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ON THE SURFACE DIFFUSION FLOW

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ABSTRACT

In this paper we present recent existence, uniqueness, and stability results for the motion of immersed hypersurfaces driven by surface diffusion. We provide numerical simulations for curves and surfaces that exhibit the creation of singularities. Moreover, our numerical simulations show that the flow causes a loss of embeddedness for some initially embedded configurations.

1. INTRODUCTION

The surface diffusion flow is a geometric evolution law in which the normal velocity is equal to the Laplace-Beltrami of the mean curvature. More precisely, we assume that Γ_0 is a compact closed immersed orientable hypersurface in $\mathbb{R}^n$ of class $C^{2+\beta}$. Then one is looking for a family $\Gamma = \{\Gamma(t)\ ;\ t \geq 0\}$ of smooth immersed orientable hypersurfaces satisfying the following evolution equation

$$V(t) = \Delta_{\Gamma(t)} H_{\Gamma(t)}\,, \qquad \Gamma(0) = \Gamma_0\,. \tag{1.1}$$

Here $V(t)$ denotes the velocity in the normal direction of Γ at time t, while $\Delta_{\Gamma(t)}$ and $H_{\Gamma(t)}$ stand for the Laplace-Beltrami operator and the mean curvature of $\Gamma(t)$, respectively. Both the normal velocity and the curvature depend on the choice of the orientation, however, (1.1) does not, and so we are free to choose whichever one we like. In particular, if $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$ we always choose the outer normal, so that $V(t)$ is positive if $\Omega(t)$ grows, and so that $H_{\Gamma(t)}$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$.

The surface diffusion flow (1.1) was first proposed by Mullins (Mu57) to model the dynamics for the motion of the surface of a crystal when all mass transport is by curvature driven diffusion along the surface. It has also been examined in a more general mathematical and physical context by Davì and Gurtin (DG90), and by Cahn and Taylor (CT94). More recently, Cahn, Elliott, and Novick-Cohen (CEN96) showed by formal asymptotics that the surface diffusion flow is the singular limit of the zero level set of the solution to the

Cahn-Hilliard equation with a concentration dependent mobility. The Cahn-Hilliard equation is a model describing the phase separation and coarsening phenomena in a quenched binary alloy. The surface diffusion model describes the coarsening phenomena after the distinct phases have already been well established, and is derived from the Cahn-Hilliard model with a concentration dependent mobility as a limit as the thickness of the mushy region between the two phases approaches zero. In the case of constant mobility in the Cahn-Hilliard equation, Alikakos, Bates, and Chen (ABC94) proved that the motion of the singular limit is governed by the Mullins-Sekerka model (also called the Hele-Shaw model with surface tension), rigorously establishing a result that was formally derived by Pego (Pe89).

Due to the local nature of the evolution we may assume the hypersurface Γ_0 to be connected. However, unlike the well-studied mean curvature flow (Hu84; GH86; Gr87), the surface diffusion flow does not enjoy the luxury of a maximum principle, as the equations are of fourth order. In particular, several disjoint components may collide, which is easily seen by putting a stationary sphere into the path of a moving surface.

The motion given by (1.1) has some interesting geometrical features. Assume that Γ is a smooth solution to (1.1) and let $A(t)$ denote the area of $\Gamma(t)$. Then the function A is smooth and we find for its derivative (see e.g. (La80, Theorem 4) or (GH86, p. 70))

$$\begin{aligned} \frac{1}{n-1}\frac{d}{dt}A(t) &= \int_{\Gamma(t)} V(t) H_{\Gamma(t)}\, d\sigma = \int_{\Gamma(t)} [\Delta_{\Gamma(t)} H_{\Gamma(t)}] H_{\Gamma(t)}\, d\sigma \qquad (1.2) \\ &= -\int_{\Gamma(t)} |\mathrm{grad}_{\Gamma(t)} H_{\Gamma(t)}|^2_{\Gamma(t)} d\sigma \le 0\,. \end{aligned}$$

