\alpha(E) \leqslant \gamma\, A_\alpha(Q)$.

Proof. Suppose that $B_\alpha[|f|^2](w) B_\alpha[|f|^{-2}](w) \leqslant M$, for all $w \in \mathbb{D}$. Let E be a subset of Q with $A_\alpha(E) \leqslant \gamma\, A_\alpha(Q)$. Applying the inequality of Cauchy-Schwarz and Lemma 4.8 we have

$$\begin{aligned} A_\alpha(Q \setminus E)^2 &= \left(\int_{Q\setminus E} |f \circ \varphi_w|\, |f \circ \varphi_w|^{-1} \, dA_\alpha\right)^2 \\ &\leqslant \left(\int_{Q\setminus E} |f \circ \varphi_w|^2 \, dA_\alpha\right) \left(\int_{Q\setminus E} |f \circ \varphi_w|^{-2} \, dA_\alpha\right) \\ &\leqslant \left(\int_{Q\setminus E} |f \circ \varphi_w|^2 \, dA_\alpha\right) \left(\int_{Q} |f \circ \varphi_w|^{-2} \, dA_\alpha\right) \\ &\leqslant \left(\int_{Q\setminus E} |f \circ \varphi_w|^2 \, dA_\alpha\right) C\, A_\alpha(Q)^2 \left(\int_{Q} |f \circ \varphi_w|^{2} \, dA_\alpha\right)^{-1} \\ &= C\, A_\alpha(Q)^2 \left\{1 - \frac{\mu^{(\alpha)}_w(E)}{\mu^{(\alpha)}_w(Q)}\right\}. \end{aligned}$$

It follows that

$$\frac{\mu_w^{(\alpha)}(E)}{\mu_w^{(\alpha)}(Q)} \leqslant 1 - \frac{1}{C}\left(1 - \frac{A_\alpha(E)}{A_\alpha(Q)}\right)^2 \leqslant \delta,$$

if we put $\gamma = 1 - (1-\gamma)^2/C$. □

The Dyadic Maximal Function. The dyadic maximal operator $\mathcal{M}_\alpha$ is defined by

$$(\mathcal{M}_\alpha f)(w) = \sup_{w \in Q} \frac{1}{A_\alpha(Q)} \int_Q |f| \, dA_\alpha,$$

where the supremum is over all dyadic rectangles Q that contain w. The maximal function is of weak-type (1,1) and the maximal function is greater than the dyadic maximal function, so the dyadic maximal function of any continuous integrable function is finite on $\mathbb{D}$. In particular, if $f \in A^2_\alpha$ satisfies the invariant A_2-condition, then the dyadic maximal function $\mathcal{M}_\alpha |f|^2$ is always finite. This can also be seen directly as follows. Given a point $w \in \mathbb{D}$, there is a number $0 < R < 1$ such that all but a finite number of dyadic rectangles containing the point w lie inside the closed disk $\bar{D}(0,R) = \{z \in \mathbb{C} : |z| \leqslant R\}$. If $f \in A^2_\alpha$ and Q is a dyadic rectangle containing w inside the disk $\bar{D}(0,R)$, then

$$\frac{1}{A_\alpha(Q)} \int_Q |f(z)|^2 \, dA_\alpha(z) \leqslant \max\{|f(z)|^2 : |z| \leqslant R\}.$$

If $Q_1, \dots, Q_m$ are dyadic rectangles containing w *not* contained in the disk $\bar{D}(0,R)$, then

$$\mathcal{M}_\alpha |f|^2(w) \leqslant \max\{|f(z)|^2 : |z| \leqslant R\} + \max_{1 \leqslant j \leqslant m} \frac{1}{|Q_j|} \int_{Q_j} |f(z)|^2 \, dA(z) < \infty.$$

This proves that the dyadic function of $|f|^2$ is finite on $\mathbb{D}$.

The principal fact about the dyadic maximal function is the Calderon-Zygmund decomposition formulated in the next theorem. We will need the notion of "doubling" of dyadic rectangles in its proof. Suppose that $n \geqslant 1$ and m, k are positive integers such that $m, k \leqslant 2^n$. The double of $Q = Q_{n,m,k}$, denoted by $2Q$, is defined by

$$2Q = Q_{n-1,[(m+1)/2],[(k+1)/2]},$$

where $[\ell]$ denotes the greatest integer less than or equal to ℓ.

Doubling Property. The following figures shows a dyadic rectangle Q and its double $2Q$.

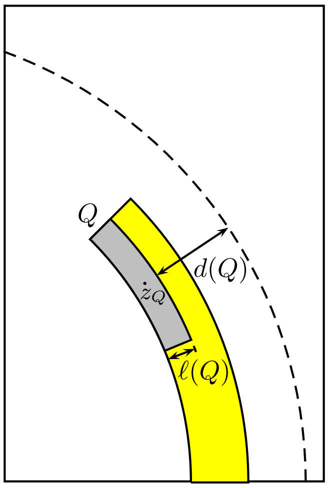

Figure 3: Dyadic rectangle and its double

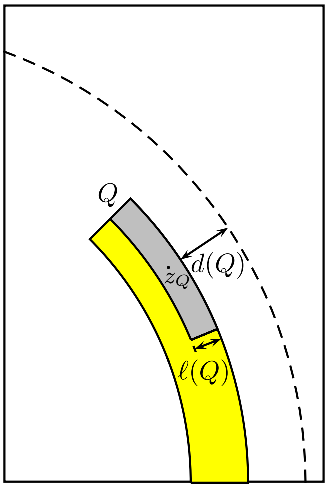

Figure 4: Dyadic rectangle and its double

Using (4.3) as well as $d(2Q) = d(Q) - \frac{1}{2}\ell(Q)$ and $\ell(2Q) = 2\ell(Q)$, an elementary calculation shows that

$$\frac{|2Q|}{|Q|} \leqslant 8, \tag{4.10}$$

for every proper dyadic rectangle Q in the unit disk. We will show that this doubling property extends to the weighted measures A_α. We first prove two elementary lemmas.

