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Continuous maximal regularity on uniformly regular Riemannian manifolds

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Abstract. We establish continuous maximal regularity results for parabolic differential operators acting on sections of tensor bundles on uniformly regular Riemannian manifolds M. As an application, we show that solutions to the Yamabe flow on M instantaneously regularize and become real analytic in space and time. The regularity result is obtained by introducing a family of parameter-dependent diffeomorphisms acting on functions on M in conjunction with maximal regularity and the implicit function theorem.

1. Introduction

It is the main purpose of this paper to introduce a basic theory of continuous maximal regularity for parabolic differential operators acting on sections of tensor bundles on a class of Riemannian manifolds called *uniformly regular Riemannian manifolds* in this paper.

This concept was first introduced by Amann in [3]. These manifolds may be noncompact. As a special case, any complete manifold (M, g) without boundary and with bounded geometry (i.e., M has positive injectivity radius and all covariant derivatives of its curvature tensor are bounded) is a uniformly regular Riemannian manifold, see [3, Example 2.1(f)]. The class of uniformly regular Riemannian manifolds is large enough in the sense that it satisfies most of the geometric conditions imposed by other authors in the study of geometric evolution problems. In this paper and a subsequent one [36], we mainly focus on two kinds of geometric evolution problems, namely the evolution of metrics and the evolution of surfaces driven by their curvatures. Numerous results have been formulated for geometric evolution equations over compact closed manifolds, which are special cases of uniformly regular Riemannian manifolds. Nowadays, there is increased interest in generalizing these results for noncompact manifolds or manifolds with boundary. Most of the achievements in this research line are formulated for complete manifolds with certain restrictions on their curvatures. Within the

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class of uniformly regular Riemannian manifolds, we are able to relax many of these constraints. One typical instance is the Yamabe flow, a well-known geometric evolution problem. We will show in Sect. 5 that this problem possesses a local solution as long as the manifold is conformally uniformly regular. Under appropriate assumptions on the background metric g_0, we will in addition show that solutions instantaneously regularize and become real analytic in space and time.

For a densely embedded Banach couple $E_1 \overset{d}{\hookrightarrow} E_0$, we consider the following abstract linear inhomogeneous parabolic equation:

$$\begin{cases} \partial_t u(t) + Au(t) = f(t), \quad t \in \dot{I}, \\ u(0) = u_0 \end{cases} \tag{1.1}$$

on $I := [0, T]$. Given $I \in \{[0, T], [0, T)\}$, $\dot{I} := I \backslash \{0\}$. (E_0, E_1) is then called a pair of continuous maximal regularity for $A \in \mathcal{L}(E_1, E_0)$ if

$$(\partial_t + A, \gamma_0) \in \mathcal{L} \text{ is } (\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\gamma), \tag{1.2}$$

that is, $(\partial_t + A, \gamma_0)$ is a linear isomorphism between the indicated spaces, where $\gamma_0 u := u(0)$, $E_\gamma := (E_0, E_1)^0_{\gamma,\infty}$, and $(\cdot, \cdot)^0_{\gamma,\infty}$ denotes the continuous interpolation method, see Sect. 2.2. The reader may refer to (3.7) for the definitions of $\mathbb{E}_0(I)$ and $\mathbb{E}_1(I)$. Equation (1.2) implies that problem (1.1) admits for each $(f, u_0) \in \mathbb{E}_0(I) \times E_\gamma$ a unique solution u which has the best possible regularity, i.e., there is no loss of regularity as $\partial_t u$ and Au have the same regularity as f. The theory of continuous maximal regularity provides a general and flexible tool for the analysis of nonlinear parabolic equations, including fully nonlinear problems. In some cases, it can be viewed as a substitute for the well-known Nash–Moser iteration approach to fully nonlinear parabolic equations. In addition to providing existence and uniqueness of solutions, maximal regularity theory combined with the implicit function theorem renders a powerful tool to establishing further regularity results for solutions of nonlinear parabolic problems and studying geometric properties of the semiflows generated. Here, we refer to [16,18,20,21,23] for a list of related work. The theory of maximal regularity has been well-formulated in Euclidean spaces and used on compact closed manifolds. However, not until recently was L_p-maximal regularity theory established on noncompact manifolds [6], where the author uses retraction–coretraction systems of Sobolev spaces to translate the problem onto Euclidean spaces. We will make use of a similar building block to establish a theory of continuous maximal regularity on uniformly regular Riemannian manifolds without boundary. Here, we should like to mention that Amann's result [6] is of much greater generality and remarkably more technical, as maximal regularity results for parabolic boundary value problems on manifolds with boundary are obtained. Nevertheless, our result is distinctive in two respects. Firstly, it is the first maximal regularity theory for Hölder continuous functions on uniformly regular manifolds. Secondly, our result is formulated for tensor-valued problems. This theory complements the work in [20,21] for the compact case. Although we will not consider