Hence the motion driven by surface diffusion is area decreasing. In fact, the surface diffusion flow is a gradient flow for the area functional (Ma97b), if interpreted in the sense of Fife (Fi91a; Fi91b). Additionally, if Γ consists of embedded hypersurfaces which enclose a region $\Omega(t)$, and if $\mathrm{Vol}(t)$ denotes the volume of $\Omega(t)$, then the derivative of the smooth function Vol is given by

$$\frac{d}{dt}\mathrm{Vol}(t) = \int_{\Gamma(t)} V(t)\, d\sigma = \int_{\Gamma(t)} \Delta_{\Gamma(t)} H_{\Gamma(t)}\, d\sigma = 0\,,$$

thus the motion driven by surface diffusion is also volume preserving in the embedded case.

Clearly every compact surface of constant mean curvature is an equilibrium for (1.1), and the converse is also true by Liouville's Theorem. In the embedded case this leaves only the spheres, see (Al56), while in the immersed case there are many such surfaces, as for example the Wente tori (We86).

2. EXISTENCE OF SOLUTIONS

The surface diffusion flow was studied by Baras, Duchon, and Robert (BDR84) for strip-like domains in two space dimensions. They prove global existence for rather weak assumptions on the initial data. Also in two dimensions, the surface diffusion flow for closed embedded curves was analyzed by Elliott and Garcke (EG97) who show local existence and regularization for C^4-initial curves, but not uniqueness. Polden (Po96) and, independently, Giga and Ito (GI97) show short-term existence and uniqueness for immersed H^4-initial curves. The general case of immersed hypersurfaces in any space dimension is considered by the authors (EMS97). We show uniqueness and local existence of smooth solutions for the motion of any immersed $C^{2+\beta}$-initial hypersurface in $\mathbb{R}^n$.

In order to give precise results, let us introduce the following notation. Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ. The following results have been proved in (EMS97).

Theorem 1 *Assume that $0 < \beta < 1$, and let Γ_0 be a compact closed immersed orientable hypersurface in $\mathbb{R}^n$ belonging to the class $h^{2+\beta}$.*

(a) *The surface diffusion flow (1.1) has a unique local classical solution $\Gamma = \{\Gamma(t)\ ;\ t \in [0,T)\}$ for some $T > 0$. Each hypersurface $\Gamma(t)$ is of class C^∞ for $t \in (0,T)$. Moreover, the mapping $[t \mapsto \Gamma(t)]$ is continuous on $[0,T)$ with respect to the $h^{2+\beta}$-topology and smooth on $(0,T)$ with respect to the C^∞-topology.*

(b) *Suppose that the initial hypersurface Γ_0 is a $h^{2+\beta}$-graph in the normal direction over some smooth immersed hypersurface Σ. Then the mapping $\varphi := [(t,\Gamma_0) \mapsto \Gamma(t)]$ induces a smooth local semiflow on a open subset of $h^{2+\beta}(\Sigma)$.*

As mentioned in the introduction, every embedded equilibrium of the surface diffusion flow is a Euclidean sphere. However, none of these equilibria is isolated, since in every neighborhood of a fixed sphere there is a continuum of further spheres. Thus the dynamics of the flow generated by (1.1) is even near spheres difficult to analyze. For curves, it is shown in (EG97) that small embedded perturbations of circles exist globally in time, and furthermore, assuming global existence, that any closed curve will become circular under this evolution. The methods used in (EG97) are restricted to the evolution of curves. For higher space dimensions, if the initial surface is embedded and close to a sphere, we prove analogously, but with entirely different methods, that the solution exists globally and converges exponentially fast to some sphere (EMS97).

Theorem 2 *Let S be a fixed Euclidean sphere and let $\mathcal{M}$ denote the set of all spheres which are sufficiently close to S. Then $\mathcal{M}$ attracts all embedded solutions which are $h^{2+\beta}(S)$-close to $\mathcal{M}$ at an exponential rate. In particular, if Γ_0 is sufficiently close to S in $h^{2+\beta}(S)$ then Γ exists globally and converges exponentially fast to some sphere in $\mathcal{M}$ enclosing the same volume as Γ_0. The convergence is in the C^k-topology for every initial hypersurface Γ_0 which is in a sufficiently small $h^{2+\beta}(S)$-neighborhood $W = W(k)$ of S, where $k \in \mathbb{N}$ is a fixed number.*

It is interesting to note that the volume-preserving mean curvature flow and the Mullins-Sekerka model share many properties with the surface diffusion flow (1.1). They all preserve the enclosed volume, decrease the area of the surface, and for all three the invariant manifold $\mathcal{M}$ of spheres is exponentially attracting (ES96; ES98).