Lemma 4.11. *For every dyadic rectangle in the unit disk other than $\mathbb{D}$ the following inequalities hold:*

$$\tfrac{1}{2}(1 - |z_Q|) < 1 - |z_{2Q}| < \tfrac{3}{2}(1 - |z_Q|).$$

Proof. If $2Q$ is closer to the unit circle, as in figure 3, then

$$1 - |z_Q| = 1 - |z_{2Q}| + \ell(Q)/2.$$

Clearly $1 - |z_{2Q}| < 1 - |z_Q|$. Since $\ell(Q) < 1 - |z_Q|$ we also have

$$1 - |z_{2Q}| = 1 - |z_Q| - \ell(Q)/2 > 1 - |z_Q| - (1 - |z_Q|)/2 = (1 - |z_Q|)/2.$$

Thus

$$\tfrac{1}{2}(1 - |z_Q|) < 1 - |z_{2Q}| < 1 - |z_Q|.$$

If $d(2Q) = d(Q)$, as in figure 4, then

$$1 - |z_{2Q}| = 1 - |z_Q| + \ell(Q)/2.$$

Clearly $1 - |z_{2Q}| > 1 - |z_Q|$. Since $\ell(Q) < 1 - |z_Q|$ we also have

$$1 - |z_{2Q}| = 1 - |z_Q| + \ell(Q)/2 < 1 - |z_Q| + \tfrac{1}{2}(1 - |z_Q|) = \tfrac{3}{2}(1 - |z_Q|).$$

Thus

$$(1 - |z_Q|) < 1 - |z_{2Q}| < \tfrac{3}{2}(1 - |z_Q|).$$

This completes the proof. $\square$

That the functions $(1-|z|^2)^\alpha$ are approximately constant on pseudohyperbolic disks is well-know. The following lemma gives concrete bounds.

Lemma 4.12. *Let $w \in \mathbb{D}$, $0 < r < 1$, and let α be a real number. Then*
$$\left(\frac{1-r}{1+r}\right)^{|\alpha|}(1-|w|^2)^\alpha \leqslant (1-|z|^2)^\alpha \leqslant \left(\frac{1+r}{1-r}\right)^{|\alpha|}(1-|w|^2)^\alpha,$$
for all $z \in D(w,r)$.

This lemma is easily proved using (2.3) and standard estimates.

The following proposition shows that doubling property (4.10) extends to the weighted cases.

Proposition 4.13. *If $-1 < \alpha < \infty$, then there exists a constant $N_\alpha < \infty$ such that*
$$\frac{A_\alpha(2Q)}{A_\alpha(Q)} \leqslant N_\alpha$$
for every dyadic squares Q in the unit disk which is not equal to $\mathbb{D}$.

Proof. Let Q be a dyadic square other than $\mathbb{D} = Q_{0,1,1}$, and let $2Q$ denote its double. There are three cases to consider.

Case 1. $d(2Q) > 0$. By Proposition 4.7 we have $2Q \subset D(z_{2Q}, R)$. Using Lemma 4.12 we get
$$\begin{aligned} A_\alpha(2Q) &= (\alpha+1)\int_{2Q}(1-|z|^2)^\alpha\,dA(z) \\ &\leqslant (\alpha+1)\left(\frac{1+R}{1-R}\right)^{|\alpha|}(1-|z_{2Q}|^2)^\alpha\int_{D(z_{2Q},R)}dA(z) \\ &= (\alpha+1)\left(\frac{1+R}{1-R}\right)^{|\alpha|}(1-|z_{2Q}|^2)^\alpha|2Q|. \end{aligned}$$
Since also $d(Q) > 0$ we also have
$$A_\alpha(Q) \geqslant (\alpha+1)\left(\frac{1-R}{1+R}\right)^{|\alpha|}(1-|z_Q|^2)^\alpha|Q|.$$
Thus
$$\frac{A_\alpha(2Q)}{A_\alpha(Q)} \leqslant \left(\frac{1+R}{1-R}\right)^{2|\alpha|}\frac{(1-|z_{2Q}|^2)^\alpha}{(1-|z_Q|^2)^\alpha}\,\frac{|2Q|}{|Q|},$$
and that this is bounded above follows from (4.10) as well as Lemma 4.11.