parabolic boundary value problems in this paper, we nevertheless introduce function spaces on uniformly regular Riemannian manifolds with boundary. This setup may prove useful for further studies of geometric evolution equations, say the Yamabe flow, on manifolds with boundary. There are a few results dealing with (semilinear) parabolic equations on noncompact complete Riemannian manifolds under various curvature assumptions, which are based on heat kernel estimates, e.g., Zhang [41,42], Mazzucato and Nistor [30], Punzo [33], Bandle et al. [10]. The approach developed in [6] and in this paper does not rely on heat kernel estimates and is, thus, not limited to second-order equations. It can be applied to a wide array of nonlinear parabolic equations, including quasilinear (and even fully nonlinear) equations.

A serious challenge in the development of a general and useful theory of function spaces on uniformly regular manifolds turns out to be the availability of interpolation results. This difficulty can be overcome by the interpolation result [37, Section 1.18.1] in the case of Sobolev spaces, but is surprisingly difficult for Hölder spaces, which are natural candidates for the spaces E_0, E_1 in (1.1). Thanks to the work of Amann in [3,4], we are able to build up the theory via a linear isomorphism $\boldsymbol{f}$ defined in Sect. 2.2.

Section 2 is the step stone to the theory of continuous maximal regularity. Section 2.1 is of preparatory character, wherein we state the geometric assumptions and some basic concepts on tensor bundles and connections from manifold theory. In the first half of the subsequent subsection, we introduce Hölder continuous tensor fields and the corresponding retraction–coretraction theory of these spaces. This work, as aforementioned, was accomplished by Amann in his two consecutive papers [3,4]. It paves the path for the interpolation, embedding, pointwise multiplication and differentiation theorems for Hölder continuous tensor fields in the second half of the same section and Sect. 2.3.

The main theorem of this paper, Theorem 3.3, is then formulated in Sect. 3. Its proof relies on the retraction–coretraction system established in Sect. 2.2 and a careful estimate of the lower order terms via interpolation theory. A resolvent estimate for so-called $(\mathcal{E}, \vartheta; l)$-elliptic operators acting on tensor bundles is presented therein. Following the well-known semigroup theory and Da Prato et al.'s work, we then prove a continuous maximal regularity theory on uniformly regular Riemannian manifolds. We will present the theory in a general format, that is to say, we establish maximal regularity for so-called *normally elliptic* differential operators acting on tensor fields, with minimal regularity assumptions on the coefficients of the differential operators. One of the reasons for considering tensor fields, rather than scalar functions, lies in the fact that this general result is used in a forthcoming paper [36] to prove analyticity of solutions to the Ricci flow.

In Sect. 4, we give a short introduction to a parameter-dependent translation technique on manifolds, which combined with maximal regularity theory serves as a very beneficial tool for establishing regularity of solutions to parabolic equations. The idea of employing a localized translation in conjunction with the implicit function theorem

was initiated by Escher et al. [22]. Through the retraction–coretraction systems, we can thus introduce an analogy for functions over manifolds. We refer the reader to [36] for more information on this technique.

