Maximal regularity for degenerate evolution equations with an exponential weight function

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Dedicated to the memory of Günter Lumer

Abstract. In this contribution we consider degenerate evolution equations on the real line that have the distinguished feature that they contain an exponential weight function in front of the time derivative.

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1. Introduction

In this contribution we consider degenerate evolution equations on the real line that have the distinguished feature that they contain an exponential weight function. More precisely, we consider evolution equations of the type

$$e^{sx}\partial_t u + h(\partial_x)u = f, \quad u(0) = u_0, \tag{1.1}$$

where $s > 0$ is a fixed number, $x \in \mathbb{R}$ and $u = u(t,x)$. Here $h(\partial_x)$ is a pseudo-differential operator whose symbol $h = h(i\xi)$ is meromorphic in a vertical strip around the imaginary axis and satisfies appropriate growth conditions.

Our interest is motivated by problems that arise from elliptic or parabolic equations on angles and wedges, and by free boundary problems with moving contact lines. To describe the class of symbols we have in mind, let us consider the case of dynamic boundary conditions. It can be shown that the boundary symbol for the Laplace equation $\Delta u = 0$ on an angle $G = \{(r\cos\phi, r\sin\phi);\ r > 0\ \phi \in (0,\alpha)\}$ in $\mathbb{R}^2$ with Dirichlet condition $u = 0$ on $\phi = \alpha$ and dynamic boundary condition $\partial_t u + \partial_\nu u = g$ on $\phi = 0$ is given by

$$\partial_t e^x + \psi_0(-(\partial_x + \beta)^2), \quad \psi_0(z) = \sqrt{z}\coth(\alpha\sqrt{z}), \quad z \in \mathbb{C}.$$

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Here $\beta \in \mathbb{R}$ is a parameter that will ultimately determine the weight function corresponding to the angle α. Similarly, if one considers the one-phase Mullins-Sekerka problem in two dimensions with boundary intersection and prescribed contact angle $\alpha \in (0, \pi]$, one is led to the boundary symbol

$$\partial_t e^{3x} - \psi_1(-(\partial_x + \beta)^2)(\partial_x + \beta + 1)(\partial_x + \beta + 2),$$

where this time $\psi_1(z) = \sqrt{z}\tanh(\alpha\sqrt{z})$. The free boundary problem for the stationary Stokes equations with boundary contact and prescribed contact angle in two dimensions leads to

$$\partial_t e^x + \psi(\partial_x + \beta),$$

where

$$\psi(z) = (1+z)\frac{\cos(2\alpha z) - \cos(2\alpha)}{\sin(2\alpha z) + z\sin(2\alpha)},$$

in the slip case and

$$\psi(z) = \frac{(1+z)}{4}\frac{\sin(2\alpha z) - z\sin(2\alpha)}{z^2\sin^2(\alpha) - \cos^2(\alpha z)}$$

in the non-slip case. This motivates the study of equations of the type (1.1) and its parametric form

$$\nu e^{sx}u + h(\partial_x)u = f, \tag{1.2}$$

where $s > 0$, $\nu \in \mathbb{C}$.

It is our goal to identify function spaces such that the operators in (1.1) and (1.2) become topological isomorphisms between these spaces, i.e. to obtain optimal solvability results. We will do this in the framework of L_p-spaces. Our main tools are recent results on sums of sectorial operators, their $\mathcal{H}^\infty$-calculi, and $\mathcal{R}$-boundedness of associated operator families, see for instance [1, 2, 3, 4, 6, 7, 9].

Once this goal is achieved, one can go on to study symbols of higher dimensional or time-dependent problems. The symbols for the Mullins-Sekerka problem in higher dimensions, for the Stefan problem with surface tension, and for the free boundary of the non-steady Stokes problem will be the subject of future work.

The case where h is a second order polynomial is studied in detail in [8], and an application to a parabolic evolution equation in a wedge domain with dynamic boundary conditions is given.

Observe that equations (1.1) and (1.2) are highly degenerate, due to the presence of the exponentials. Therefore they are not directly accessible by standard methods for pseudo-differential operators. Moreover, the basic ingredients of these symbols, namely e^x and ∂_x, do not commute. Still, there is a close relation between these operators. In fact, e^{sx} is an eigenfunction of ∂_x with eigenvalue s, or to put it in a different way, the commutator between e^{sx} and ∂_x is se^{sx}. It is this relation we base our approach on. It allows us to apply abstract results on sums of non-commuting operators.

The plan for this paper is as follows. In Section 2 we introduce the symbol class $\mathcal{M}^r_{a,b}$. Our first main result, Theorem 2.5, states that parametric symbols of the form (1.2) lead to sectorial operators in $L_p(\mathbb{R})$ which admit a bounded $\mathcal{H}^\infty$-calculus. This result is used in Section 3 to show that problem (1.1) generates a bounded, strongly continuous, analytic semigroup on $L_p(\mathbb{R})$ for every symbol $h \in \mathcal{M}^r_{a,b}$, see Theorem 3.1 We can also show that the degenerate evolution equation (1.1) enjoys L_p-maximal regularity, provided h is replaced by $\omega_0 + h$ with an appropriate nonnegative number ω_0, see Proposition 3.2. We pose the open question whether or not ω_0 can in fact be chosen to be zero, and we answer this question in the affirmative in case that $p = 2$. Finally, in Section 5 we study some of the functions introduced above, and we characterize values of β so that the associated symbol h_β belongs to the symbol class $\mathcal{M}^r_{a,b}$.

In order to keep this paper short, we refer to [2, 7] for the definitions and for background material on sectorial operators, their $\mathcal{H}^\infty$-calculus, and the concept of $\mathcal{R}$-boundedness. For the reader's convenience, we will include a recent result on an $\mathcal{H}^\infty$-calculus for the sum of non-commuting operators. For this, we consider two sectorial operators A and B and we assume that A and B satisfy the Labbas-Terreni commutator condition [5], which reads as follows.

$$\begin{cases} 0 \in \rho(A). \text{ There are constants } c > 0,\ \ 0 \le \alpha < \beta < 1, \\ \psi_A > \phi_A,\ \psi_B > \phi_B,\ \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi-\psi_A},\ \mu \in \Sigma_{\pi-\psi_B} \\ \|A(\lambda + A)^{-1}[A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1}A^{-1}]\| \le c/(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{cases} \tag{1.3}$$

Assuming this condition we have the following generalization of a result by Kalton-Weis [3] on sums of operators to the non-commuting case, see [7].