Case 2. $d(2Q) = 0$ and $d(Q) > 0$. By the Proposition 4.7, $Q \subset D(z_Q, R)$. Then
$$A_\alpha(Q) \geqslant (\alpha+1)\left(\frac{1-R}{1+R}\right)^{|\alpha|}(1-|z_Q|^2)^\alpha|Q|.$$
Since Q is near the boundary, $|z_Q| \geqslant 1/4$, and it follows from formula (4.3) that $|Q| \geqslant (1-|z_Q|^2)^2$, thus
$$A_\alpha(Q) \geqslant (\alpha+1)\left(\frac{1-R}{1+R}\right)^{|\alpha|}(1-|z_Q|^2)^{\alpha+2}.$$
By (4.4)
$$A_\alpha(2Q) = 4^{1+\alpha}|z_{2Q}|^{1+\alpha}(1-|z_{2Q}|)^{\alpha+2} \leqslant 4^{1+\alpha}(1-|z_{2Q}|)^{\alpha+2}.$$

Combining the last inequalities we have

$$\frac{A_\alpha(2Q)}{A_\alpha(Q)} \leqslant \frac{4^{\alpha+1}}{\alpha+1}\left(\frac{1+R}{1-R}\right)^{|\alpha|}\left(\frac{1-|z_{2Q}|}{1-|z_Q|}\right)^{2+\alpha},$$

which is bounded by Lemma 4.11.

Case 3. $d(2Q)=0$ and $d(Q)=0$. In this case, by (4.4)

$$A_\alpha(Q) = 4^{1+\alpha}|z_Q|^{1+\alpha}(1-|z_Q|)^{2+\alpha} \geqslant (1-|z_Q|)^{2+\alpha}$$

(since $|z_Q| \geqslant 1/2$). Hence

$$\frac{A_\alpha(2Q)}{A_\alpha(Q)} \leqslant 4^{1+\alpha}\left(\frac{1-|z_{2Q}|}{1-|z_Q|}\right)^{2+\alpha},$$

which is bounded by Lemma 4.11. This proves the doubling property. □

The following theorem should be compared with Lemma 1 in Section IV.3 (p. 150) of Stein's book [13].

Calderon-Zygmund Decomposition Theorem. *Let $-1<\alpha<\infty$ and f be locally integrable on $\mathbb{D}$, let $t>0$, and suppose that $\Omega=\{z\in\mathbb{D} : \mathcal{M}_\alpha f(z)>t\}$ is not equal to $\mathbb{D}$. Then Ω may be written as the disjoint union of dyadic rectangles $\{Q_j\}$ with*

$$t < \frac{1}{A_\alpha(Q_j)}\int_{Q_j}|f|\,dA_\alpha < N_\alpha\, t,$$

where N_α is as in Proposition 4.13.

Proof. Suppose that $w\in\Omega$, that is, $\mathcal{M}_\alpha f(w)>t$. Then there exists a dyadic rectangle Q containing w such that

$$\frac{1}{A_\alpha(Q)}\int_Q |f|\,dA_\alpha > t.$$

Now, if $z\in Q$, then

$$\mathcal{M}_\alpha f(z) \geqslant \frac{1}{A_\alpha(Q)}\int_Q |f|\,dA_\alpha > t,$$

and it follows $z\in\Omega$. This proves that $Q\subset\Omega$. It follows that $\Omega=\bigcup_j Q_j$. We may assume that the Q_j are *maximal* dyadic rectangles. Since $Q=Q_j$ is not equal to $\mathbb{D}$, by maximality its double $2Q$ is not contained in Ω. This means that $2Q$ contains a point z which is not in Ω. Since $\mathcal{M}_\alpha f(z)\leqslant t$, we obtain

$$\frac{1}{A_\alpha(2Q)}\int_{2Q}|f|\,dA_\alpha \leqslant \mathcal{M}_\alpha f(z) \leqslant t,$$

and hence

$$\int_Q |f|\,dA_\alpha \leqslant \int_{2Q}|f|\,dA_\alpha \leqslant tA_\alpha(2Q).$$

It follows that

$$\frac{1}{A_\alpha(Q)}\int_Q |f|\,dA_\alpha \leqslant t\frac{A_\alpha(2Q)}{A_\alpha(Q)} \leqslant N_\alpha\, t,$$

completing the proof. □

Before we prove the reversed Hölder inequality (Theorem 4.1), we need one more preliminary result for the dyadic maximal function:

Proposition 4.14. *If $f \in A^2_\alpha$, then*
(i) $|f|^2 \leqslant \mathcal{M}_\alpha |f|^2$ *on* $\mathbb{D}$, *and*

(ii) $\|f\|^2_\alpha \leqslant \mathcal{M}_\alpha |f|^2(0) \leqslant (4/3)^{2+\alpha} \|f\|^2_\alpha$.

Proof. (i) In fact, we will prove that if g is continuous on $\mathbb{D}$, then $|g(w)| \leqslant \mathcal{M}_\alpha g(w)$ for every $w \in \mathbb{D}$. Fix $w \in \mathbb{D}$. Let Q_0 be any dyadic rectangle containing w such that $\bar{Q}_0 \subset \mathbb{D}$. Since function g is uniformly continuous on Q_0, given $\epsilon > 0$, there is a $\delta > 0$ such that $|g(z) - g(w)| < \varepsilon$ whenever $z, w \in Q_0$ are such that $|z - w| < \delta$. If necessary, subdividing Q_0 a number of times, there exists a dyadic rectangle Q containing w with diameter less than δ. Then

$$|g(w)| \leqslant |g(z)| + |g(w) - g(z)| \leqslant |g(z)| + \varepsilon$$

for all $z \in Q$. This implies that

$$|g(w)| \leqslant \frac{1}{A_\alpha(Q)} \int_Q |g(z)| \, dA_\alpha(z) + \varepsilon \leqslant \mathcal{M}_\alpha g(w) + \varepsilon.$$

