Differential and Integral Equations Volume 14, Number 8, August 2001, Pages 1005–1014

THE WILLMORE FLOW NEAR SPHERES

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Abstract. The Willmore flow leads to a quasilinear evolution equation of fourth order. We study existence, uniqueness and regularity of solutions. Moreover, we prove that solutions exist globally and converge exponentially fast to a sphere, provided that they are initially close to a sphere.

1. Introduction

Let Γ_0 be a compact, closed, immersed, orientable surface in $\mathbb{R}^3$. The Willmore flow consists in finding a family $\Gamma = \{\Gamma(t)\,;\, t \geq 0\}$ of smooth, closed, immersed, orientable hypersurfaces in $\mathbb{R}^3$ which evolve according to the law

$$V = \Delta H + 2H(H^2 - K), \qquad \Gamma(0) = \Gamma_0. \tag{1.1}$$

Here V denotes the normal velocity of Γ, while Δ, H and K stand for the Laplace–Beltrami operator, the mean curvature, and the Gauss curvature of $\Gamma(t)$, respectively. That is, $H = \frac{1}{2}(k_1 + k_2)$ and $K = k_1 k_2$, where k_1, k_2 are the principal curvatures of $\Gamma(t)$. If $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$ we choose the orientation induced by the outer normal, so that V is positive if $\Omega(t)$ grows, and so that H is positive if $\Gamma(t)$ is a sphere.

It is easy to verify that any sphere in $\mathbb{R}^3$ is an equilibrium for the Willmore flow. In this short note we shall study the stability of these equilibria.

In general, any equilibrium of (1.1), that is, any closed smooth surface that satisfies the equation

$$\Delta H + 2H(H^2 - K) = 0, \tag{1.2}$$

is called a Willmore surface [15, p. 282]. There has been much interest over the last years in characterizing Willmore surfaces in $\mathbb{R}^3$, as well as in the

[1]The research of the author has been partially supported by the Vanderbilt University Research Council and by NSF Grant DMS-9801337.

Accepted for publication July 2000.
AMS Subject Classifications: 35K55, 53A05, 58G99.

case that the ambient space is $\mathbb{S}^3$, the 3-dimensional unit sphere in $\mathbb{R}^4$; see for instance [11, 15] and the references cited therein.

Willmore surfaces arise as the critical points of the functional

$$W(f) := \int_{f(M)} H^2\, dS; \tag{1.3}$$

see [14] or [15, Section 7.4]. Here, M denotes a smooth, closed, orientable surface and $f : M \to \mathbb{R}^3$ is a smooth immersion of M into $\mathbb{R}^3$. Associated with this functional is a variational problem: Given a smooth, closed, orientable surface M_g of genus g determine the infimum $W(M_g)$ of $W(f)$ over all immersions $f : M_g \to \mathbb{R}^3$ and classify all manifolds $f(M_g)$ which minimize W. It is known that $W(M) \geq 4\pi$ for any surface M and that the minimum is attained if and only if $f(M)$ is embedded as a round sphere [15, Theorem 7.7.2]. A characterization of all Willmore immersions $f : \mathbb{S}^2 \to \mathbb{R}^3$ has been obtained in [3]. The possible values of $W(f)$ are $4\pi n$ where $n = 1$, or $n \geq 4$ even, or $n \geq 9$ odd. For the torus T^2 there is the long-standing "Willmore Conjecture" asserting that $W(T^2) = 2\pi^2$. The value $2\pi^2$ is attained by those tori of revolution whose generating circles have ratio $1/\sqrt{2}$; see for instance [15, p. 274]. It is known that there exist embedded Willmore surfaces in $\mathbb{R}^3$ of arbitrary genus. Such surfaces arise, for instance, as the images of embedded minimal surfaces in $\mathbb{S}^3$ under stereographic projection of $\mathbb{S}^3$ into $\mathbb{R}^3$. Examples of compact embedded Willmore surfaces that are not stereographic projections of compact embedded minimal surfaces in $\mathbb{S}^3$ were first found in [10]. Lastly we mention that all surfaces of genus g that are absolute minimizers of the functional W (if they exist) are necessarily embedded as Willmore surfaces. We refer to [11, 15] and the references therein for more details and interesting results.