Theorem 1 constitutes a precise local existence and uniqueness result for classical solutions to (1.1) starting out as immersed hypersurfaces. The evolution (1.1) has in fact a quasilinear parabolic structure. This structure effects, for example, a parabolic regularization of the flow φ since we are allowed to choose initial surfaces Γ_0 of class $h^{2+\beta}$, although $\Delta_{\Gamma_0} H_{\Gamma_0}$ is for such Γ_0 in general not a classical function. It also provides the foundation for the study of the qualitative behavior of the semiflow φ. Our approach for proving existence, uniqueness, and regularity of solutions is based on the general theory of Amann (Am93; Am95) for quasilinear parabolic evolution equations.

The proof of Theorem 2 consists of two steps. We first show that the semiflow φ admits a stable $(n+1)$-dimensional local center manifold $\mathcal{M}^c$. This means, in particular, that $\mathcal{M}^c$ is a locally invariant manifold and that $\mathcal{M}^c$ contains all small global solutions of φ. In a second step we then prove that $\mathcal{M}^c$ coincides with the manifold $\mathcal{M}$ of the theorem. It is well-known that local center manifolds are generally not unique. However since each local center manifold of the surface diffusion flow consists of equilibria only this forces uniqueness. Under suitable spectral assumptions for the linearization the existence of center manifolds is well-known for finite-dimensional dynamical systems. The corresponding construction for quasilinear infinite-dimensional semiflows (e.g. for φ) is considerably more involved. The basic technical tool here is the theory of maximal regularity, due to G. Da Prato and P. Grisvard (DG79), see also (Am95; An90; Lu95). In particular, these results allow to treat (1.1) as a fully-nonlinear perturbed linear evolution equation, see (DL88; Lu95; Si95).

3. NUMERICAL SIMULATIONS

In this section we will consider various initial configurations which display several phenomena of the flow. All statements in this section are to be understood as being based on numerical simulations, and not on analytical proofs. One of our examples is an embedded curve that loses and regains embeddedness twice, and ultimately approaches a circle. The resulting evolution exists for all time. On the other hand, some immersed curves develop singularities. For example, we provide evidence that the surface diffusion flow shrinks a figure-eight to a point in finite time. Also, a two-dimensional surface shaped like a dumbbell with a sufficiently thin neck will pinch off, thereby forming an essential singularity since the curvature becomes infinite. A surface shaped like an erythrocyte (a red blood cell) with a thin enough center will cease to be embedded and become immersed, similar to the behavior of a dumbbell curve in two dimensions. For dumbbell curves, this behavior was conjectured by Elliot and Garcke (EG97), numerically established by the authors (EMS97), and proven by Giga and Ito (GI97). This situation is in clear contrast to the mean curvature flow, where the maximum principle prevents creation of self-intersections.

Our approach for analytically proving existence and uniqueness of solutions involves parameterizing the hypersurfaces as graphs over some fixed reference hypersurface. The evolution law is then recast as an equation for functions defined on this reference manifold. For the numerical scheme we compute the linearizations of the operators involved and set up a semi-implicit finite difference equation. The main idea then is to choose the reference manifold to be equal to the initial hypersurface, which simplifies the equations, and to compute only one time step, and then to consider the newly computed manifold as a new reference manifold. This is the approach taken in (EMS97), implemented numerically for curves in $\mathbb{R}^2$. A different theoretical background can be given to the numerical scheme by considering the gradient flow structure of the surface diffusion flow (Ma97b), see also (Ma97a). The relevant definition of gradient flow (Fi91a; Fi91b) establishes a gradient for the area of an evolving hypersurface. This definition of the gradient of the area at a given time t depends on the hypersurface at that time t. In other words, the space with respect to which one computes the gradient is constantly changing. Implementation of a finite difference scheme based on this gradient flow structure leads then naturally to a scheme for functions defined on a constantly changing reference manifold, and one is lead to the same numerical scheme. The implementation for surfaces in $\mathbb{R}^3$ is given in (Ma97b).