Theorem 1.1. *Suppose $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{R}S(X)$ and suppose that (1.3) holds for some angles $\psi_A > \phi^\infty_A$, $\psi_B > \phi^R_B$ with $\psi_A + \psi_B < \pi$.*
Then there is a number $\omega_0 \ge 0$ such that $\omega_0 + A + B$ is invertible and sectorial with angle $\phi_{\omega_0+A+B} \le \max\{\psi_A, \psi_B\}$. Moreover, if in addition $B \in \mathcal{R}H^\infty(X)$ and $\psi_B > \phi^{R\infty}_B$, then $\omega_0 + A + B \in \mathcal{H}^\infty(X)$ and $\phi^\infty_{\omega_0+A+B} \le \max\{\psi_A, \psi_B\}$.

2. Parametric Symbols

In this section we consider the parametric problem

$$\nu e^{sx} u + h(\partial_x)u = f, \tag{2.1}$$

where $f \in L_p(\mathbb{R})$ for $1 < p < \infty$, $\nu \in \Sigma_\theta$, $s \in \mathbb{R}$, $s \neq 0$, and $h(\partial_x)$ is a pseudo-differential operator whose symbol h belongs to the class $\mathcal{M}^r_{a,b}$ defined below. We study the unique solvability of (2.1) in $L_p(\mathbb{R})$ with optimal regularity. This means that we are looking for a unique solution u of (2.1) such that $e^{sx}u \in L_p(\mathbb{R})$ and $u \in H^r_p(\mathbb{R})$, where $r \in \mathbb{R}$ denotes the order of the symbol $h(z)$. It is an important objective to obtain estimates for the solutions that are uniform in $\nu \in \Sigma_\theta$.

We introduce now the class of symbols. For this purpose we consider the vertical strip

$$S_{(a,b)} = \{z \in \mathbb{C} : \ a < \operatorname{Re} z < b\} \quad \text{where} \quad 0 \in (a,b).$$

Definition 2.1. Let $r \geq 1$ be a fixed number.
Then h is said to belong to the class $\mathcal{M}^r_{a,b}$ if

(i) $h(z)$ is a meromorphic function defined on the strip $S_{(a,b)}$,
(ii) $h(z)/|z|^r \to 1$ as $|z| \to \infty$, $z \in S_{(a,b)}$,
(iii) there are constants $C, N > 0$ such that

$$|zh'(z)| \leq C(1 + |z|^r), \quad z \in S_{(a,b)}, \quad |z| \geq N,$$

(iv) h has no poles on the line $i\mathbb{R}$,
(v) there exists a number $c_0 > 0$ such that $\operatorname{Re} h(i\xi) \geq c_0$ for all $\xi \in \mathbb{R}$.

Remark 2.2. The following properties are easy consequences of Definition 2.1.

(a) Suppose h satisfies (i)–(ii) in Definition 2.1. Then h has only finitely many poles in $S_{(a,b)}$.
(b) Suppose h satisfies (i)–(ii) and (iv)–(v) in Definition 2.1. Let

$$\theta_h := \sup\{|\arg h(i\xi)| : \xi \in \mathbb{R}\}.$$

Then $\theta_h < \pi/2$.

In the next proposition, we study some mapping properties of $h(\partial_x)$ and we derive an expression for the commutator $[e^{sx}, h(\partial_x)]$.

Proposition 2.3. *Let $r > 0$ and $1 < p < \infty$. Suppose $0, -s \in (a,b)$ and suppose that*

(i) *$g : S_{(a,b)} \to \mathbb{C}$ is meromorphic,*
(ii) *there are positive constants C and N such that*

$$|g(z)| + |zg'(z)| \leq C(1 + |z|^r), \quad z \in S_{(a,b)}, \ |z| \geq N,$$

(iii) *g has no poles on the lines $i\mathbb{R}$ and $i\mathbb{R} - s$.*

Let $g(\partial_x)$ and $g(\partial_x - s)$ be the pseudo-differential operators defined by

$$g(\partial_x)u := \mathcal{F}^{-1}(g(i\xi)\mathcal{F}u), \quad g(\partial_x - s)u := \mathcal{F}^{-1}(g(i\xi - s)\mathcal{F}u), \quad u \in \mathcal{S}(\mathbb{R}),$$

respectively, where $\mathcal{F}$ denotes the Fourier transform, and $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decaying functions. Then

(a) *the operators $g(\partial_x)$ and $g(\partial_x - s)$ are well-defined and*

$$g(\partial_x), \ g(\partial_x - s) \in \mathcal{B}(H^r_p(\mathbb{R}), L_p(\mathbb{R})).$$

(b) *For any function $v \in H^r_p(\mathbb{R})$ such that $e^{sx}v \in H^r_p(\mathbb{R})$ we have the identity*

$$e^{sx} g(\partial_x)v(x) = g(\partial_x - s)e^{sx}v(x) + e^{sx}\sum_k \int_{\mathbb{R}} p_k(x-y)e^{z_k(x-y)}v(y)dy,$$

for $x \in \mathbb{R}$, where z_k denote the finitely many poles with order m_k of g in the strip $S_{(-s,0)}$ and $p_k(x)$ are polynomials of order $m_k - 1$.

Proof. (a) Let m_σ be defined by $m_\sigma(\xi) = h(i\xi - \sigma)/(1+|\xi|^2)^{r/2}$ for $\xi \in \mathbb{R}$ and $\sigma = 0, s$. It is not difficult to see that m_σ satisfies $\sup_{\xi\in\mathbb{R}}(|m_\sigma(\xi)| + |\xi m'_\sigma(\xi)|) < \infty$, and the assertion follows from Mikhlin's multiplier theorem.