Therefore

$$|g(w)| \leqslant \mathcal{M}_\alpha g(w),$$

as desired.
(ii) Since $\mathbb{D}$ is a dyadic rectangle and A_α is a probability measure, we have

$$\mathcal{M}_\alpha |f|^2(0) \geqslant \frac{1}{A_\alpha(\mathbb{D})} \int_{\mathbb{D}} |f|^2 \, dA_\alpha = \|f\|^2_\alpha.$$

Suppose $f \in A^2_\alpha$. If Q is a dyadic rectangle other than $\mathbb{D}$ containing 0, then $Q \subset D(0, 1/2)$. Then for each z in the unit disk, $f(z) = \langle f, K^{(\alpha)}_z \rangle_\alpha$ and the inequality of Cauchy-Schwarz imply

$$|f(z)|^2 \leqslant \|f\|^2_\alpha \, \|K^{(\alpha)}_z\|^2_\alpha = \frac{1}{(1 - |z|^2)^{2+\alpha}} \|f\|^2_\alpha \leqslant (4/3)^{2+\alpha} \|f\|^2_\alpha,$$

for all $z \in D(0, 1/2)$. Since $Q \subset D(0, 1/2)$ it follows that

$$\frac{1}{A_\alpha(Q)} \int_Q |f|^2 \, dA_\alpha \leqslant (4/3)^{2+\alpha} \|f\|^2_\alpha.$$

We conclude that

$$\|f\|^2_\alpha \leqslant \mathcal{M}_\alpha |f|^2(0) \leqslant (4/3)^{2+\alpha} \|f\|^2_\alpha,$$

as desired. □

We are now ready to prove the reversed Hölder inequality contained in Theorem 4.1.

Proof of Theorem 4.1. First we prove that for some constant $C_M > 0$,

$$\int_{\mathbb{D}} |f|^{2+\varepsilon} \, dA_\alpha \leqslant C_M \left(\int_{\mathbb{D}} |f|^2 \, dA_\alpha \right)^{(2+\varepsilon)/2}.$$

Let m be a positive integer such that the constant N_α of Proposition 4.13 satisfies $N_\alpha \leqslant 2^{m-1}$. For each integer $k \geqslant 0$, set

$$E_k = \{ z \in \mathbb{D} : \mathcal{M}_\alpha |f|^2(z) > 2^{mk+\alpha} \|f\|^2_\alpha \}.$$

By Proposition 4.14 (ii) we have $\mathcal{M}_\alpha|f|^2(0) \leqslant (4/3)^{2+\alpha}\,\|f\|_\alpha^2 \leqslant 2^{mk+\alpha}\,\|f\|_\alpha^2$, for every positive integer k, so the set E_k does not contain 0. Fix $k \geqslant 1$. By the Calderon-Zygmund Decomposition Theorem, $E_k = \bigcup_j Q_j$, where Q_j are disjoint dyadic rectangles in E_k that satisfy

$$2^{mk+\alpha}\,\|f\|_\alpha^2 < \frac{1}{A_\alpha(Q_j)} \int_{Q_j} |f|\, dA_\alpha < 2^{mk+\alpha}\, N_\alpha \|f\|_\alpha^2,$$

thus

$$A_\alpha(Q_j) \leqslant 2^{-mk-\alpha}\,\|f\|_\alpha^{-2} \int_{Q_j} |f|\, dA_\alpha,$$

and

$$\int_{Q_j} |f|\, dA_\alpha < 2^{mk+\alpha}\, N_\alpha \|f\|_\alpha^2\, A_\alpha(Q_j).$$

Let Q be a maximal dyadic rectangle in E_{k-1}. Summing over all such $Q_j \subset Q$ gives that

$$A_\alpha(E_k \cap Q) = \sum_{j: Q_j \subset Q} A_\alpha(Q_j) \leqslant 2^{-mk-\alpha}\,\|f\|_\alpha^{-2} \int_Q |f|^2\, dA_\alpha,$$

since the Q_j are disjoint and their union is E_k. On the other hand, by maximality the double $2Q$ is not contained in E_{k-1}, and as in the proof of the Calderon-Zygmund Decomposition Theorem it follows that

$$\begin{aligned}\int_Q |f|^2\, dA_\alpha &\leqslant 2^{m(k-1)+\alpha}\, N_\alpha\, \|f\|_\alpha^2\, A_\alpha(Q)\\ &\leqslant 2^{m(k-1)+\alpha}\, 2^{m-1}\, \|f\|_\alpha^2\, A_\alpha(Q)\\ &= 2^{mk+\alpha-1}\, \|f\|_\alpha^2\, A_\alpha(Q).\end{aligned}$$

Hence

$$A_\alpha(E_k \cap Q) \leqslant \tfrac{1}{2} A_\alpha(Q).$$

Now by Lemma 4.9 there exists a $0 < \delta < 1$ such that

$$\mu_\alpha(E_k \cap Q) \leqslant \delta \mu_\alpha(Q),$$

where $d\mu_\alpha = |f|^2\, dA_\alpha$. Taking the union over all maximal dyadic rectangles Q in E_{k-1} gives

$$\mu_\alpha(E_k) \leqslant \delta \mu_\alpha(E_{k-1}),$$

and therefore

$$\mu_\alpha(E_k) \leqslant \delta^k \mu_\alpha(E_0) \leqslant \delta^k \|f\|_\alpha^2.$$