3.1. A figure-eight

One can make perfect sense of the enclosed signed area of a figure-eight, which is for a symmetric figure-eight equal to zero. As the evolution decreases the length of the curve, and preserves the enclosed area, it can be expected that the limiting figure has zero area and zero length. In fact, Polden (Po96) shows analytically that a symmetric figure-eight either develops a singularity, or has to shrink in finite time to a point. His argument starts with equation (1.2) for curves (writing L for length instead of A, and κ for curvature instead of H), and then uses Poincaré's inequality (the curvature has average zero), Hölder's inequality, and finally the Fenchel-Fary inequality:

$$\begin{aligned}\frac{dL}{dt} &= -\int_{\Gamma(t)} |\nabla\kappa|^2\,ds \le -\left(\frac{2\pi}{L}\right)^2 \int_{\Gamma(t)} \kappa^2\,ds\\ &\le -\left(\frac{2\pi}{L}\right)^2 \frac{1}{L}\left(\int_{\Gamma(t)} |\kappa|\,ds\right)^2 \le -\frac{16\pi^4}{L^3}\,.\end{aligned}$$

Hence one obtains

$$\frac{dL^4}{dt} \le -64\pi^4\,,$$

so that the length of the curve reduces to zero in finite time, assuming no singularity arises sooner. This shrinking to a point is exactly what happens for the symmetric figure-eight. The argument in fact applies to any curve for which the curvature has average zero, so that evolutions starting with such a curve exist only for a finite time, at least as far as classical solutions are concerned.

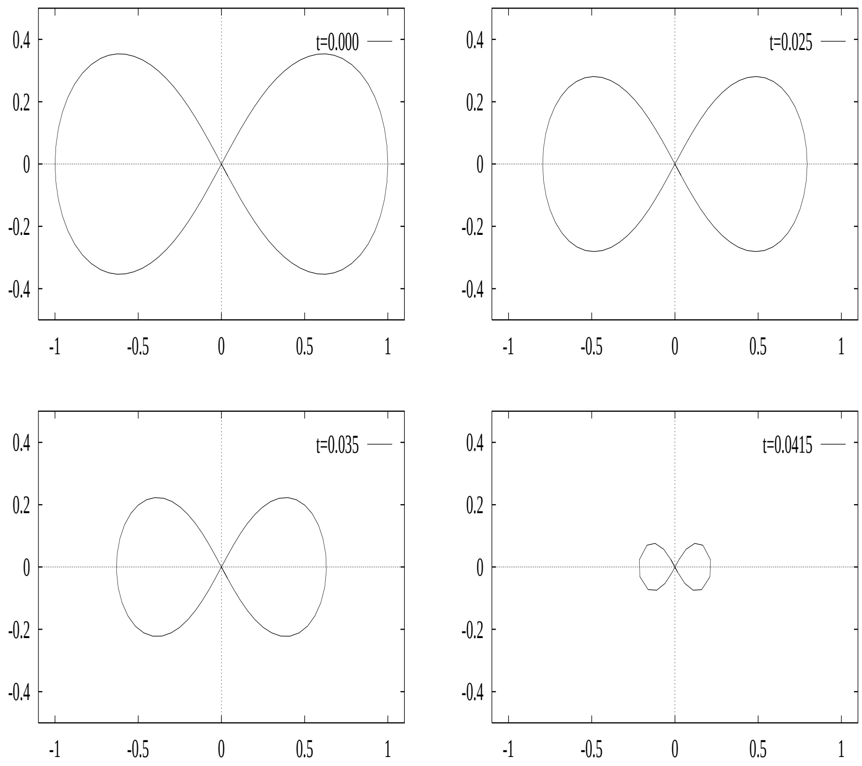


Fig. 1 The figure-eight $r(\theta) = \sqrt{\cos(\theta)}$.

