Self-intersections for the Willmore flow

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Abstract

We prove that the Willmore flow can drive embedded surfaces to self-intersections in finite time.

1 Introduction

In this paper we consider the Willmore flow in three space dimensions. We prove that embedded surfaces can be driven to a self-intersection in finite time. This situation is in strict contrast to the behavior of hypersurfaces under the mean curvature flow, where the maximum principle prevents self-intersections, but very much analogous to the surface diffusion flow.

The Willmore flow is a geometric evolution law in which the normal velocity of a moving surface equals the Laplace-Beltrami of the mean curvature plus some lower order terms. More precisely, we assume in the following that $\Gamma_0$ is a closed compact immersed and orientable surface in $\mathbb{R}^3$. Then the Willmore flow is governed by the law

$$V(t) = \Delta_{\Gamma(t)} H_{\Gamma(t)} + 2 H_{\Gamma(t)} (H^2 - K_{\Gamma(t)}) , \quad \Gamma(0) = \Gamma_0 . \quad (1.1)$$

Here $\Gamma = \{ \Gamma(t) ; t \geq 0 \}$ is a family of smooth immersed orientable surfaces, $V(t)$ denotes the velocity of $\Gamma$ in the normal direction at time $t$, while $\Delta_{\Gamma(t)}$, $H_{\Gamma(t)}$, and $K_{\Gamma(t)}$ stand for the Laplace-Beltrami operator, the mean curvature, and the Gauss curvature of $\Gamma(t)$, respectively.

The evolution law (1.1) does not depend on the local choice of the orientation. However, if $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$ we always choose the outer normal, so that $V(t)$ is positive if $\Omega(t)$ grows, and so that $H_{\Gamma(t)}$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$.

Any equilibrium of (1.1), that is, any closed smooth surface that satisfies the equation

$$\Delta H + 2 H (H^2 - K) = 0 \quad (1.2)$$

is called a Willmore surface [18, p. 282]. There has been much interest over the last years in characterizing Willmore surfaces, see for instance [15, 18] and the references cited therein. Willmore surfaces arise as the critical points of the
functional
\[ W(f) := \int_{f(M)} H^2 \, dS, \tag{1.3} \]
see [18, Section 7.4]. Here, \( M \) denotes a smooth closed orientable surface and \( f : M \to \mathbb{R}^3 \) is a smooth immersion of \( M \) into \( \mathbb{R}^3 \). Associated with this functional is a variational problem: Given a smooth closed orientable surface \( M_g \) of genus \( g \) determine the infimum \( W(M_g) \) of \( W(f) \) over all immersions \( f : M_g \to \mathbb{R}^3 \) and classify all manifolds \( f(M_g) \) which minimize \( W \). We refer to [4, 8, 14, 15, 17, 18] and the references therein for more details and interesting results.

The Willmore flow is the \( L^2 \)-gradient flow for the functional (1.3) on the moving boundary, see for example [7], and also [10] for related work on gradient flows. Thus the Willmore flow has the distinctive property that it evolves surfaces in such a way as to reduce the total quadratic curvature. To be more precise, we show that the flow decreases the total quadratic curvature for any \( C^{2+\beta} \) initial surface \( \Gamma_0 \).

**Proposition 1.** Let \( 0 < \beta < 1 \) and let \( \Gamma_0 \) be a closed compact immersed orientable surface that is \( C^{2+\beta} \)-smooth. Then

\[ \int_{\Gamma(t)} H^2(t) \, d\mu \leq \int_{\Gamma_0} H^2(0) \, d\mu, \quad 0 \leq t \leq T, \]

where \( [0, T] \) denotes the interval of existence guaranteed in the existence theorem of [16], and where \( H(t) \) denotes the mean curvature of \( \Gamma(t) \).

To the best of our knowledge, the result of Proposition 1 is new (under the given assumptions).

Next we show that the flow can force \( \Gamma(t) \) to lose embeddedness in order to decrease the total quadratic curvature.

**Theorem 2.** Let \( 0 < \beta < 1 \) be fixed.

There exist a closed embedded surface \( \Sigma_0 \in C^{2+\beta} \), a constant \( T_0 > 0 \), numbers \( t_0, t_1 \in (0, T_0) \) with \( t_0 < t_1 \), and a \( C^{2+\beta} \)-neighborhood \( U_0 \) of \( \Sigma_0 \) such that

(a) the Willmore flow (1.1) has a unique classical solution \( \Gamma = \{ \Gamma(t); t \in [0, T_0] \} \) for all \( \Gamma_0 \in U_0 \),

(b) \( \Gamma(t) \) ceases to be embedded for every \( t \in (t_0, t_1) \) and every \( \Gamma_0 \in U_0 \),

(c) each surface \( \Gamma(t) \) is of class \( C^\infty \) for \( t \in (0, T_0) \) and smooth in \( t \in (0, T_0) \).