(b) Let $v \in \mathcal{D}(\mathbb{R})$ be a test function. Then by definition of the pseudo-differential operator $g(\partial_x)$ we have

$$g(\partial_x)v(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} g(i\xi)\mathcal{F}v(\xi)d\xi, \quad x \in \mathbb{R}.$$

Note that by assumption (ii) there are only finitely many poles z_k in the strip $S_{(-s,0)}$. Multiplying with e^{sx} and applying the residue theorem yields

$$\begin{aligned} e^{sx}g(\partial_x)v(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{(s+i\xi)x} g(i\xi)\mathcal{F}v(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} g(i\xi - s)\mathcal{F}v(\xi + is)d\xi + e^{sx}\sum_k \operatorname{Res}[e^{zx}g(z)\mathcal{F}v(-iz)]_{z=z_k} \\ &= g(\partial_x - s)e^{sx}v(x) + e^{sx}\sum_k \int_{\mathbb{R}} e^{z_k(x-y)}p_k(x-y)v(y)dy, \end{aligned}$$

where the $p_k(x)$ are polynomials of order $m_k - 1$ corresponding to the order m_k of the pole of $g(z)$ at $z = z_k$. The assertion now follows from an approximation argument. □

Next we state a result on kernel bounds for $h(\partial_x)^{-1}$ which is also of independent interest.

Proposition 2.4. *Suppose $r \geq 1$ and*

(i) *$h : S_{(-d,d)} \to \mathbb{C}$ is holomorphic for some $d > 0$,*
(ii) *there are positive constants c, C such that*

$$|h(z)| \geq c(|z|^r + 1) \quad \text{and} \quad |h(z)| + |zh'(z)| \leq C(1 + |z|^r), \ z \in S_{(-d,d)}.$$

Then

(a) *the operator $h(\partial_x)$ is well-defined and*

$$h(\partial_x) \in \mathit{Isom}(H^r_p(\mathbb{R}), L_p(\mathbb{R})).$$

(b) *$h(\partial_x)^{-1}$ is a convolution operator with kernel k, where $e^{\delta|\cdot|}k \in L_1(\mathbb{R})$ for some $\delta > 0$.*

Proof. (a) Mikhlin's theorem implies that $h(\partial_x)$ is a well-defined invertible operator with domain $H^r_p(\mathbb{R})$.

(b) The kernel of $h(\partial_x)^{-1}$ is given by the inverse Fourier transform of $h(i\xi)^{-1}$, i.e.

$$k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\frac{d\xi}{h(i\xi)}, \quad x \in \mathbb{R}.$$

Shifting the path of integration by $2\delta < d$ to the left or to the right, we obtain by Cauchy's theorem

$$e^{\pm 2\delta x}k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\frac{d\xi}{h(i\xi \mp 2\delta)}, \quad x \in \mathbb{R}.$$

Plancherel's theorem then yields $e^{2\delta|x|}k \in L_2(\mathbb{R})$. Using that $e^{\delta|x|}k = e^{-\delta|x|}e^{2\delta|x|}k$ we obtain from Hölder's inequality that $e^{\delta|x|}k \in L_1(\mathbb{R})$. □

We will now state our main result for problem (2.1). Before doing so, we introduce the following spaces

$$\begin{aligned} X_0 :=& L_p(\mathbb{R}), \\ X_1 :=& H^r_p(\mathbb{R}) \cap \{v \in L_p(\mathbb{R}) \ : e^{sx}v \in L_p(\mathbb{R})). \end{aligned} \tag{2.2}$$

Theorem 2.5. *Let $1 < p < \infty$, $r \geq 1$, and $a, b \in \mathbb{R}$ with $0, -s \in (a, b)$. Suppose the symbol h belongs to the class $\mathcal{M}^r_{a,b}$ and let θ_h be as in Remark* 2.2. *Then*

(a) $(\nu e^{sx} + h(\partial_x)) \in Isom(X_1, X_0)$ *for each* $\nu \in \Sigma_{\pi-\theta_h}$.

(b) *For each $\theta > \theta_h$ there is a positive number M_θ such that*

$$\|(\nu e^{sx} + h(\partial_x))^{-1}\|_{\mathcal{B}(L_p, H^r_p)} + \|\nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}\|_{\mathcal{B}(L_p)} \leq M_\theta, \tag{2.3}$$

for every $\nu \in \Sigma_{\pi-\theta}$.

(c) $(\nu e^{sx} + h(\partial_x)) \in \mathcal{H}^\infty(L_p(\mathbb{R}))$ *for each* $\nu \in \Sigma_{\pi-\theta_h}$.

Proof. (1) Let $\theta > \theta_h$ be fixed and choose $\nu \in \Sigma_{\pi-\theta}$. Let A be the operator in $X_0 = L_p(\mathbb{R})$ defined by means of $(Au)(x) = \nu e^{sx}u(x),\ x \in \mathbb{R}$, for

$$u \in D(A) = \{u \in L_p(\mathbb{R}) :\ e^{sx}u \in L_p(\mathbb{R})\}.$$

A is a multiplication operator, hence it is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus with angle $\phi^\infty_A = \phi_A = |\arg \nu| \leq \pi - \theta$. Next we introduce the operator B in X_0 given by

$$Bu = h(\partial_x)u, \quad u \in D(B) = H^r_p(\mathbb{R}).$$

As in the proofs of Proposition 2.3 and Proposition 2.4 we obtain from Mikhlin's theorem that B is well-defined, invertible, sectorial, and admits a bounded $\mathcal{H}^\infty$-calculus with angle $\phi^\infty_B = \theta_h$.

We would now like to apply Theorem 1.1 to the sum $A + B$. For this purpose we have to check the commutator condition (1.3). In order to do so, it turns out to be convenient to first remove the poles of h in the strip $\bar{S}_{(-s,0)}$, decomposing h as $h = h_1 + h_2$, where h_1 is holomorphic in $S_{(-s-\varepsilon,\varepsilon)}$ and h_2 is rational and bounded at infinity. By adding a sufficiently large constant to h_1 (and subtracting it off from h_2) we can assume that $\operatorname{Re} h_1(i\xi - \sigma) \geq c_0 > 0$ for all $\sigma \in [0, s]$, and also that $\theta_{h_1} \leq \theta_h$. Therefore, the operators $h_1(\partial_x)$ and $h_1(\partial_x - s)$ have the same properties as B. In particular, the parabolicity condition $\phi^\infty_A + \phi^\infty_{h_1(\partial_x)} \leq \pi - \theta + \theta_h < \pi$ is satisfied. For $\eta > 0$ fixed we obtain from Proposition 2.3(b), with $g = (\mu + h_1)^{-1}$ and $(a, b) = (-s - \varepsilon, \varepsilon)$, that

$$\begin{aligned}(\eta + A)(\lambda + \eta + A)^{-1}&[(\eta + A)^{-1}, (\mu + h_1(\partial_x))^{-1}]\\ &= (\lambda + \eta + A)^{-1}[(\mu + h_1(\partial_x))^{-1}, A](\eta + A)^{-1}\\ &= -(\lambda + \eta + A)^{-1}(\mu + h_1(\partial_x - s))^{-1}[h_1(\partial_x) - h_1(\partial_x - s)]\\ &\qquad\qquad \cdot (\mu + h_1(\partial_x))^{-1}A(\eta + A)^{-1}.\end{aligned}$$