Now, using Proposition 4.14, we have

$$\begin{aligned}\int_{\mathbb{D}} |f|^{2+\varepsilon}\, dA_\alpha &\leqslant \int_{\mathbb{D}} (\mathcal{M}_\alpha|f|^2)^{\varepsilon/2}\, |f|^2\, dA_\alpha \\ &= \int_{\{\mathcal{M}_\alpha|f|^2 \leqslant 2^\alpha \|f\|_\alpha^2\}} (\mathcal{M}_\alpha|f|^2)^{\varepsilon/2}\, |f|^2\, dA_\alpha \\ &\qquad + \sum_{k=0}^{\infty} \int_{E_k \setminus E_{k+1}} (\mathcal{M}_\alpha|f|^2)^{\varepsilon/2}\, |f|^2\, dA_\alpha \\ &\leqslant 2^\alpha \|f\|_\alpha^\varepsilon \|f\|_\alpha^2 + \sum_{k=0}^{\infty} 2^{(m(k+1)+\alpha)\varepsilon/2} \|f\|_\alpha^\varepsilon\, \mu_\alpha(E_k) \\ &\leqslant 2^\alpha \|f\|_\alpha^{2+\varepsilon} + \sum_{k=0}^{\infty} 2^{(mk+m+\alpha)\varepsilon/2} \delta^k\, \|f\|_\alpha^{2+\varepsilon} \\ &\leqslant 2^\alpha \|f\|_\alpha^{2+\varepsilon} + 2^{(m+\alpha)\varepsilon/2}\, \|f\|_\alpha^{2+\varepsilon} \sum_{k=0}^{\infty} (2^{m\varepsilon/2}\delta)^k \\ &= \left(2^\alpha + \frac{2^{(m+\alpha)\varepsilon/2}}{1 - 2^{m\varepsilon/2}\delta}\right) \|f\|_\alpha^{2+\varepsilon},\end{aligned}$$

if $2^{m\varepsilon/2}\delta < 1$. Put $\varepsilon_M = 2\ln(1/(1+\delta))/(m\ln 2)$. If $0 < \varepsilon < \varepsilon_M$, then $2^{m\varepsilon/2} < 1/(1+\delta)$, thus $2^{m\varepsilon/2}/(1-2^{m\varepsilon/2}\delta) < 1$. So, if $C_M = 2^\alpha + 2^{\alpha\varepsilon_M/2}$, then for $0 < \varepsilon < \varepsilon_M$ we have shown that

$$\int_{\mathbb{D}} |f|^{2+\varepsilon}\, dA_\alpha \leqslant C_M \left(\int_{\mathbb{D}} |f|^2\, dA_\alpha\right)^{(2+\varepsilon)/2}.$$

For a fixed $w \in \mathbb{D}$, by Möbius-invariance of the Berezin transform we also have

$$M_\alpha = \sup_{z \in \mathbb{D}} B_\alpha[|f \circ \varphi_w|^2](z) B_\alpha[|f \circ \varphi_w|^{-2}](z).$$

Applying the above argument to the function $|f \circ \varphi_w|^2$ we obtain

$$\int_{\mathbb{D}} |f \circ \varphi_w|^{2+\varepsilon}\, dA_\alpha \leqslant C_M \left(\int_{\mathbb{D}} |f \circ \varphi_w|^2\, dA_\alpha\right)^{(2+\varepsilon)/2},$$

that is,

$$B_\alpha[|f|^{2+\varepsilon}](w) \leqslant C_M \left(B_\alpha[|f|^2](w)\right)^{(2+\varepsilon)/2},$$

as desired. □

Note that Theorem 4.1 combined with Theorem 1.2 gives a proof of Theorem 1.3.

Proof of Theorem 1.3. If $f \in A^2_\alpha$ satisfies the condition

$$\sup_{w \in \mathbb{D}} B_\alpha[|f|^2](w) B_\alpha[|f|^{-2}](w) < \infty,$$

then by the reversed Hölder inequality of Theorem 4.1, for some $\varepsilon > 0$,

$$\sup_{w \in \mathbb{D}} B_\alpha[|f|^{2+\varepsilon}](w)\, B_\alpha[|f|^{-(2+\varepsilon)}](w) < \infty,$$

for all $w \in \mathbb{D}$. By Theorem 1.2, $T_f T_{\overline{1/f}}$ is bounded on A^2_α. □

5. Invertible Toeplitz Products

In this section we will completely characterize the bounded invertible Toeplitz products $T_f T_{\bar{g}}$ on the weighted Bergman space A^2_α. We have the following result:

Theorem 5.1. *Let $-1 < \alpha < \infty$ and let $f, g \in A^2_\alpha$. Then: $T_f T_{\bar{g}}$ is bounded and invertible on A^2_α if and only if* $\sup\{B_\alpha[|f|^2](w)\, B_\alpha[|g|^2](w) : w \in \mathbb{D}\} < \infty$ *and* $\inf\{|f(w)|\,|g(w)| : w \in \mathbb{D}\} > 0$.