In this paper, we show that the Willmore flow (1.1) admits a unique local smooth solution for any initial surface $\Gamma_0 \in C^{2+\beta}$. Moreover, it is shown that solutions that start out close to spheres with respect to the $C^{2+\beta}$-topology exist globally and converge exponentially fast to a sphere.

In order to state the main results we introduce some notation. Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$. Here $BUC^s(U)$ stands for the Banach space of all functions which are bounded and uniformly Hölder continuous of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ. It is known that $BUC^t(\Sigma)$ is continuously embedded in $h^s(\Sigma)$ whenever $t > s$.

Theorem 1.1. *Assume that* $0 < \beta < 1$, *and let* Γ_0 *be a compact, closed, immersed, orientable hypersurface in* $\mathbb{R}^n$ *belonging to the class* $h^{2+\beta}$.

(a) *The Willmore flow* (1.1) *has a unique local classical solution* $\Gamma = \{\Gamma(t) : t \in [0,T)\}$ *for some* $T > 0$. *Each hypersurface* $\Gamma(t)$ *is of class* C^∞ *for* $t \in (0,T)$. *Moreover, the mapping* $[t \mapsto \Gamma(t)]$ *is continuous on* $[0,T)$ *with respect to the* $h^{2+\beta}$*-topology and smooth on* $(0,T)$ *with respect to the* C^∞*-topology.*

(b) *Suppose that the initial hypersurface* Γ_0 *is a* $h^{2+\beta}$*-graph in the normal direction over some smooth, immersed, orientable hypersurface* Σ. *Then the mapping* $[(t,\Gamma_0) \mapsto \Gamma(t)]$ *induces a smooth local semiflow on an open subset of* $h^{2+\beta}(\Sigma)$.

The Willmore flow resembles the surface diffusion flow, which is governed by the law

$$V = \Delta H, \qquad \Gamma(0) = \Gamma_0. \tag{1.4}$$

This evolution equation has been studied in [8]. It should be mentioned that the surface diffusion flow preserves the volume of the region enclosed by $\Gamma(t)$ and reduces the surface area of $\Gamma(t)$. This is no longer true for the Willmore flow. Moreover, the only equilibria of (1.4) are surfaces with constant mean curvature (which leaves only the spheres in the case of embedded surfaces). This provides some evidence that the dynamics for the Willmore flow might be much more complex than for the surface diffusion flow. Nevertheless, we prove that spheres still are exponentially attracting for (1.1), as they are for (1.4).

Theorem 1.2. *Let* S *be a fixed Euclidean sphere and let* $\mathcal{M}$ *denote the set of all spheres which are sufficiently close to* S. *Then* $\mathcal{M}$ *attracts all solutions which are* $h^{2+\beta}(S)$*-close to* $\mathcal{M}$ *at an exponential rate. In particular, if* Γ_0 *is sufficiently close to* S *in* $h^{2+\beta}(S)$ *then* $\Gamma(t)$ *exists globally and converges exponentially fast to some sphere in* $\mathcal{M}$. *The convergence is in the* C^k*-topology for a fixed* $k \in \mathbb{N}$.

It would be interesting to know if a corresponding result also holds for the tori of revolution described above. It is not known whether or not the Willmore flow can develop singularities in finite time. In [9] a lower bound on the lifespan of a smooth solution is given which depends on how much the curvature of the initial surface is concentrated in space.

2. Local existence of solutions

This section is devoted to the proof of Theorem 1.1. In the following, we assume that Σ is a smooth, compact, closed, immersed, oriented hypersurface in $\mathbb{R}^n$, and that Γ_0 is close to this fixed reference manifold. Let ν be the unit normal field on Σ commensurable with the chosen orientation. Then we can find $a > 0$ and an open covering $\{U_l : l = 1, \dots, m\}$ of Σ such that $X_l : U_l \times (-a, a) \to \mathbb{R}^n$, $X_l(s, r) := s + r\nu(s)$ is a smooth diffeomorphism onto its image $\mathcal{R}_l := \mathrm{im}(X_l)$; that is,

$$X_l \in \mathit{Diff}^{\infty}(U_l \times (-a, a), \mathcal{R}_l), \qquad 1 \le l \le m.$$