After importing all the theoretic tools, in Sect. 5, we thus can present an application to the Yamabe flow. The reader may refer to Sect. 5 for a brief historic account of this problem. It has been proved that the normalized Yamabe flow on compact manifolds admits a unique global and smooth solution, for smooth initial data, see [40]. We will show in this paper that this solution exists analytically for all positive times. Less is known about the Yamabe flow on noncompact manifolds. To the best of the authors' knowledge, all available results in this direction require the underlying manifold to have bounded curvatures, or even to be of some explicit expression, see [7] and [14]. We will formulate an existence and regularity result for the Yamabe flow without asking for boundedness of the curvatures.

In the rest of this introductory section, we give the precise definition of uniformly regular Riemannian manifolds and present the existence of a localization system, which plays a key role in the retraction–coretraction theory established in Sect. 2.2. After that, we briefly list some notations that we shall use throughout.

1.1. Assumptions on manifolds

In this section, we list some background information on manifolds, which provide the basis for the Hölder, little Hölder spaces and tensor fields on Riemannian manifolds to be introduced below. This fundamental work was first introduced in [3] and [4].

Let (M, g) be a C^∞-Riemannian manifold of dimension m with or without boundary endowed with g as its Riemannian metric such that its underlying topological space is separable. An atlas $\mathfrak{A} := (\mathsf{O}_\kappa, \varphi_\kappa)_{\kappa\in\mathfrak{K}}$ for M is said to be normalized if

$$\varphi_\kappa(\mathsf{O}_\kappa) = \begin{cases} \mathbb{B}^m, & \mathsf{O}_\kappa \subset \mathring{\mathsf{M}}, \\ \mathbb{B}^m \cap \mathbb{H}^m, & \mathsf{O}_\kappa \cap \partial\mathsf{M} \neq \emptyset, \end{cases}$$

where $\mathbb{H}^m$ is the closed half space $\mathbb{R}^+ \times \mathbb{R}^{m-1}$ and $\mathbb{B}^m$ is the unit Euclidean ball centered at the origin in $\mathbb{R}^m$. We put $\mathbb{B}^m_\kappa := \varphi_\kappa(\mathsf{O}_\kappa)$ and $\psi_\kappa := \varphi_\kappa^{-1}$.

The atlas $\mathfrak{A}$ is said to have *finite multiplicity* if there exists $K \in \mathbb{N}$ such that any intersection of more than K coordinate patches is empty. Put

$$\mathfrak{N}(\kappa) := \{\tilde\kappa \in \mathfrak{K} : \mathsf{O}_{\tilde\kappa} \cap \mathsf{O}_\kappa \neq \emptyset\}.$$

The finite multiplicity of $\mathfrak{A}$ and the separability of M imply that $\mathfrak{A}$ is countable. If two real-valued functions f and g are equivalent in the sense that $f/c \le g \le cf$ for some $c \ge 1$, then we write $f \sim g$.

An atlas $\mathfrak{A}$ is said to fulfill the *uniformly shrinkable* condition, if it is normalized and there exists $r \in (0, 1)$ such that $\{\psi_\kappa(r\mathbb{B}^m_\kappa) : \kappa \in \mathfrak{K}\}$ is a cover for M.

Following Amann [3,4], we say that (M, g) is a *uniformly regular Riemannian manifold* if it admits an atlas $\mathfrak{A}$ such that

(R1) $\mathfrak{A}$ is uniformly shrinkable and has finite multiplicity.
(R2) $\|\varphi_\eta \circ \psi_\kappa\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$, $\eta \in \mathfrak{N}(\kappa)$, and $k \in \mathbb{N}_0$.
(R3) $\psi_\kappa^* g \sim g_m$, $\kappa \in \mathfrak{K}$. Here g_m denotes the Euclidean metric on $\mathbb{R}^m$ and $\psi_\kappa^* g$ denotes the pull-back metric of g by ψ_κ.
(R4) $\|\psi_\kappa^* g\|_{k,\infty} \le c(k)$, $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}_0$.