Since $|h_1(i\xi) - h_1(i\xi - s)| \sim |\xi|^{r-1}$ we see that the function m defined by

$$m(\xi) := \frac{h_1(i\xi) - h_1(i\xi - s)}{(1 + \xi^2)^{(r-\delta)/2}}$$

belongs to $L_2(\mathbb{R})$, and also that $m' \in L_2(\mathbb{R})$ for each $\delta \in (0, 1/2)$. This implies that m is the Fourier transform of an L_1-function and it follows that

$$(h_1(\partial_x) - h_1(\partial_x - s)) \in \mathcal{B}(H_p^{r-\delta}(\mathbb{R}), L_p(\mathbb{R})).$$

Hence we obtain the estimate

$$\begin{aligned}&\|(\eta + A)(\lambda + \eta + A)^{-1}[(\eta + A)^{-1}, (\mu + h_1(\partial_x))^{-1}]\|\\ &\le C(|\lambda| + \eta)^{-1}|\mu|^{-1}\|h_1(\partial_x) - h_1(\partial_x - s)\|_{\mathcal{B}(H_p^{r-\delta}, L_p)}\|(\mu + h_1(\partial_x))^{-1}\|_{\mathcal{B}(L_p, H_p^{r-\delta})}\\ &\le C_\eta(1 + |\lambda|)^{-1}|\mu|^{-(1+\delta/r)},\end{aligned}$$

and (1.3) holds with $\alpha = 0$, $\beta = \delta/r$, and $\psi_A > \phi_A$, $\psi_B > \phi_B$ such that $\psi_A + \psi_B < \pi$. Thus by Theorem 1.1 and [7, Remark 2.1] there is a sufficiently large ω_0 such that $\omega_0 + A + h_1(\partial_x)$ is invertible, sectorial, and belongs to $\mathcal{H}^\infty(X_0)$ with angle less than $\max\{\psi_A, \psi_B\}$. Since $h_2(\partial_x)$ is bounded, the same results hold for

$$\omega_0 + A + B = \omega_0 + A + h_1(\partial_x) + h_2(\partial_x),$$

possibly at the expense of enlarging ω_0. This implies in particular that $A+B$ with domain

$$D(A + B) = D(A) \cap D(B) = X_1$$

is closed.

In the remaining part of the proof we want to remove ω_0 by means of a Fredholm type argument. Suppose we know that $\omega + A + B$ is injective and has closed range for all $\omega \in [0, \omega_0]$. Then these operators are semi-Fredholm, hence their index is well-defined and constant, by the well-known result on the continuity of the Fredholm index. Now, for $\omega = \omega_0$ this index is zero since $\omega + A + B$ is bijective as proved above. Then it must be zero for all $\omega \in [0, \omega_0]$, hence the operators $\omega + A + B$ must also be surjective since they are injective. We can then conclude from [2, Proposition 2.7] that $A+B$ is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus as well.

(2) Let us first consider the easiest case $p = 2$. Suppose $u \in D(A) \cap D(B)$ satisfies

$$\nu e^{sx}u + \omega u + h(\partial_x)u = f.$$

Taking the inner product with u in $L_2(\mathbb{R})$ yields

$$\nu\|e^{sx/2}u\|_2^2 + \omega\|u\|_2^2 + (h(\partial_x)u|u) = (f|u).$$

By means of Plancherel's theorem we have

$$(h(\partial_x)u|u) = (\mathcal{F}(h(\partial_x)u)|\mathcal{F}u) = (h(i\xi)\mathcal{F}u|\mathcal{F}u),$$

and by taking real parts we obtain

$$c_0\|u\|_2^2 \le \mathrm{Re}\,(h(\partial_x)u|u) \le \|f\|_2\|u\|_2,$$

provided $\mathrm{Re}\,\nu \ge 0$. This implies the a-priori bound

$$\|u\|_2 \le c_0^{-1}\|f\|_2,$$

which is independent of $\omega \ge 0$ and $\mathrm{Re}\,\nu \ge 0$, i.e. injectivity and closed range of $\omega + A + B$ follow. In the case of a general angle $\theta > \theta_h$ we set $\rho = \tan\theta_h$. Then

$$|\mathrm{Im}\,h(i\xi)| \le \rho \mathrm{Re}\,h(i\xi), \quad \xi \in \mathbb{R}.$$

Taking real parts we get this time

$$\mathrm{Re}\,\nu\|e^{sx/2}u\|_2^2 + \omega\|u\|_2^2 + \int_{\mathbb{R}} \mathrm{Re}\,h(i\xi)|\mathcal{F}u|^2\,d\xi \le \|f\|_2\|u\|_2,$$

and taking imaginary parts we obtain

$$|\mathrm{Im}\,\nu|\,\|e^{sx/2}u\|_2^2 - \int_{\mathbb{R}} |\mathrm{Im}\,h(i\xi)||\mathcal{F}u|^2\,d\xi \le \|f\|_2\|u\|_2.$$

Thus

$$(|\mathrm{Im}\,\nu|+(\varepsilon+\rho)\mathrm{Re}\,\nu)\|e^{sx/2}u\|_2^2+\int_{\mathbb{R}} \big((\varepsilon+\rho)\mathrm{Re}\,h(i\xi)-|\mathrm{Im}\,h(i\xi)\big)|\mathcal{F}u|^2\,d\xi \le c\|f\|_2\|u\|_2.$$

For $|\mathrm{Im}\,\nu| + (\varepsilon+\rho)\mathrm{Re}\,\nu \ge 0$ we may now conclude that

$$\|u\|_2 \le (1+\rho+\varepsilon)/c_0\varepsilon)\|f\|_2.$$

The assumptions $|\arg\nu| \le \pi - \theta$ and $\theta > \theta_h$ allow for such a choice of $\varepsilon > 0$. Hence in any case we have shown that $\omega + A + B$ is injective and has closed range for all $\omega \ge 0$, which completes the proof of the theorem for $p = 2$.