This can be done by selecting the open sets $U_l \subset \Sigma$ in such a way that they are embedded in $\mathbb{R}^n$ instead of only immersed, and then taking $a > 0$ sufficiently small so that each of the U_l has a tubular neighborhood of radius a. It follows that $\mathcal{R} := \cup \mathcal{R}_l$ consists of those points in $\mathbb{R}^n$ with distance less than a to Σ. Let $\beta \in (0, 1)$ be fixed. Then we choose numbers α, $\beta_0 \in (0, 1)$ with $\alpha < \beta_0 < \beta$. Let

$$W := \{\rho \in h^{2+\beta_0}(\Sigma)\,;\, \|\rho\|_\infty < a\}. \tag{2.1}$$

Given any $\rho \in W$ we obtain a compact, oriented, immersed manifold Γ_ρ of class $h^{2+\beta_0}$ by means of the following construction:

$$\Gamma_\rho := \bigcup_{l=1}^{m} \mathrm{Im}\left(X_l : U_l \to \mathbb{R}^n\,,\ [s \mapsto X_l(s, \rho(s))]\,\right).$$

Thus Γ_ρ is a graph in the normal direction over Σ and ρ is the signed distance between Σ and Γ_ρ. For convenience we introduce the mapping

$$\theta_\rho \,:\, \Sigma \to \Gamma_\rho, \qquad \theta_\rho(s) := X_l(s, \rho(s)) \text{ for } s \in U_l\,, \quad \rho \in W.$$

It follows that θ_ρ is a well-defined global $(2+\beta_0)$–diffeomorphism from Σ onto Γ_ρ. In the following Δ_{Γ_ρ}, H_{Γ_ρ} and K_{Γ_ρ} denote the Laplace-Beltrami operator, the mean curvature and the Gauss curvature of Γ_ρ, respectively. Let θ^*_ρ denote the pull-back operator induced by the diffeomorphism θ_ρ. We can then define the transformed operators Δ_ρ, $H(\rho)$ and $K(\rho)$:

$$\Delta_\rho := \theta^*_\rho \Delta_{\Gamma_\rho} [\theta^*_\rho]^{-1}, \quad H(\rho) := \theta^*_\rho H_{\Gamma_\rho}, \quad K(\rho) := \theta^*_\rho K_{\Gamma_\rho}.$$

The Willmore flow (1.1) can now be expressed as an evolution equation for the distance function ρ over the fixed reference manifold Σ,

$$\partial_t \rho = G(\rho)\,, \qquad \rho(0) = \rho_0, \tag{2.2}$$

where $G(\rho)$ is given by

$$G(\rho) := L_\rho\Big(\Delta_\rho H(\rho) + 2H(\rho)\big(H^2(\rho) - K(\rho)\big)\Big) \tag{2.3}$$

for $\rho \in W \cap h^{4+\alpha}(\Sigma)$. L_ρ is a factor that comes in by calculating the normal velocity in terms of ρ; see [8] for more details.

In order to state our next result, let E_1 and E_0 be Banach spaces with $E_1 \hookrightarrow E_0$, and let $\mathcal{H}(E_1, E_0)$ be the set of all bounded linear operators $A \in \mathcal{L}(E_1, E_0)$ which have the additional property that $-A$, considered as an unbounded operator in E_0, generates a strongly continuous analytic semigroup on E_0. It can be shown that $\mathcal{H}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$; cf. [2, Theorem 1.3.1]. It is always assumed that $\mathcal{H}(E_1, E_0)$ carries the corresponding relative topology.

Lemma 2.1. *There exist mappings*

$$A \in C^\infty(W, \mathcal{H}(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))), \quad F \in C^\infty(W, h^{\beta_0}(\Sigma))$$

such that $G(\rho) = -A(\rho)\rho + F(\rho)$, $\rho \in W \cap h^{4+\alpha}(\Sigma)$.