3.2. A spiral

A spiral exhibits the phenomenon that it undergoes several stages under the surface diffusion flow, switching back and forth between being embedded and being immersed. The area enclosed by the curve remains constant, provided one counts those patches twice where the curve overlaps while being immersed.

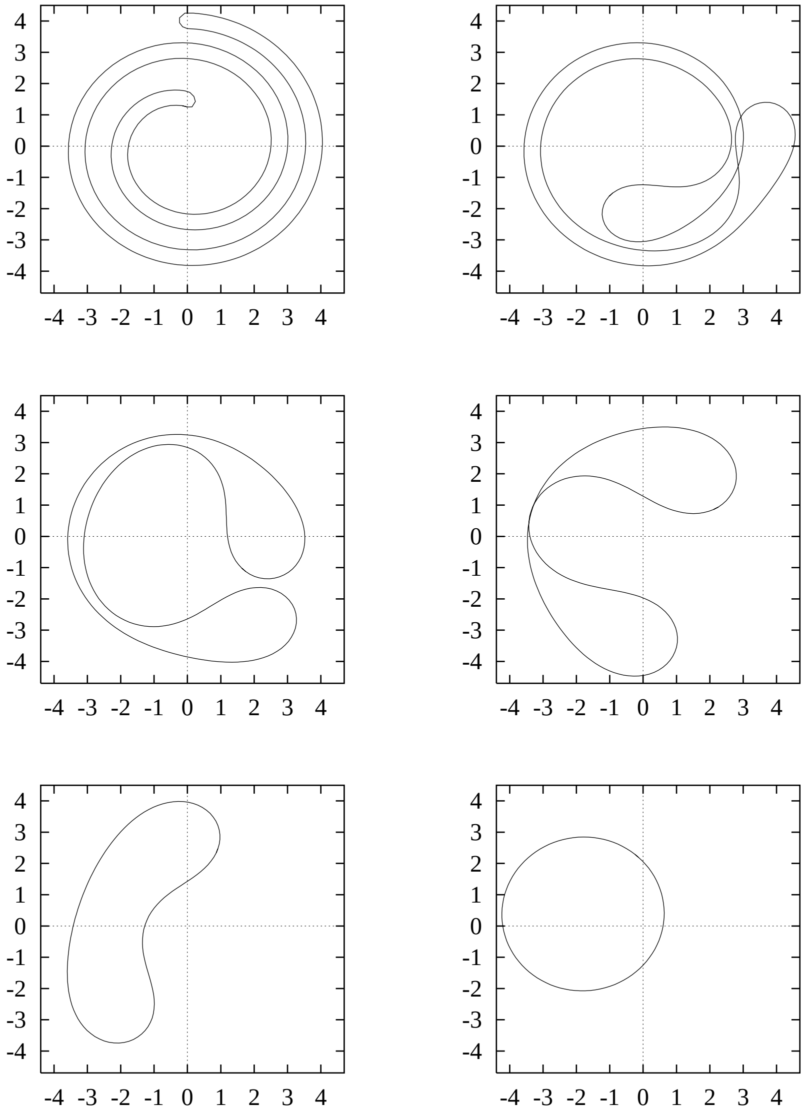


Fig. 2 A spiral modeled on $r = \sqrt{\theta}$. The times of the snapshots are at $t = 0, 3, 10, 17.5, 25, 40$, top-left to bottom-right.

3.3. An erythrocyte

As demonstrated by the previous example, certain curves can lose their embeddedness and become immersed, see also (EG97; EMS97; GI97). Here we provide an example of a surface that exhibits the same effect. The snapshot at $t = 0.003$ shows maximal overlap (this is the second graph below), the surface will then start to separate again, and will ultimately evolve into a sphere.