It should be noted that the neighborhood \( U_0 \) of Theorem 2 also contains \( C^\infty \)-surfaces that will be driven to a self-intersection in finite time. Our approach relies on results and techniques in [6, 12, 16], and we follow closely the original argument in [12].

Lastly we mention that numerical simulations [13] seem to indicate that the Willmore flow can drive immersed surfaces to topological changes in finite time.
2 The mathematical setting

We first introduce some notations. Given an open set $U \subset \mathbb{R}^3$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^t(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order $s$. If $\Sigma$ is a (sufficiently) smooth submanifold of $\mathbb{R}^3$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for $\Sigma$. It is known that $BUC^t(\Sigma)$ is continuously embedded in $h^s(\Sigma)$ whenever $t > s$. In the following, we assume that $\Sigma$ is a smooth compact closed immersed oriented surface in $\mathbb{R}^3$.

Let $\nu$ be the unit normal field on $\Sigma$ commensurable with the chosen orientation. Then we can find $a > 0$ and an open covering $\{U_i ; i = 1, \ldots, m\}$ of $\Sigma$ such that

$$X_i : U_i \times (-a, a) \to \mathbb{R}^3, \quad X_i(s, r) := s + r\nu(s),$$

is a smooth diffeomorphism onto its image $\mathcal{R}_i := \text{im}(X_i)$, that is,

$$X_i \in \text{Diff}^\infty(U_i \times (-a, a), \mathcal{R}_i), \quad 1 \leq i \leq m.$$

This can be done by selecting the open sets $U_i \subset \Sigma$ in such a way that they are embedded in $\mathbb{R}^3$ instead of only immersed, and then taking $a > 0$ sufficiently small so that each of the $U_i$ has a tubular neighborhood of radius $a$. It follows that $\mathcal{R} := \bigcup \mathcal{R}_i$ consists of those points in $\mathbb{R}^3$ with distance less than $a$ to $\Sigma$. Let $\beta \in (0, 1)$ be fixed. Then we choose numbers $\alpha, \beta_1 \in (0, 1)$ with $\alpha < \beta_1 < \beta$. Let

$$W := \{ \rho \in h^{2+\beta_1}(\Sigma) : \|\rho\|_\infty < a \}, \quad (2.1)$$

Given any $\rho \in W$ we obtain a compact oriented immersed manifold $\Gamma_\rho$ of class $h^{2+\beta_1}$ by means of the following construction:

$$\Gamma_\rho := \bigcup_{i=1}^m \text{Im} \left( X_i : U_i \to \mathbb{R}^3, \ [s \mapsto X_i(s, \rho(s))] \right). \quad (2.2)$$

Thus $\Gamma_\rho$ is a graph in normal direction over $\Sigma$ and $\rho$ is the signed distance between $\Sigma$ and $\Gamma_\rho$. On the other hand, every compact immersed oriented manifold $\Gamma$ that is a smooth graph over $\Sigma$ and that is contained in $\mathcal{R}$ can be obtained in this way. For convenience we introduce the mapping

$$\theta_\rho : \Sigma \to \Gamma_\rho, \quad \theta_\rho(s) := X_i(s, \rho(s)) \text{ for } s \in U_i, \quad \rho \in W.$$

It follows that $\theta_\rho$ is a well-defined global $(2+\beta_1)$-diffeomorphism from $\Sigma$ onto $\Gamma_\rho$.

The Willmore flow (1.1) can now be expressed as an evolution equation for the distance function $\rho$ over the fixed reference manifold $\Sigma$,

$$\partial_t \rho = G(\rho), \quad \rho(0) = \rho_0. \quad (2.3)$$

Here $G(\rho) := L_\rho \theta_\rho^*(\Delta_{\Gamma_\rho} H_{\Gamma_\rho} + 2H_{\Gamma_\rho}(H_{\Gamma_\rho}^2 - K_{\Gamma_\rho}))$ for $\rho \in h^{3+\alpha}(\Sigma) \cap W$, while $\Delta_{\Gamma_\rho}$, $H_{\Gamma_\rho}$, and $K_{\Gamma_\rho}$ are the Laplace-Beltrami operator, the mean curvature, and
the Gauss curvature of $\Gamma_\rho$, respectively, and $L(\rho)$ is a factor that comes in by calculating the normal velocity in terms of $\rho$, see [6] for more details. We are now ready to state the following existence result for solutions of (2.3).