Proof. It has been shown in [8, Lemma 2.1] that there exist mappings

$$A \in C^\infty(W, \mathcal{H}(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))), \quad F_1 \in C^\infty(W, h^{\beta_0}(\Sigma))$$

such that

$$L_\rho \Delta_\rho H(\rho) = -A(\rho)\rho + F_1(\rho), \quad \rho \in W \cap h^{4+\alpha}(\Sigma).$$

By using well-known representations of $H(\rho)$ and $K(\rho)$ in local coordinates we also infer that the mapping

$$[\rho \mapsto F_2(\rho) := 2L_\rho H(\rho)\big(H^2(\rho) - K(\rho)\big)] : W \to h^{\beta_0}(\Sigma)$$

is smooth. The assertion follows by setting $F(\rho) := F_1(\rho) + F_2(\rho)$. ☐

Lemma 2.1 shows that the Willmore flow leads to a parabolic quasilinear evolution equation of fourth order. We can now use the general results of H. Amann for parabolic quasilinear equations to obtain existence and uniqueness of solutions to (2.2).

Proposition 2.2. **(a)** *Let* $\rho_0 \in W_\beta := W \cap h^{2+\beta}(\Sigma)$ *be given. Then there exists a positive constant* $T = T(\rho_0) > 0$ *such that* (2.2) *has a unique maximal solution* $\rho(\cdot, \rho_0) \in C([0,T), W_\beta) \cap C^\infty((0,T), C^\infty(\Sigma))$.

(b) *The map* $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ *defines a smooth local semiflow on* W_β.

Proof. The assertions follow from the results in [1, Section 12] and a bootstrapping argument; see [8] for more details. ☐

It is now evident that Theorem 1.1 follows from Proposition 2.2.

3. Global existence

In order to prove Theorem 1.2 we fix a Euclidean sphere S and set $\Sigma = S$ in the construction of Section 2. In order to simplify the notation we will assume that $S = \mathbb{S}^2$, the unit sphere centered at 0. It follows from Lemma 2.1 that the mapping $G : W \cap h^{4+\alpha}(S) \to h^\alpha$, $\rho \mapsto G(\rho)$ is smooth. Hence we can consider the derivative $A := G'(0)$ of G at $\rho = 0$.

Lemma 3.1. $A = -\frac{1}{2}\Delta_S(\Delta_S + 2)$, *with* Δ_S *the Laplace-Beltrami operator on* S.

Proof. In the following we fix $g \in h^{4+\alpha}(S)$. Let $\varepsilon_0 > 0$ be sufficiently small such that $\|\varepsilon_0 g\|_\infty < a$, where the number a was introduced in Section 2. It follows that $G(\varepsilon g)$ is well-defined for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The Fréchet derivative $G'(0)g$ can be calculated as

$$G'(0)g = \frac{d}{d\varepsilon} G(\varepsilon g)\big|_{\varepsilon=0}, \tag{3.1}$$

and it immediately follows from [8, Lemma 2.1] that

$$\frac{d}{d\varepsilon}\Delta_{\varepsilon g} H(\varepsilon g)\big|_{\varepsilon=0} = -\frac{1}{2}\Delta_S(\Delta_S + 2)g. \tag{3.2}$$

Moreover,

$$\frac{d}{d\varepsilon} H(\varepsilon g)\big|_{\varepsilon=0}(H^2(0) - K(0)) = 0 \tag{3.3}$$

since $H^2(0) - K(0) = 0$. We claim that

$$\frac{d}{d\varepsilon}\big(H^2(\varepsilon g) - K(\varepsilon g)\big)\big|_{\varepsilon=0} = 0. \tag{3.4}$$

For this we first note that $4H^2 - 2K = |A|^2$, where $|A|^2$ is the total curvature; that is, $|A|^2 = k_1^2 + k_2^2$, with k_1 and k_2 the principal curvatures. Hence we also have the relation $H^2 - K = \frac{1}{2}|A|^2 - H^2$. We shall now introduce local coordinates on S. Let $V \subset \mathbb{R}^2$ be open and $\varphi : V \to U$ be a smooth parametrization of an open chart $U \subset S$. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we set

$$\phi_\varepsilon : V \to \mathbb{R}^n, \quad \phi_\varepsilon(x) := \varphi(x) + \varepsilon g(\varphi(x))\nu(\varphi(x)),$$

where ν denotes the outward-pointing unit-normal field on S. Thus ϕ_ε provides a $h^{4+\alpha}$-parametrization of the manifold $M(\varepsilon) := \mathrm{im}(\phi_\varepsilon)$. Let

$$g_{ij}(\varepsilon) := (\partial_i\phi_\varepsilon | \partial_j\phi_\varepsilon), \qquad h_{ij}(\varepsilon) := -(\partial_i\partial_j\phi_\varepsilon | N(\varepsilon))$$