(3) We next prove injectivity for all $p \in (1,\infty)$. Suppose $u \in X_1$ satisfies

$$\nu e^{sx}u + \omega u + h(\partial_x)u = 0.$$

Then $u, e^{sx}u \in L_p(\mathbb{R})$ implies that $e^{\sigma x}u \in L_p(\mathbb{R})$ for all $\sigma \in [0,s]$. But this gives

$$e^{-\varepsilon x}u = -e^{-\varepsilon x}(\omega + h(\partial_x))^{-1}\nu e^{sx}u = -(\omega + h(\partial_x+\varepsilon))^{-1}\nu e^{(s-\varepsilon)x}u,$$

where $\varepsilon > 0$ is so small that $\mathrm{Re}\,h(i\xi+\sigma) \ge c_0/2$ for all $\xi \in \mathbb{R}$ and $0 \le \sigma \le \varepsilon$. It follows that $e^{\sigma x}u \in L_p(\mathbb{R})$ for all $\sigma \in [-\varepsilon, s]$. Using the Sobolev embedding $H_p^r(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$ and Hölder's inequality we get

$$\int_{\mathbb{R}} |u|^2\,dx \le \|u\|_\infty(\int_{\mathbb{R}} e^{-\varepsilon p'|x|}\,dx)^{1/p'}(\int_{\mathbb{R}} e^{\varepsilon p|x|}|u|^p\,dx)^{1/p}$$

and we conclude that $u \in L_2(\mathbb{R})$. Uniqueness in $L_2(\mathbb{R})$ now implies $u = 0$, i.e. $\omega + A + B$ is injective in $L_p(\mathbb{R})$ for all $\omega \geq 0$.

(4) Closedness of the ranges is more involved for $p \neq 2$ since we cannot refer to Parseval's theorem. Moreover, B will in general not be accretive in $L_p(\mathbb{R})$. So assume to the contrary that $R(\omega + A + B)$ is not closed in $L_p(\mathbb{R})$, for some $\omega \geq 0$. Then there is a sequence $(u_n) \subset D(A) \cap D(B)$ with

$$\|u_n\|_p = 1 \text{ and } f_n := (\omega + A + B)u_n \to 0 \text{ in } L_p(\mathbb{R}) \text{ as } n \to \infty.$$

Since $\omega_0 + A + B$ is invertible by step (1) this implies that u_n is bounded in $H_p^r(\mathbb{R})$ and $e^{sx}u$ is bounded in $L_p(\mathbb{R})$. By reflexivity of these spaces there exists a function $u \in H_p^r(\mathbb{R}) \cap L_p(\mathbb{R}, e^{psx}dx)$ and a subsequence (w.l.o.g. the full sequence) such that

$$u_n \rightharpoonup u \;\text{ in } H_p^r(\mathbb{R}), \quad Bu_n \rightharpoonup Bu \;\text{ in } L_p(\mathbb{R}) \quad \text{and } e^{sx}u_n \rightharpoonup e^{sx}u \;\text{ in } L_p(\mathbb{R}).$$

The function u then satisfies $\nu e^{sx}u + \omega u + h(\partial_x)u = 0$. Hence $u = 0$ by the uniqueness result proved in the previous step.

We want to show $u_n \to 0$ in $L_p(\mathbb{R})$ which gives a contradiction to $\|u_n\|_p = 1$. To achieve this we use the embedding $H_p^r(\mathbb{R}) \hookrightarrow BUC^\alpha(\mathbb{R})$ for $\alpha = r - 1/p > 0$. Since u_n converges weakly to 0 in $L_p(\mathbb{R})$ and is relatively compact in $C(\mathbb{R})$ w.r.t the topology of uniform convergence on compact sets by the Arzela-Ascoli theorem, we may conclude that $u_n \to 0$ locally uniformly. Let $a \in \mathbb{R}$ be a fixed number. Then given any $\varepsilon > 0$ there exists numbers $b > a$ and $k \in \mathbb{N}$ such that for any $n \geq k$

$$\begin{aligned}\int_a^\infty |u_n|^p dx &\leq e^{-sbp}\int_b^\infty |u_n e^{sx}|^p dx + \int_a^b |u_n|^p dx \\ &\leq e^{-sbp}\sup_n |u_n e^{sx}|_p^p + (b-a)\sup\{|u_n(x)|^p :\; x \in [a,b],\; n \geq k\} \leq \varepsilon.\end{aligned}$$

Hence $\int_a^\infty |u_n|^p dx \to 0$ as $n \to \infty$ for each $a \in \mathbb{R}$.

We will now apply Proposition 2.4 to $\omega + h(z)$ and we find that its inverse has a kernel k such that $e^{\delta|x|}k \in L_1(\mathbb{R})$ for $\delta > 0$ sufficiently small. This yields

$$u_n = (\omega + h(\partial_x))^{-1}(f_n - \nu e^{sx}u_n) = k * f_n - k * \nu e^{sx}u_n =: k * f_n - v_n.$$

Observe that $e^{-\delta x}v_n = (e^{-\delta x}k)*(\nu e^{(s-\delta)x}u_n)$. Since $e^{(s-\delta)x}u_n$ is uniformly bounded in $L_p(\mathbb{R})$ with respect to n and $e^{-\delta x}k \in L_1(\mathbb{R})$ we conclude that $e^{-\delta x}v_n$ is also uniformly bounded in $L_p(\mathbb{R})$. Let $\varepsilon > 0$ be given. Then we can find numbers $a \in \mathbb{R}$ and $k \in \mathbb{N}$ such that

$$\begin{aligned}\Big(\int_{-\infty}^a |u_n|^p dx\Big)^{1/p} &\leq \|k\|_1\|f_n\|_p + e^{\delta a}\Big(\int_{-\infty}^a |e^{-\delta x}v_n|^p dx\Big)^{1/p} \\ &\leq \|k\|_1\|f_n\|_p + e^{\delta a}\|e^{-\delta x}v_n\|_p \leq \varepsilon\end{aligned}$$

whenever $n \geq k$. (This can be done by choosing first a sufficiently negative and then k sufficiently large.) Hence $u_n \to 0$ in $L_p(\mathbb{R})$, and so the range of $\omega + A + B$ must be closed for each $\omega \geq 0$.