Proof. "$\Longrightarrow$" Suppose that $T_f T_{\bar{g}}$ is bounded and invertible on A^2_α. By Theorem 1.1 there exists a constant M such that

$$B_\alpha[|f|^2](w)\, B_\alpha[|g|^2](w) \leqslant M, \tag{5.2}$$

for all $w \in \mathbb{D}$. Note that

$$T_f T_{\bar{g}} k_w = \overline{g(w)} f k_w^{(\alpha)}.$$

Thus

$$\|T_f T_{\bar{g}} k_w^{(\alpha)}\|_2^2 = |g(w)|^2 \|f k_w^{(\alpha)}\|_2^2 = |g(w)|^2\, B_\alpha[|f|^2](w),$$

so the invertibility of $T_f T_{\bar{g}}$ yields

$$|g(w)|^2\, B_\alpha[|f|^2](w) \geqslant \delta_1 > 0 \tag{5.3}$$

for some constant δ_1 and for all $w \in \mathbb{D}$. Since also $T_g T_{\bar{f}} = (T_f T_{\bar{g}})^*$ is bounded and invertible, there also is a constant δ_2 such that

$$|f(w)|^2\, B_\alpha[|g|^2](w) \geqslant \delta_2 > 0 \tag{5.4}$$

for all $w \in \mathbb{D}$. Putting $\delta = \delta_1 \delta_2$, it follows from (5.2), (5.3) and (5.4) that

$$\delta \leqslant |f(w)|^2 |g(w)|^2\, B_\alpha[|f|^2](w) B_\alpha[|g|^2](w) \leqslant M |f(w)|^2 |g(w)|^2,$$

and thus

$$|f(w)|\,|g(w)| \geqslant \frac{\delta^{1/2}}{M^{1/2}},$$

for all $w \in \mathbb{D}$.

"$\Longleftarrow$" Suppose that

$$M = \sup\{B_\alpha[|f|^2](w)\, B_\alpha[|g|^2](w) : w \in \mathbb{D}\} < \infty,$$

and

$$\eta = \inf\{|f(w)|\,|g(w)| : w \in \mathbb{D}\} > 0.$$

By the inequality of Cauchy-Schwarz,

$$|f(w)|^2 \leqslant B_\alpha[|f|^2](w),$$

for all $w \in \mathbb{D}$, thus $|f(w)|\,|g(w)| \leqslant M^{1/2}$, for all $w \in \mathbb{D}$. So, fg is a bounded function on $\mathbb{D}$. Note that f and g cannot have zeros in $\mathbb{D}$. Since $|g(z)|^2 \geqslant \eta^2 |f(z)|^{-2}$, for all $z \in \mathbb{D}$, we have

$$B_\alpha[|g|^2](w) \geqslant \eta^2 B_\alpha[|f|^{-2}](w),$$

for all $w \in \mathbb{D}$. Consequently

$$M \geqslant B_\alpha[|f|^2](w)\, B_\alpha[|g|^2](w) \geqslant \eta^2 B_\alpha[|f|^2](w)\, B_\alpha[|f|^{-2}](w),$$

so that

$$B_\alpha[|f|^2](w)\, B_\alpha[|f|^{-2}](w) \leqslant M/\eta^2,$$

for all $w \in \mathbb{D}$. This means that f satisfies the (M_2) condition. By Theorem 1.3 the Toeplitz product $T_f T_{\overline{1/f}}$ is bounded on A^2_α. Since fg is bounded on $\mathbb{D}$, the operator $T_{\overline{fg}}$ is bounded on A^2_α. It follows that $T_f T_{\overline{g}} = T_f T_{\overline{1/f}} T_{\overline{fg}}$ is bounded on A^2_α.

The function $\psi = 1/(f\bar{g})$ is bounded on $\mathbb{D}$, so that the operator T_ψ is bounded on A^2_α. Using that

$$T_f T_{\bar{g}} T_\psi = I = T_\psi T_f T_{\bar{g}},$$

we conclude that $T_f T_{\bar{g}}$ is invertible on A^2_α. □

6. Fredholm Toeplitz Products

In this section we will completely characterize the bounded invertible Toeplitz products $T_f T_{\bar{g}}$ on A^2_α. We have the following result:

Theorem 6.1. *Let $-1 < \alpha < \infty$ and let f and g be in A^2_α. Then: $T_f T_{\bar{g}}$ is a bounded Fredholm operator on A^2_α if and only if $B_\alpha[|f|^2]\, B_\alpha[|g|^2]$ is bounded on $\mathbb{D}$ and the function $|f|\,|g|$ is bounded away from zero near $\partial\mathbb{D}$.*

The latter condition simply means that there exists a number r with $0 < r < 1$ such that $\inf\{|f(z)|\,|g(z)| : r < |z| < 1\} > 0$.

In the proof of the above theorem we will need the following lemma.

Lemma 6.2. *Let $-1 < \alpha < \infty$. Suppose that $f \in A^2_\alpha$ has a finite number of zeros. Let b denote the Blaschke product of the zeros of f and $F = f/b$. Then there exists a constant C_α, only depending on α, such that*

$$B_\alpha[|F|^2](w) \leqslant C_\alpha\, B_\alpha[|f|^2](w),$$

for all w in $\mathbb{D}$.