denote the components of the first and the second fundamental forms on $M(\varepsilon)$. Of course, $N(\varepsilon)$ stands for the unit-normal field of $M(\varepsilon)$ with the orientation induced by ν. We write $[g^{ij}(\varepsilon)]$ for the inverse matrix of $[g_{ij}(\varepsilon)]$

and we set, as is usual in differential geometry, $h^i_j(\varepsilon) := g^{ik}(\varepsilon)h_{kj}(\varepsilon)$ and $h^{ij}(\varepsilon) := g^{jk}(\varepsilon)h^i_k(\varepsilon)$. Finally, we just write g^{ij} for $g^{ij}(0)$ and so on. The total curvature $|A(\varepsilon)|^2$ expressed in local coordinates is given by $|A(\varepsilon)|^2 = h^j_i(\varepsilon)h^i_j(\varepsilon)$, whereas the mean curvature is given by $H(\varepsilon) = \frac{1}{2}g^{ij}(\varepsilon)h_{ij}(\varepsilon)$. The following formulas are well-known; see for instance [15, p. 281]:

$$\begin{aligned} &\frac{d}{d\varepsilon}g^{ik}(\varepsilon)\big|_{\varepsilon=0} = -2h^{ik}g, \qquad \frac{d}{d\varepsilon}h_{kj}(\varepsilon)\big|_{\varepsilon=0} = -(\nabla_j\nabla_k - h^l_k h_{jl})g \\ &\frac{d}{d\varepsilon}H(\varepsilon)\big|_{\varepsilon=0} = -\frac{1}{2}(g^{ij}\nabla_i\nabla_j + h^j_i h^i_j)g, \end{aligned} \tag{3.5}$$

where ∇_j denote covariant derivatives. We obtain from the first two equations

$$\begin{aligned} \frac{d}{d\varepsilon}h^i_j(\varepsilon)\big|_{\varepsilon=0} = \frac{d}{d\varepsilon}g^{ik}(\varepsilon)h_{kj}(\varepsilon)\big|_{\varepsilon=0} &= [-2h^{ik}h_{kj} - g^{ik}(\nabla_j\nabla_k - h^l_k h_{jl})]g \\ &= -[g^{ik}\nabla_j\nabla_k + h^i_k h^k_j]g. \end{aligned}$$

It is now easy to compute the derivative of $h^j_i(\varepsilon)h^i_j(\varepsilon)$ with the result

$$\frac{1}{2}\frac{d}{d\varepsilon}h^j_i(\varepsilon)h^i_j(\varepsilon)\big|_{\varepsilon=0} = -(h^{ij}\nabla_i\nabla_j + h^j_i h^i_k h^k_j)g.$$

Altogether we obtain in local coordinates

$$\frac{d}{d\varepsilon}\big(\frac{1}{2}|A(\varepsilon)|^2 - H^2(\varepsilon)\big)\Big|_{\varepsilon=0} = [-(h^{ij}\nabla_i\nabla_j + h^j_i h^i_k h^k_j) + H(g^{ij}\nabla_i\nabla_j + h^j_i h^i_j)]g. \tag{3.6}$$

This formula is, of course, true for any immersed manifold M in $\mathbb{R}^2$ for variations with respect to the vector field $g\nu$. In the particular case $M = \mathbb{S}^2$ we obtain $h_{ij} = -(\partial_i\partial_j\varphi|\nu) = -(\partial_i\partial_j\varphi|\varphi) = (\partial_i\varphi|\partial_j\varphi) = g_{ij}$. We conclude that $h^i_j = \delta^i_j$ and that $h^{ij} = g^{ij}$, where δ^i_j denotes the Kronecker symbol. Thus it follows that the expression on the right side of formula (3.6) vanishes, and we have proved claim (3.4). It remains to observe that $L_0 = 1$ and $\frac{d}{d\varepsilon}L_{\varepsilon g}|_{\varepsilon=0} = 0$; see [8, p. 1425] for a representation of L_ρ. Lemma 3.1 now follows from (3.1)–(3.4). □