(5) Finally we prove the estimate (2.3) by a scaling argument. Let τ_a denote the translation group on $L_p(\mathbb{R})$, i.e. $(\tau_a v)(x) = v(x+a)$, and observe that $h(\partial_x)$ commutes with this group. Then with $a = \frac{1}{s}\ln|\nu|$ and $\vartheta = \arg\nu$ we have

$$\nu e^{sx}u(x) + h(\partial_x)u(x) = f(x), \quad x \in \mathbb{R},$$

if and only if

$$e^{i\vartheta}e^{sx}\tau_{-a}u + h(\partial_x)\tau_{-a}u = \tau_{-a}f.$$

Setting $T_\vartheta = e^{i\vartheta}e^{sx}(e^{i\vartheta}e^{sx} + h(\partial_x))^{-1}$ this gives the representation

$$\nu e^{sx}u = \nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}f = \tau_a T_\vartheta \tau_{-a}f.$$

The family $\{T_\vartheta\}_{\vartheta\in[-\theta,\theta]} \subset \mathcal{B}(L_p(\mathbb{R}))$ is continuous in ϑ, hence uniformly bounded. Since the translations are isometries on $L_p(\mathbb{R})$ we obtain the estimate

$$\|\nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}\|_{\mathcal{B}(L_p(\mathbb{R}))} \le \sup_{|\vartheta|\le\theta}\|T_\vartheta\|_{\mathcal{B}(L_p(\mathbb{R}))} < \infty. \tag{2.4}$$

This proves estimate (2.3) since $h(\partial_x) \in \mathcal{B}(H^r_p(\mathbb{R}), L_p(\mathbb{R}))$ is an isomorphism. □

3. The evolution equation

By means of the transformation $v(x) = e^{sx}u(x)$, (2.1) is equivalent to the parametric problem

$$\nu v + h(\partial_x)e^{-sx}v = f. \tag{3.1}$$

We thus consider the new operator C on $X = L_p(\mathbb{R})$ given by

$$Cv = h(\partial_x)(e^{-sx}v), \quad v \in \mathsf{D}(C) = \{v \in L_p(\mathbb{R}) :\ e^{-sx}v \in H^r_p(\mathbb{R})\}. \tag{3.2}$$

We have the following result.

Theorem 3.1. *Let the assumptions of Theorem* 2.5 *be satisfied. Then C is sectorial with $\phi_C = \theta_h < \pi/2$. Hence $-C$ is the generator of a bounded analytic C_0-semigroup on X.*

Proof. It is clear that C is densely defined, since $\mathcal{D}(\mathbb{R}) \subset \mathsf{D}(C)$. Observing that

$$\nu(\nu + C)^{-1} = \nu e^{sx}(\nu e^{sx} + h(\partial_x))^{-1}, \quad \nu \in \Sigma_{\pi-\theta_h}, \tag{3.3}$$

it follows from Theorem 2.5(b) that C is sectorial with angle θ_h. This shows that $-C$ is the generator of a bounded analytic C_0-semigroup in $X = L_p(\mathbb{R})$. The ergodic theorem $X = N(C) \oplus \overline{R(C)}$ shows also that the range of C is dense in X since obviuosly C is injective. □

We pose the question whether the Cauchy problem

$$\dot v + Cv = f, \quad v(0) = 0, \tag{3.4}$$

has maximal L_p-regularity. This is not clear from Theorem 2.5, but the first step of its proof shows that (3.4) has in fact maximal L_p-regularity if C is replaced by $C_\omega = (\omega + h(\partial_x))e^{-sx}$ with $\omega \ge \omega_0$, where ω_0 is an appropriate nonnegative number. It is an interesting open question whether ω_0 can be chosen to be 0.

Due to the transformation $u(t,x) = e^{-sx}v(t,x)$ is is clear that every solution of the Cauchy problem (3.4) is also a solution of the following degenerate Cauchy-problem

$$\begin{aligned} e^{sx}\partial_t u(t,x) + h(\partial_x)u(t,x) &= f(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) &= 0. \end{aligned} \tag{3.5}$$

Thanks to Theorem 3.1 we know that problem (3.4) admits a unique solution v for an appropriate function f, and hence, problem (3.5) also admits a unique solution (whose regularity properties can be deduced from the regularity properties of v via the transformation $u = e^{-sx}v$).

It is an open problem whether or not (3.5) has maximal regularity. In that direction, we can only prove the following weaker result.

Proposition 3.2. *Let the assumptions of Theorem* 2.5 *be satisfied. Then there exists a non-negative number* ω_0 *such that*

$$\begin{aligned} e^{sx}\partial_t u(t,x) + \omega u(t,x) + h(\partial_x)u(t,x) &= f(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) &= 0, \end{aligned} \tag{3.6}$$

admits a unique solution u *with maximal* L_p*-regularity for every* $\omega \geq \omega_0$. *That is, for each* $f \in L_p(J \times \mathbb{R})$, *problem* (3.6) *admits a unique solution* $u \in L_p(J, H_p^r(\mathbb{R}))$ *such that* $e^{sx}\partial_t u \in L_p(J \times \mathbb{R})$ *where* $J = (0,T)$. *There is a constant* $M = M_\omega > 0$, *independent of* f, *such that*

$$\|e^{sx}\partial_t u\|_{L_p(J\times\mathbb{R})} + \|u\|_{L_p(J,H_p^r(\mathbb{R}))} \leq M\|f\|_{L_p(J\times\mathbb{R})}.$$

Moreover, the operator $L = \partial_t e^{sx} + \omega + h(\partial_x)$ *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $L_p(J \times \mathbb{R})$ *for* $\omega \geq \omega_0$.

Proof. Repeating step (1) of the proof of Theorem 2.5 in $L_p(J\times\mathbb{R}) = L_p(J, L_p(\mathbb{R}))$ with A replaced by $A = \partial_t e^{sx}$, we obtain a number ω_0 such that the operator

$$\omega_0 + \partial_t e^{sx} + h(\partial_x),$$

with natural domain, is invertible and admits a bounded $\mathcal{H}^\infty$-calculus. Propositions 1.3.(iv) and 2.7 in [2] imply that this is also true for any $\omega \geq \omega_0$. □

On the other hand, we do obtain maximal L_p-regularity for problem (3.5) in case that $X = L_2(\mathbb{R})$. This is the statement of the next theorem.