Proof. Choose $0 < R < 1$ be such that $|b(z)| > 1/\sqrt{2}$, for all $R < |z| < 1$. Suppose $w \in \mathbb{D}$. Then

$$\begin{aligned} B_\alpha[|f|^2](w) &= \int_{\mathbb{D}} |f(\varphi_w(z))|^2\, dA_\alpha(z) \\ &= \int_{\mathbb{D}} |b(\varphi_w(z))|^2\, |F(\varphi_w(z))|^2\, dA_\alpha(z) \\ &\geqslant \frac{1}{2} \int_{R<|\varphi_w(z)|<1} |F(\varphi_w(z))|^2\, dA_\alpha(z). \end{aligned}$$

By a change-of-variable,

$$\int_{R<|\varphi_w(z)|<1} |F(\varphi_w(z))|^2\, dA_\alpha(z) = \int_{R<|z|<1} |F(z)|^2\, \frac{(1-|w|^2)^{2+\alpha}}{|1-\bar{w}z|^{4+2\alpha}}\, dA_\alpha(z).$$

Now, if h is analytic on $\mathbb{D}$, then

$$\int_{\mathbb{D}} |h(z)|^2\, dA_\alpha(z) \leqslant \frac{\alpha+1}{(1-R^2)^{\alpha+1}} \int_{R<|z|<1} |h(z)|^2\, dA_\alpha(z). \tag{6.3}$$

It is enough to prove inequality (6.3) for monomials $h(z) = z^n$. Integration by parts shows that

$$\begin{aligned} \int_{R<|z|<1} |z|^{2n}\, dA_\alpha(z) &= \int_{R^2}^1 x^n (1-x)^\alpha\, dx \\ &= \frac{R^{2n}(1-R^2)^{\alpha+1}}{\alpha+1} + \frac{n}{\alpha+1} \int_{R^2}^1 x^{n-1}(1-x)^{\alpha+1}\, dx \\ &\geqslant \frac{R^{2n}(1-R^2)^{\alpha+1}}{\alpha+1}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{|z|\leqslant R} |z|^{2n}\, dA_\alpha(z) &\leqslant R^{2n}\left\{1-\frac{(1-R^2)^{\alpha+1}}{\alpha+1}\right\}\\ &= \frac{R^{2n}(1-R^2)^{\alpha+1}}{\alpha+1}\left\{\frac{\alpha+1}{(1-R^2)^{\alpha+1}}-1\right\}\\ &\leqslant \left\{\frac{\alpha+1}{(1-R^2)^{\alpha+1}}-1\right\}\int_{R<|z|<1}|z|^{2n}\, dA_\alpha(z).\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{D}} |z|^{2n}\, dA_\alpha(z) &= \int_{|z|\leqslant R}|z|^{2n}\, dA_\alpha(z)+\int_{R<|z|<1}|z|^{2n}\, dA_\alpha(z)\\ &\leqslant \frac{\alpha+1}{(1-R^2)^{\alpha+1}}\int_{R<|z|<1}|z|^{2n}\, dA_\alpha(z),\end{aligned}$$

proving inequality (6.3).

Applying the above estimate to the function

$$h(z)=F(z)\frac{(1-|w|^2)^{1+\alpha/2}}{(1-\bar{w}z)^{2+\alpha}},$$

we see that

$$\begin{aligned}&\int_{R<|z|<1}|F(z)|^2\,\frac{(1-|w|^2)^{2+\alpha}}{|1-\bar{w}z|^{4+\alpha}}\, dA_\alpha(z)\\ &\qquad\geqslant \frac{(1-R^2)^{\alpha+1}}{\alpha+1}\int_{\mathbb{D}}|F(z)|^2\,\frac{(1-|w|^2)^{2+\alpha}}{|1-\bar{w}z|^{4+\alpha}}\, dA_\alpha(z)\\ &\qquad\geqslant \frac{(1-R^2)^{\alpha+1}}{\alpha+1}\, B_\alpha[|F|^2](w).\end{aligned}$$

Thus

$$B_\alpha[|f|^2](w)\geqslant \frac{1}{2}\frac{(1-R^2)^{\alpha+1}}{\alpha+1}\, B_\alpha[|F|^2](w),$$

so that

$$B_\alpha[|F|^2](w)\leqslant C_\alpha\, B_\alpha[|f|^2](w),$$

with $C_\alpha=2(\alpha+1)/(1-R^2)^{\alpha+1}$, for all $w\in\mathbb{D}$. □

Proof of Theorem 6.1. "$\Longrightarrow$" If $T_fT_{\bar{g}}$ is bounded on A^2_α, then there is an M such that $B_\alpha[|f|^2]B_\alpha[|g|^2]\leqslant M$ on $\mathbb{D}$. If $T_fT_{\bar{g}}$ is Fredholm, then $T_fT_{\bar{g}}+\mathcal{K}$ is invertible in the Calkin algebra. Thus there exist a bounded operator V and a compact operator S such that

$$VT_fT_{\bar{g}}=I+S.$$

Using that $T_fT_{\bar{g}}k_w^{(\alpha)}=\overline{g(w)}fk_w^{(\alpha)}$ we have

$$\begin{aligned}\|V\|\,|g(w)|\,B_\alpha[|f|^2](w)^{1/2} &= \|V\|\|T_fT_{\bar{g}}k_w^{(\alpha)}\|_\alpha\\ &\geqslant \|VT_fT_{\bar{g}}k_w^{(\alpha)}\|_\alpha\\ &\geqslant \|k_w^{(\alpha)}\|_\alpha-\|Sk_w^{(\alpha)}\|_\alpha\\ &= 1-\|Sk_w^{(\alpha)}\|_\alpha.\end{aligned}$$