Remark 3.2. If $S = r\mathbb{S}^2$, a sphere of radius r centered at the origin, then the same proof also shows that

$$A = -\frac{1}{2}\Delta_S\big(\Delta_S + \frac{2}{r^2}\big).$$

Lemma 3.3. *The spectrum of A consists of a sequence $\{\mu_k : k \in \mathbb{N}\}$ of real eigenvalues with $\mu_k < \mu_{k-1} < \cdots < \mu_1 < \mu_0 = 0$. μ_0 is an eigenvalue of* *geometric multiplicity* 4 *and* $\ker(A) = \mathrm{span}\{1, Y_1, Y_2, Y_3\}$, *where* Y_1, Y_2, Y_3 *are spherical harmonics of degree 1 in* $\mathbb{R}^3$.

Proof. This follows as in [8, Lemma 3.2] □

Proof of Theorem 1.2. The proof follows along the lines of the corresponding proof in [8]. For the reader's convenience we shall indicate some steps of the proof. We first show that the nonlinear equation (2.2) admits a stable 4-dimensional local center manifold $\mathcal{M}^c$. This implies, in particular, that $\mathcal{M}^c$ contains all small global solutions of (2.2). In a second step we then prove that $\mathcal{M}^c$ coincides with the manifold $\mathcal{M}$ of the theorem. It is well-known that local center manifolds are not unique in general. However, since each local center manifold of (2.2) consists of equilibria this implies uniqueness in our case.

Under suitable spectral assumptions for the linearization, the existence of center manifolds is well-known for finite-dimensional dynamical systems. The corresponding construction for quasilinear infinite-dimensional semiflows is considerably more involved. The basic technical tool here is the theory of maximal regularity, due to G. Da Prato and P. Grisvard [5]. In particular, these results allow us to treat (2.2) as a fully nonlinear perturbation of a linear evolution equation; see [6, 13].

(i) In a first step we briefly sketch the construction of a locally invariant center manifold $\mathcal{M}^c$ over $N := \ker(A)$. Let $Y_0 := |S|^{-1}\mathbf{1}$, and let $Pg := \sum_{k=0}^{3}(g|Y_k)Y_k$ for $g \in h^r(S)$. Then P is a continuous projection of $h^r(S)$ onto N parallel to $\ker(P)$, and it is easy to verify that P commutes with A; that is, $PAg = APg = 0$ for every $g \in h^{4+\alpha}(S)$. Therefore, $N = \mathrm{im}(P)$ and $\ker(P)$ are complementary subspaces of $h^{4+\alpha}(S)$ that reduce A. To simplify the notation we write $\pi^c = P$ and $\pi^s = (1-P)$, and we define $h_s^{4+\alpha}(S) := \pi^s(h^{4+\alpha}(S))$. It follows that $\sigma(\pi^c A) = \{0\}$ and $\sigma(\pi^s A) \subset (-\infty, 0)$. We can now apply Theorem 4.1 in [13]; see also [6]. These results imply that, given $m \in \mathbb{N}^*$, there exists an open neighborhood U_0 of 0 in N and a mapping

$$\gamma \in C^m(U_0, h_s^{4+\alpha}(S)) \quad \text{with} \quad \gamma(0) = 0,\ \partial\gamma(0) = 0$$

such that $\mathcal{M}^c := \mathrm{graph}(\gamma)$ is a locally invariant manifold for the semiflow generated by the quasilinear evolution equation (2.2). $\mathcal{M}^c$ is a 4-dimensional submanifold of $h^{4+\alpha}(S)$. Moreover, $\mathcal{M}^c$ attracts solutions of (2.2) that start in a sufficiently small $h^{2+\beta}(S)$-neighborhood $W_0 \subset W_\beta$ of 0 at an exponential rate, and $\mathcal{M}^c$ contains all small equilibria of (2.2); see [13, Theorems 4.1 and 5.8].