Theorem 3.3. *Let the assumptions of Theorem* 2.5 *be satisfied. Then for each* $f \in L_p(J, L_2(\mathbb{R}))$, *problem* (3.6) *admits a unique solution* $u \in L_p(J, H_2^r(\mathbb{R}))$ *such that* $e^{sx}\partial_t u \in L_p(J, L_2(\mathbb{R}))$. *There is a constant* $M > 0$, *independent of* f, *such that*

$$\|e^{sx}\partial_t u\|_{L_p(J,L_2(\mathbb{R}))} + \|u\|_{L_p(J,H_2^r(\mathbb{R}))} \leq M\|f\|_{L_p(J,L_2(\mathbb{R}))}.$$

The operator $L = \partial_t e^{sx} + h(\partial_x)$ *admits a bounded* $\mathcal{H}^\infty$*-calculus on* $L_p(J, L_2(\mathbb{R}))$.

Proof. Let $X = L_2(\mathbb{R})$. According to Theorem 3.1 we know that the operator C is sectorial with $\phi_C = \theta_h$. Since X is a Hilbert space, we have that C is, in addition, also $\mathcal{R}$-sectorial with $\phi_C^R = \theta_h$, see for instance [2, Remark 3.2.(3)]. This implies that the Cauchy problem (3.4) has maximal L_p-regularity, see for instance [2, Theorem 4.4]. Since $\omega + h(z)$ satisfies the same assumptions as $h(z)$ for each $\omega \geq 0$ we deduce that the Cauchy problem (3.4) with C replaced by C_ω also has maximal L_p-regularity. That is, for each $f \in L_p(J, X)$, with $X = L_2(\mathbb{R})$, there is a unique solution $v \in H_p^1(J, X)$ of (3.4), and there is a positive constant $M = M(\omega)$ independent of f such that

$$\|\dot{v}\|_{L_p(J,X)} + \|C_\omega v\|_{L_p(J,X)} \leq M\|f\|_{L_p(J,X)}, \quad f \in L_p(J, X).$$

Going to (3.6) via the transformation $u = e^{-sx}v$ yields a unique solution of (3.6) and the estimate

$$\|e^{sx}\partial_t u\|_{L_p(J,X)} + \|(\omega + h(\partial_x))u\|_{L_p(J,X)} \leq M\|f\|_{L_p(J,X)}, \quad f \in L_p(J, X).$$

Since $\omega + h(\partial_x) \in \mathcal{B}(H_p^r(\mathbb{R}), L_p(\mathbb{R}))$ is an isomorphism for each $\omega \geq 0$, this yields invertibility of the operators $\omega + \partial_t e^{sx} + h(\partial_x)$ on $L_p(J, L_2(\mathbb{R}))$ with natural domain, for each $\omega \geq 0$. As in Theorem 3.1 we obtain that there is a number $\omega_0 \geq 0$ such that $\omega_0 + \partial_t e^{sx} + h(\partial_x)$ admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(J, L_2(\mathbb{R}))$. Using [2, Proposition 2.7] we conclude that $\partial_t e^{sx} + h(\partial_x)$ is invertible, sectorial and admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(J, L_2(\mathbb{R}))$, and this completes the proof. □

4. Examples

In this section we discuss some of the examples introduced in Section 1.

(i) We first consider the symbol of the Laplace equation in an angle $G = \{(r\cos\phi, r\sin\phi) : r > 0,\ \phi \in (0, \alpha)\}$ with homogeneous Dirichlet condition on $\phi = \alpha$ and dynamic boundary condition $\partial_t u + \partial_\nu u = g$ on $\phi = 0$. Then we obtain a problem of the form (1.1) with $s = 1$ and

$$h_\beta(z) = \psi_0(-(z + \beta)^2) \quad \text{with} \quad \psi_0(\zeta) = \sqrt{\zeta}\coth(\alpha\sqrt{\zeta}).$$

Since the function $\coth(\zeta)$ is odd, ψ_0 is a meromorphic function on $\mathbb{C}$ with poles in $\{\zeta_k = -r_k^2 = -k^2(\pi/\alpha)^2 : k \in \mathbb{N}\}$. Since $\coth\zeta \to 1$ for $|\zeta| \to \infty$, $|\arg(\zeta)| \leq \theta < \pi/2$, it is easy to see that $h_\beta(z)/|z| \to 1$ as $|z| \to \infty$, in any strip $S_{(a,b)}$. In particular $r = 1$ and h satisfies (i)–(iii) of Definition 2.1 in $S_{(a,b)}$ for all $a < b$. Next we determine the values of β which are admissible. The parabola $P_\beta = \{-(i\xi + \beta)^2 : \xi \in \mathbb{R}\}$ passes through a pole of ψ_0 if and only if $|\beta| = r_k$ for some $k \in \mathbb{N}$. Therefore Definition 2.1(iv) is satisfied if and only if $|\beta| \neq r_k$ for all $k \in \mathbb{N}$. To check Definition 2.1(v), we compute the real part of $h_\beta(i\xi)$, to the result

$$\operatorname{Re} h_\beta(i\xi) = \frac{|\xi|\sinh(2\alpha|\xi|) + \beta\sin(2\alpha\beta)}{\cosh(2\alpha|\xi|) - \cos(2\alpha\beta)}.$$

This shows that the real part of $h_\beta(i\xi)$ is strictly positive for all values of $\xi \in \mathbb{R}$ if and only if $\operatorname{Re} h_\beta(0) > 0$, which in turn is equivalent to $|\beta|\cot(\alpha|\beta|) > 0$. This yields the range

$$|\beta| \in [0, \pi/2\alpha) \bigcup_{k\geq 1} (k\pi/\alpha, (k+1/2)\pi/\alpha).$$

(ii) If in (i) we change the Dirichlet condition on $\phi = \alpha$ into a Neumann condition then the function h becomes

$$h_\beta(z) = \psi_1(-(z+\beta)^2) \quad \text{with} \quad \psi_1(\zeta) = \sqrt{\zeta}\tanh(\alpha\sqrt{\zeta}).$$

Here we have again a meromorphic function, $s = r = 1$, but the poles are this time in $\{\zeta_k = -s_k^2 = -(2k+1)^2(\pi/2\alpha)^2 : k \in \mathbb{N}_0\}$. The admissible values of β then are $|\beta| \neq s_k$ for $k \in \mathbb{N}_0$. For the real part of $h_\beta(i\xi)$ we get

$$\operatorname{Re} h_\beta(i\xi) = \frac{|\xi|\sinh(2\alpha|\xi|) - \beta\sin(2\alpha\beta)}{\cosh(2\alpha|\xi|) + \cos(2\alpha\beta)}.$$

Thus the real part of $h_\beta(i\xi)$ is in this case strictly positive for all $\xi \in \mathbb{R}$ if and only if $\operatorname{Re} h_\beta(0) > 0$ which in turn is equivalent to $|\beta|\tan(\alpha|\beta|) < 0$. This yields the range $|\beta| \in \cup_{k\geq 1}(k\pi/\alpha, (k+1/2)\pi/\alpha)$.