Since S is compact on A^2_α and $k^{(\alpha)}_w \to 0$ weakly on A^2_α, we have $\|Sk^{(\alpha)}_w\|_\alpha \to 0$ as $|w| \to 1^-$, so there exists an $0 < r_1 < 1$ such that $\|Sk^{(\alpha)}_w\|_\alpha < 1/2$, for all $r_1 < |w| < 1$. The above inequality shows that

$$|g(w)|^2\, B_\alpha[|f|^2](w) \geqslant M_1 \left(= \tfrac{1}{2}\|V\|^{-1}\right),$$

for all $r_1 < |w| < 1$. Since also $T_g T_{\bar f} = (T_f T_{\bar g})^*$ is Fredholm, there is a positive constant M_2 and a number r_2 with $0 < r_2 < 1$ such that

$$|f(w)|^2 B_\alpha[|g|^2](w) \geqslant M_2,$$

for all $r_2 < |w| < 1$. Thus

$$M_1 M_2 \leqslant |f(z)|^2 |g(z)|^2 B_\alpha[|f|^2](z) B_\alpha[|g|^2](z) \leqslant M |f(z)|^2 |g(z)|^2,$$

and hence

$$|f(z)|^2 |g(z)|^2 \geqslant M_1 M_2 / M,$$

for all $\max\{r_1, r_2\} < |z| < 1$.

"$\Longleftarrow$" Suppose that

$$|f(z)||g(z)| \geqslant \delta > 0, \tag{*}$$

for all $0 < r < |z| < 1$. Inequality (*) implies that f and g have no zeros in the annulus $\{z : r < |z| < 1\}$. Let b_1 and b_2 denote the (finite) Blaschke products of the zeros of f and g respectively. Then $F = f/b_1$ and $G = g/b_2$ are zero free, and by (*) we have

$$|F(z)|\,|G(z)| \geqslant \delta |b_1(z)|\,|b_2(z)|,$$

for all $r < |z| < 1$. The function on the right is positive and continuous on annulus $\{z : \frac{1}{2}(1+r) \leqslant |z| \leqslant 1\}$, thus has a positive minimum. So putting $\rho = \frac{1}{2}(1+r)$, we have

$$|F(z)|\,|G(z)| \geqslant \eta',$$

for all $\rho < |z| < 1$. Then

$$|G(z)| \geqslant \eta'\, |F(z)|^{-1},$$

for all $\rho < |z| < 1$. Note that

$$\eta'' = \inf\{|F(z)|\,|G(z)| : |z| \leqslant \rho\} > 0.$$

If we take $\eta = \min\{\eta', \eta''\}$, then

$$|G(z)| \geqslant \eta\, |F(z)|^{-1},$$

for all $z \in \mathbb{D}$. By Lemma 6.2

$$B_\alpha[|F|^2](z) \leqslant C_\alpha\, B_\alpha[|f|^2](z),$$

and

$$B_\alpha[|G|^2](z) \leqslant C_\alpha\, B_\alpha[|g|^2](z),$$

for all $z \in \mathbb{D}$. Thus

$$B_\alpha[|F|^2](z) B_\alpha[|G|^2](z) \leqslant M',$$

for all $z \in \mathbb{D}$. As before we conclude that

$$B_\alpha[|F|^2](z)\, B_\alpha[|F|^{-2}](z) \leqslant \frac{M'}{\eta^2},$$

for all $z \in \mathbb{D}$, so F satisfies condition (M_2). By Theorem 1.3 the Toeplitz product $T_F T_{1/\bar F}$ is bounded. As in the proof of Theorem 5.1 it follows that $T_F T_{\bar G}$ is bounded. This implies that

$$T_f T_{\bar g} = T_{b_1} T_F T_{\bar G} T_{\bar b_2}$$

is bounded.

Since $1/(F\bar{G})$ is bounded, the Toeplitz operator $T_{1/(F\bar{G})}$ is bounded, and it follows that $T_F T_{\bar{G}}$ is invertible. Since $T_{\bar{b}_2}$ is Fredholm, there is a bounded operator V_2 on A^2_α and a compact operator S_2 on A^2_α such that $T_{\bar{b}_2} V_2 = I + S_2$. It follows that

$$T_f T_{\bar{g}} V_2 = T_{b_1} T_F T_{\bar{G}} + T_{b_1} T_F T_{\bar{G}} S_2,$$

thus

$$T_f T_{\bar{g}} V_2 (T_F T_{\bar{G}})^{-1} = T_{b_1} + T_{b_1} T_F T_{\bar{G}} S_2 (T_F T_{\bar{G}})^{-1}.$$

Using that also T_{b_1} is Fredholm, there is a bounded operator V_1 on A^2_α and a compact operator S_1 on A^2_α such that $T_{b_1} V_1 = I + S_1$. Then

$$T_f T_{\bar{g}} V_2 (T_F T_{\bar{G}})^{-1} S_1 = I + S_1 + T_{b_1} T_F T_{\bar{G}} S_2 (T_F T_{\bar{G}})^{-1}.$$

Hence $T_f T_{\bar{g}} + \mathcal{K}$ is right-invertible in the Calkin algebra. Similarly $T_f T_{\bar{g}} + \mathcal{K}$ is left-invertible in the Calkin algebra, so that $T_f T_{\bar{g}}$ is Fredholm. □

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