(ii) Step (i) shows that $\mathcal{M}^c$ contains all small equilibria of (2.2). We show that $\mathcal{M}^c$ in fact coincides with $\mathcal{M}$ near 0. Suppose that S' is a sphere which is sufficiently close to S. Let (z_1, z_2, z_3) be the coordinates of its center and let r be its radius. Recall that $S \subset \mathbb{R}^3$ is the unit sphere centered at the origin and let $z_0 := 1 - r$. If ρ measures the distance of S to S' in the normal direction with respect to S, then it can be verified by some elementary geometric considerations that

$$(1 + z_0)^2 = \sum_{k=1}^{3} \big((1 + \rho)Y_k - z_k\big)^2.$$

Here we used that the spherical harmonics $Y_k,\ k = 1, 2, 3$, are given as the restrictions of the harmonic coordinate functions $[x \mapsto x_k]$. Let $Y_0 := \mathbf{1}$. Solving the above identify for ρ, we obtain that S' can be parametrized over S by the distance function

$$\rho(z) = \sum_{k=1}^{3} z_k Y_k - Y_0 + \sqrt{\Big(\sum_{k=1}^{3} z_k Y_k\Big)^2 + (1 + z_0)^2 - \sum_{k=1}^{3} z_k^2}, \tag{3.7}$$

where $z := (z_0, \dots, z_3) \in \mathbb{R}^4$. If O is a sufficiently small neighborhood of 0 in $\mathbb{R}^4$, then it is clear that any sphere S' which is close to S can be characterized by (3.7) with $z \in O$. The mapping $[z \mapsto \rho(z)] : O \to h^{4+\alpha}(S)$ is one to one and smooth. Let $\mathcal{M} := \{\rho(z) : z \in O\}$. We conclude that $\mathcal{M} \subset \mathcal{M}^c$, since $\mathcal{M}$ consists of spheres, which are equilibria for the Willmore flow. We intend to show that $\mathcal{M} = \mathcal{M}^c$. This follows, for instance, if we can verify that $\pi^c(\mathcal{M})$ is an open neighborhood of 0 in N. In order to show this we investigate the mapping $F : O \to N$, $F(z) := \pi^c \rho(z)$. It is easy to see that the partial derivatives of F with respect to z_j at $0 \in O$ are given by $\partial_{z_0} F(0) = \mathbf{1}$ and $\partial_{z_k} F(0) = Y_k$ for $1 \le k \le 3$. We conclude that the Fréchet derivative $\partial F(0)$ of F at 0 is given by $\partial F(0)h = \sum_{k=0}^{3} h_k Y_k$ for $h \in \mathbb{R}^4$. Since the set $\{Y_k : 0 \le k \le 4\}$ is a basis of N, we conclude that $\partial F(0) \in \mathcal{L}(\mathbb{R}^4, N)$ is an isomorphism. Consequently, the inverse function theorem implies that F is a smooth diffeomorphism from O onto its image $V := \mathrm{im}(F)$, provided O is small enough. Therefore, $\pi^c(\mathcal{M})$ is an open neighborhood of 0 in N which can be assumed to coincide with the open neighborhood U_0 constructed in step (i).

(iii) It follows from step (ii) that the reduced flow of (2.2) on $\mathcal{M}^c$ consists exactly of equilibria. Therefore, 0 is a stable equilibrium for the reduced flow and we conclude that 0 is also stable for the evolution equation (2.2);

see Theorem 3.3 in [12]. In particular, there exists a neighborhood W_0 of 0 in $h^{2+\beta}(S)$ such that solutions of (2.2) exist globally and converge to $\mathcal{M}^c$ exponentially fast for every initial value $\rho_0 \in W_0$.

(iv) As in [7, Theorem 6.5 and Proposition 6.6], one shows the following result. Given $k \in \mathbb{N}$ and $\omega \in (0, -\mu_1)$ there exists a neighborhood $W_0 = W_0(k, \omega)$ of 0 in $h^{2+\beta}(S)$ with the following property: Given $\rho_0 \in W_0$, the solution $\rho(\cdot, \rho_0)$ of (2.2) exists globally and there exist $T = T(k, \omega) > 0$, $c = c(k, \omega) > 0$, and a unique $z_0 = z_0(\rho_0) \in U_0$ such that

$$\|\big(\pi^c \rho(t, \rho_0), \pi^s \rho(t, \rho_0)\big) - \big(z_0, \gamma(z_0)\big)\|_{C^k} \leq c e^{-\omega t} \|\pi^s \rho_0 - \gamma(\pi^c \rho_0)\|_{h^{2+\beta}}$$

for $t > T$. According to step (ii), $(z_0, \gamma(z_0))$ is a sphere, and the proof is now complete. □

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