**Proposition 2.1.** Let $\sigma \in W$ be given.

(a) There exist a positive constant $T_0 > 0$ and a neighborhood $W_0 \subset W$ of $\sigma$ in $h^{2+\beta_1}(\Sigma)$ such that (2.3) has a unique solution

$$\rho(\cdot, \rho_0) \in C([0, T_0], W) \cap C^\infty((0, T_0) \times \Sigma) \text{ for every } \rho_0 \in W_0.$$  

(b) The map $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ defines a smooth local semiflow on $W_0$.

(c) $\rho(\cdot, \rho_0) \in C([0, T_0], h^{4+\alpha}(\Sigma)) \cap C^1([0, T_0], h^\alpha(\Sigma))$ for all $\rho_0 \in h^{4+\alpha}(\Sigma) \cap W_0$.

**Proof.** (a) and (b) follow from [16, Proposition 2.2]. Moreover, [16, Lemma 2.1] shows that the mapping $[\rho \mapsto G(\rho)] : h^{4+\alpha}(\Sigma) \cap W \to h^\alpha(\Sigma)$ is smooth and that the derivative is given by $G'(\rho) = P(\rho) + B(\rho)$, where

$$P(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad B(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad \rho \in h^{4+\alpha}(\Sigma) \cap W.$$  

In the following we fix $\rho \in h^{4+\alpha}(\Sigma) \cap W$. [16, Lemma 2.1] also shows that $P(\rho)$ generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$. A well-known perturbation result, see [1, Theorem I.1.3.1], then implies $G'(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ also generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$. It is known that the little Hölder spaces are stable under the continuous interpolation method [1, 2, 5, 9]. Therefore, the spaces $(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ form a pair of maximal regularity for $G'(\rho)$, see [1, Theorem III.3.4.1] or [2, 5, 9]. Part (c) follows now from maximal regularity results, for instance [2, Theorem 2.7]. $\square$

### 3 The proof of Proposition 1

We first note that any function in $C^{2+\beta}$ is also in $h^{2+\beta_1}$ for $\beta_1 \in (0, \beta)$. Let $\Gamma_0$ be a given surface in $\mathbb{R}^3$ that satisfies the assumptions of Proposition 1. We can find a smooth surface $\Sigma$ as in Section 2 and a function $\rho_0 \in W$ such that $\Gamma_0 = \Gamma_{\rho_0}$, where $\Gamma_{\rho_0}$ is defined in (2.2). According to Proposition 2.1(a) there exists a number $T = T(\rho_0) > 0$ such that equation (2.3) has a unique solution $\rho(\cdot, \rho_0)$ with the smoothness properties stated in the proposition. It follows from the construction in Section 2 that the family $\Gamma := \{\Gamma(t) ; 0 \leq t \leq T\}$, where $\Gamma(t) := \Gamma_{\rho(t)}$ for $0 \leq t \leq T$, is the unique classical solution for the Willmore flow (1.1). In particular, we conclude that

$$\left[ t \mapsto \int_{\Gamma(t)} H^2(t, \bar{\nu}) \, d\bar{\nu} \right] \in C^\infty((0, T), \mathbb{R}).$$
Given \( x \in \Gamma(t) \), let \( \{z(\tau, x) \in \mathbb{R}^3 \ ; \ \tau \in (-\varepsilon, \varepsilon)\} \) be an orthogonal flow line through \( x \), that is, \( z(\cdot, x) \) satisfies

\[
\begin{align*}
 z(\tau, x) & \in \Gamma(t + \tau) \text{ for } \tau \in (-\varepsilon, \varepsilon), \\
 \dot{z}(\tau) & = (VN)(t + \tau, z(\tau)) \text{ for } \tau \in (-\varepsilon, \varepsilon), \ z(0) = x,
\end{align*}
\]

where \( N(t, \cdot) \) denotes the unit normal field on \( \Gamma(t) \), and \( V(t, \cdot) \) is the normal velocity of \( \Gamma(t) \). A proof for the existence of a unique trajectory \( \{z(\tau, x) \in \mathbb{R}^3 \ ; \ \tau \in (-\varepsilon, \varepsilon)\} \) with the above properties can for instance be found in [11, Lemma 2.1]. For further use we introduce the manifold \( \mathcal{M} := \bigcup_{t \in (0, T)} \{t\} \times \Gamma(t) \). Given any smooth function \( u \) on \( \mathcal{M} \) we define

\[
\frac{d}{dt} u(t, x) := \left. \frac{d}{d\tau} u(t + \tau, z(\tau, x)) \right|_{\tau = 0}, \ (t, x) \in \mathcal{M}.
\]