(iii) We next discuss the symbol of the two-dimensional Mullins-Sekerka problem

$$h_\beta(z) = -\psi_1(-(z+\beta)^2)(z+\beta+1)(z+\beta+2),$$

with ψ_1 as in (ii), where we restrict attention to the physically relevant range $\alpha \in (0, \pi)$. Here again h is meromorphic and we have $s = r = 3$. The poles are the same as in (ii), and for the real part of $h_\beta(i\xi)$ we get the more complicated expression

$$\operatorname{Re} h_\beta(i\xi) = \frac{|\xi|\sinh(2\alpha|\xi|)(\xi^2 - 3\beta(\beta+2) - 2) + (\beta+1)\sin(2\alpha\beta)(\beta(\beta+2) - 3\xi^2)}{2(\sinh^2(\alpha\xi) + \cos^2(\alpha\beta))}.$$

For $\beta > 0$ we set $\xi_0^2 = \beta(\beta+2)/3$ to see that $\xi_0^2 - 3\beta(\beta+2) - 2 < 0$, hence $\operatorname{Re} h_\beta(i\xi_0) < 0$. If $\beta = 0$, then we also have $\operatorname{Re} h_\beta(i\xi) < 0$ for ξ sufficiently small. Thus nonnegative values of β are not admissible, and neither are small negative values of β. On the other hand, if $\beta \leq -2$ then the same choice of ξ_0 shows $\operatorname{Re} h_\beta(i\xi_0) \leq 0$, so that such values of β do also not meet (iv) of Definition 2.1. This shows that the admissible values of β are contained in the interval $(-2, 0)$. Next we look at $h_\beta(0)$ which is

$$h_\beta(0) = |\beta|\tan(\alpha|\beta|)(\beta+1)(\beta+2).$$

There are two distinguished cases, namely $-2 < \beta < -1$ and $-1 < \beta < 0$, as $h_\beta(0) = 0$ for $\beta = -1$. If $-1 < \beta < 0$ we always have the window $-\pi/2\alpha < \beta < 0$. Restricting attention to this range, a sufficient condition for $\operatorname{Re} h_\beta(i\xi) \geq c_0 > 0$ is

$$\max\{-1, -\pi/2\alpha\} < \beta < -1 + 1/\sqrt{3}.$$

In fact, we then have $\sin(2\alpha|\beta|) > 0$ as well as $3|\beta|(\beta+2) - 2 > 0$, which implies $\operatorname{Re} h_\beta(i\xi) > 0$ for all $\xi \in \mathbb{R}$. On the other hand, if ξ^2 is such that the coefficient of $\sinh(2\alpha|\xi|)|\xi|$ is negative, i.e. if $\xi^2 - 3\beta(\beta+2) - 2 < 0$, then we may estimate

$$\begin{aligned}
&\sin(2\alpha\beta)(\beta+1)(-3\xi^2 + \beta(\beta+2)) + \sinh(2\alpha|\xi|)|\xi|(\xi^2 - 3\beta(\beta+2) - 2) \\
&\leq 2\alpha|\beta|(1-|\beta|)\big[3\xi^2 + |\beta|(2-|\beta|)\big] + 2\alpha\xi^2\big[\xi^2 + 3|\beta|(2-|\beta|) - 2\big] \\
&= 2\alpha\big[\xi^4 - (2 + 6|\beta|^2 - 9|\beta|)\xi^2 + |\beta|^2(1-|\beta|)(2-|\beta|)\big].
\end{aligned}$$

The last line becomes negative for some value of $\xi^2 > 0$ if and only if

$$|\beta|^2(1-|\beta|)(2-|\beta|) < (1 + 3|\beta|^2 - 9|\beta|/2)^2,$$

which shows that the range $-0.195 \leq \beta < 0$ is forbidden. Computations with a computer algebra system suggest that there is an increasing function $\beta^*(\alpha)$ such that the range of well-posedness is given by $-1 < \beta < \beta^*(\alpha)$, and $-0.32 < \beta^*(\alpha) < -0.195$.

(iv) Finally we discuss the symbol of the stationary Stokes problem with boundary contact and prescribed contact angle in the slip case in two dimensions. This symbol reads as

$$h_\beta(z) = \psi(z+\beta) \quad \text{with} \quad \psi(\zeta) = (1+\zeta)\frac{\cos(2\alpha\zeta) - \cos(2\alpha)}{\sin(2\alpha\zeta) + \zeta\sin(2\alpha)}.$$

This symbol is much more complex than those discussed before, and we do not intend to present a complete discussion here. Obviously, $\beta = 0$ leads to a first order pole, hence neither of the intervals $[-\delta, 0]$ and $[0, \delta]$ are admissible. We want to concentrate on a neighborhood of $\beta = 1$. Computing the real part of $\psi(1+i\xi)$ leads to the expression

$$\operatorname{Re}\psi(1+i\xi) = \frac{(\cosh(2\alpha\xi) - 1)(\xi\sinh(2\alpha\xi) + \xi^2\sin(4\alpha)/2)}{\sin^2(2\alpha)(\cosh(2\alpha\xi)+1)^2 + (\cos(2\alpha)\sinh(2\alpha\xi) + \xi\sin(2\alpha))^2}.$$

This representation of $\operatorname{Re}\psi(1+i\xi)$ shows that it is strictly positive except at $\xi = 0$. Thus $\beta = 1$ is not admissible. We expand the symbol at $(\beta,\xi) = (1,0)$ to the result

$$h_\beta(i\xi) = 2\alpha(1 - \beta - i\xi) + o(|\beta - 1| + |\xi|).$$

This shows by means of a compactness argument that $\operatorname{Re} h_\beta(i\xi)$ is bounded below for $\xi \in \mathbb{R}$ when β is restricted to an interval $(\beta^*(\alpha), 1)$ with $\beta^*(\alpha) < 1$. This range of β is admissible, i.e. for such numbers β the conditions (iv) and (v) of Definition 2.1 are satisfied.

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