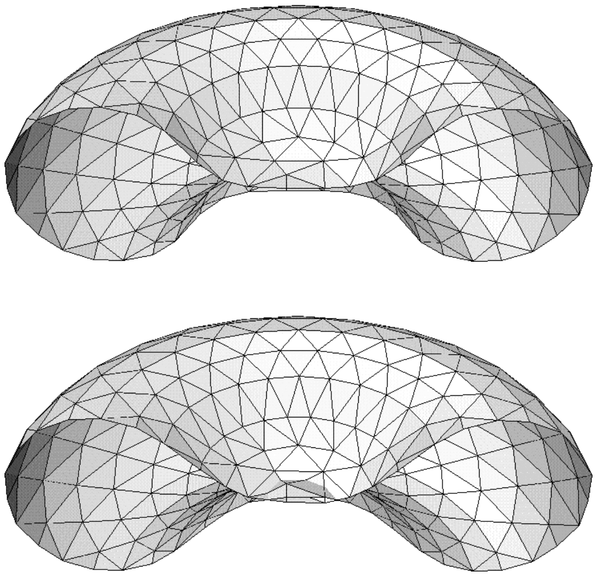
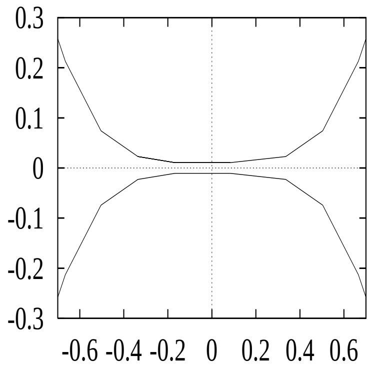


Fig. 3 This is a surface that loses its embeddedness and becomes immersed. The first graph is the starting configuration, the second is at time $t = 0.003$. Notice the small bump in the center of the second graph, where the lower half of the surface has pierced through the upper half. The main diameter of the figure is about 4 space units.



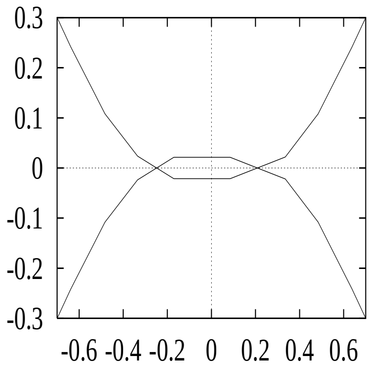


Fig. 4 These are the central portions of the cross-sections of the surfaces pictured above. The first section is the starting configuration, the diameter of the neck is 0.02, and the second section is at time $t = 0.003$, at which time the surface is immersed.

3.4. A dumbbell surface

A dumbbell with a thin neck has a maximum of the mean curvature at the neck, and hence the Laplacian of the mean curvature will be negative. This leads to an inwards motion at the neck, and hence to an increased pinching effect, so that pinching-off can be expected, and it, in fact, occurs.

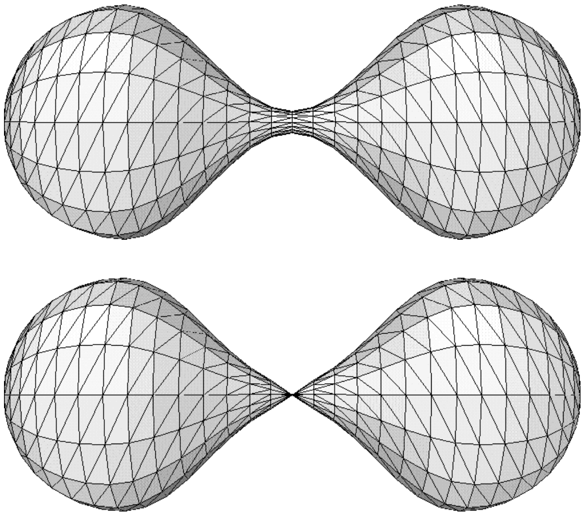


Fig. 5 A dumbbell with a sufficiently thin neck leads to a pinch-off. The first graph is the starting configuration, the second is at time $t = 0.0044$. The length of the dumbbell is about 11 space units.

This effect is well-known for motion by mean curvature and was proved by Grayson (Gr89), see also (An92) and (St96, Section 2.2). The proof uses in an essential way the maximum principle, which is unavailable for the surface diffusion flow. There is no analytical proof known to the authors for this pinching effect of the surface diffusion flow.

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