The following differentiation rule is well-known in differential geometry,

\[
\frac{d}{dt} \int_{\Gamma(t)} u(t, x) \ d\mu(x) = \int_{\Gamma(t)} \frac{d}{d\tau} u(t, x) \ d\mu(x) + 2 \int_{\Gamma(t)} (uHV)(t, x) \ d\mu(x). \quad (3.1)
\]

Let \( (t, x) \in \mathcal{M} \) be fixed and let \( \{z(\tau, x) ; \ \tau \in (-\varepsilon, \varepsilon)\} \) be a flow line through \( x \). Then one can show that

\[
\left. \frac{d}{d\tau} H^2(t + \tau, z(\tau, x)) \right|_{\tau = 0} = -H[\Delta \Gamma(t)V + (4H^2 - 2K)V](t, x), \quad (3.2)
\]

see for instance [18, Section 7.4]. If follows from (3.1)–(3.2), from the divergence theorem, and from (1.1) that

\[
\frac{d}{dt} \int_{\Gamma(t)} H^2(t) \ d\mu = -\int_{\Gamma(t)} [\Delta H + 2H(H^2 - K)] V \ d\mu \leq 0. \quad (3.3)
\]

This is true for any \( t \in (0, T) \). The mean value theorem now implies that

\[
\int_{\Gamma(t)} H^2(t) \ d\mu - \int_{\Gamma(\tau)} H^2(\tau) \ d\mu \leq 0 \quad \text{for } 0 < \tau \leq t < T.
\]

Taking the limit as \( \tau \to 0 \) and using that \( \tau \mapsto \int_{\Gamma(\tau)} H^2(\tau) \ d\mu \in C([0, T], \mathbb{R}) \), see Proposition 2.1(b), yields the assertion of Proposition 1. \( \square \)

4 The proof of Theorem 2

In order to provide a proof of Theorem 2 we now choose \( \Sigma \) to be any smooth compact closed immersed orientable surface in \( \mathbb{R}^3 \) such that its image contains the flat 2-dimensional disk \( U := \{(s, 0) \in \mathbb{R}^2 \times \mathbb{R} \ ; \ |s| \leq 1\} \) twice, and with opposite
orientations. To be precise, let \( i : \Sigma \to \mathbb{R}^3 \) be the immersion under consideration, then we ask that

\[
i^{-1}(U) = U^+ \cup U^-
\]

with \( U^+ \cap U^- = \emptyset \) and both \( U^+ \) and \( U^- \) are flat 2-dimensional disks of radius 1. Additionally we ask that \( \Sigma \setminus (U^+ \cup U^-) \) is embedded in \( \mathbb{R}^3 \). Identifying \( U^+ \) for the moment with its image \( U \) we ask that the normal on \( U^+ \) points upwards, that is, \( \nu(\cdot)|_{U^+} = e_3 \), the 3rd basis vector of \( \mathbb{R}^3 \). It follows that \( \nu(\cdot)|_{U^-} = -e_3 \).

Fig. 1 This is a possible choice of \( \Sigma \), cut in halves.

Let \( W \) be as in (2.1) and let \( \sigma \in h^{4+\alpha} \cap W \) locally be radially symmetric with regards to the centers of \( U^\pm \). This implies \( \partial_j \sigma(0) = 0 \) for \( j = 1, 2 \). Observe that \( \theta_\sigma(s) = (s, \pm \sigma(s)) \) (these are coordinates in \( \mathbb{R}^3 \)) for \( s \in U^\pm \) and that \( \theta_\sigma : U^\pm \to \theta_\sigma(U^\pm) \) is an \( h^{4+\alpha} \)-diffeomorphism. It is straightforward to compute

\[
G(\sigma)|_{U^\pm} := L(\sigma)\theta_\sigma^*(\Delta_{\Gamma_\sigma} H_{\Gamma_\sigma} + 2 H_{\Gamma_\sigma}(H^2_{\Gamma_\sigma} - K_{\Gamma_\sigma}))|_{U^\pm}
\]

in local coordinates, yielding

\[
2G(\sigma)|_{U^\pm}(0) = -\Delta^2 \sigma(0) + \sum_{j,k=1}^2 (\partial_j \partial_k \sigma(0))^2 \Delta \sigma(0) + 2 \sum_{j,k,l=1}^2 \partial_j \partial_k \sigma(0) \partial_j \partial_l \sigma(0) \partial_k \partial_l \sigma(0),
\]

where \( \Delta \) is the Laplacian in Euclidean coordinates of \( \mathbb{R}^2 \) (see [6, Section 2] for more details). Because of the radial symmetry of \( \sigma \) we have \( H^2_{\Gamma_\sigma} = K_{\Gamma_\sigma} \) at the center of the disks \( U^\pm \), so that lower order term \( \theta_\sigma^*(2 H_{\Gamma_\sigma}(H^2_{\Gamma_\sigma} - K_{\Gamma_\sigma})) \) vanishes at the center of \( U^\pm \). We will now specify one more property of \( \sigma \). We choose \( r > 0 \) small and we require that \( \sigma(s) = |s|^4 \) for \( s \in U^\pm_s = \{ s \in U^\pm \mid |s| < r \} \). If \( r \) is small enough then this is compatible with \( \sigma \in h^{4+\alpha}(\Sigma) \cap W \). We conclude that

\[
G(\sigma)|_{U^\pm}(0) = -16 < 0. \tag{4.1}
\]
It follows from Proposition 2.1 that the evolution equation (2.3) with initial value $\rho(0) = \sigma$ has a unique solution

$$
\rho^*(\sigma) \in C([0, T_0], h^{4\alpha} (\Sigma)) \cap C^1([0, T_0], h^\alpha (\Sigma)) \right)
$$

(4.2)

Next we consider the restriction $\rho^*(t, \sigma)$ on $U^\pm$ of the function $\rho(t, \sigma)$, that is, $\rho^*(t, \sigma) := \rho(t, \sigma)|_{U^\pm}$ for $0 \leq t \leq T_0$, and we set $d^\pm(t) := \rho^*(t, \sigma)(0)$, to track the position of the center. It follows from (4.2) that $d^\pm \in C^1([0, T_0])$. Moreover, using the local character of $G$, we conclude that $d^\pm$ satisfies the equation

$$
(d^\pm)'(t) = G(\rho(t, \sigma)|_{U^\pm}(0) \quad \text{for} \quad 0 \leq t \leq T_0, \quad d^\pm(0) = 0 \right)
$$

(4.3)

Equations (4.1)-(4.3) and the mean value theorem yield

$$
d^\pm(t) = -Mt + \left( \int_0^t \left( (d^\pm)'(\tau t) - (d^\pm)'(0) \right) d\tau \right) t,
$$

(4.4)

where $M := 16$. It follows from (4.4) that there exists a positive constant $\mu > 0$ and an interval $(t_0, t_1) \subset (0, T_0]$ such that $d^\pm(t, \sigma)(0) = d^\pm(t) \leq -\mu$ for $t \in (t_0, t_1)$. By Proposition 2.1(b) we can find a function $\sigma_0 \in W_0$ such that $\Sigma := \Gamma_{\sigma_0}$ is embedded and such that $\Gamma(t) := \Gamma_{\rho(t, \sigma_0)}$ is immersed for at least $t \in (t_0, t_1)$. By employing Proposition 2.1(b) once more we conclude there is a neighborhood $W(\sigma_0) \subset W_0$ of $\sigma_0$ in $h^{2+\beta_1}(\Sigma)$ such that $\Gamma_{\sigma_0}$ is still embedded, whereas $\Gamma_{\rho(t, \sigma_0)}$ is immersed for $t \in (t_0, t_1)$ and all $\rho_0 \in W(\sigma_0)$. We note that $C^{2+\beta}(\Sigma)$ is contained in $h^{2+\beta_1}(\Sigma)$ with continuous injection $j : C^{2+\beta}(\Sigma) \to h^{2+\beta_1}(\Sigma)$. Hence $U_0 := j^{-1}(W(\sigma_0))$ is a $C^{2+\beta}$-neighborhood of $\sigma_0$ and Theorem 2 follows by choosing $\Sigma_0 := \Gamma_{\sigma_0}$ and $\Gamma_0 := \Gamma_{\rho_0}$ for $\rho_0 \in U_0$. \hfill \Box

![Fig. 2](image-url) This is half of $\Gamma_0$, a surface that loses embeddedness and becomes immersed. The gap might have to be much smaller than depicted.

**Remark 4.2.** The following is the essence of the construction: $\Gamma_0$ is an immersed surface such that its image contains two opposing fourth-order paraboloids touching only at the vertex. The global symmetry of $\Gamma_0$ is irrelevant, we only need the local symmetry at the center. Locally we can compute the initial velocity of $\Gamma_0$,
and it is such as to create an overlapping of the fourth-order paraboloids. A continuity argument then guarantees the same behavior for nearby embedded surfaces, which do exist by construction of $\Gamma_\sigma$. We have chosen a fourth-order paraboloid in order to facilitate the computation of $G(\sigma)|_{\Sigma^*}$. Any other configuration that produces the same sign as in (4.1) will work as well.

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