Here, $\|u\|_{k,\infty} := \max_{|\alpha|\le k} \|\partial^\alpha u\|_\infty$, and it is understood that a constant $c(k)$, like in (R2), depends only on k. An atlas $\mathfrak{A}$ satisfying (R1) and (R2) is called a *uniformly regular atlas*. (R3) reads as

$$|\xi|^2/c \le \psi_\kappa^* g(x)(\xi,\xi) \le c|\xi|^2, \quad \text{for any } x \in \mathbb{B}^m_\kappa, \xi \in \mathbb{R}^m, \kappa \in \mathfrak{K} \text{ and some } c \ge 1.$$

We refer to [5] for examples of uniformly regular Riemannian manifolds.

Given any Riemannian manifold M without boundary, by a result of Greene [24], there exists a complete Riemannian metric g_c with bounded geometry on M, see [24, Theorem 2'] and [32, Remark 1.7]. Hence, we can always find a Riemannian metric g_c making (M, g_c) uniformly regular. However, this result is of restricted interest, since in most of the PDE problems we are forced to work with a fixed background metric whose compatibility with the metric g_c is unknown.

A uniformly regular Riemannian manifold M admits a *localization system subordinate to* $\mathfrak{A}$, by which we mean a family $(\pi_\kappa, \zeta_\kappa)_{\kappa\in\mathfrak{K}}$ satisfying:

(L1) $\pi_\kappa \in \mathcal{D}(\mathsf{O}_\kappa, [0,1])$ and $(\pi_\kappa^2)_{\kappa\in\mathfrak{K}}$ is a partition of unity subordinate to $\mathfrak{A}$.
(L2) $\zeta_\kappa := \varphi_\kappa^* \zeta$ with $\zeta \in \mathcal{D}(\mathbb{B}^m, [0,1])$ satisfying $\zeta|_{\mathrm{supp}(\psi_\kappa^* \pi_\kappa)} \equiv 1$, $\kappa \in \mathfrak{K}$.
(L3) $\|\psi_\kappa^* \pi_\kappa\|_{k,\infty} \le c(k)$, for $\kappa \in \mathfrak{K}$, $k \in \mathbb{N}_0$.

The reader may refer to [3, Lemma 3.2] for a proof. In addition to the above conditions, we will find it useful to define the following auxiliary function

$$\varpi_\kappa := \varphi_\kappa^* \varpi \text{ with } \varpi \in \mathcal{D}(\mathbb{B}^m, [0,1]) \text{ satisfying that } \varpi|_{\mathrm{supp}(\zeta)} \equiv 1. \tag{1.3}$$

Lastly, if, in addition, the atlas $\mathfrak{A}$ and the metric g are real analytic, we say that (M, g) is a *C^ω-uniformly regular Riemannian manifold*.

1.2. Notations

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For any open subset $U \subseteq \mathbb{R}^m$, we abbreviate $\mathfrak{F}^s(U, \mathbb{K})$ to $\mathfrak{F}^s(U)$, where $s \ge 0$ and $\mathfrak{F} \in \{bc, BC\}$. The precise definitions for these function spaces will be presented in Sect. 2. Similarly, $\mathfrak{F}^s(\mathsf{M})$ stands for the corresponding $\mathbb{K}$-valued spaces defined on the manifold M.

Let $\|\cdot\|_\infty$ and $\|\cdot\|_{s,\infty}$ denote the usual norm of the Banach spaces $BC(U)$, $BC^s(U)$, respectively. Likewise, their counterparts defined on M are expressed by $\|\cdot\|^{\mathsf{M}}_{\mathfrak{F}}$, where $\|\cdot\|_{\mathfrak{F}}$ stands for either of the norms defined on U.

Given $\sigma, \tau \in \mathbb{N}_0$,

$$T^\sigma_\tau \mathsf{M} := T\mathsf{M}^{\otimes\sigma} \otimes T^*\mathsf{M}^{\otimes\tau}$$

is the (σ, τ)-tensor bundle of M, where $T\mathsf{M}$ and $T^*\mathsf{M}$ are the tangent and the cotangent bundle of M, respectively. We write $\mathcal{T}^\sigma_\tau \mathsf{M}$ for the $C^\infty(\mathsf{M})$-module of all smooth sections of $T^\sigma_\tau \mathsf{M}$ and $\Gamma(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ for the set of all sections.

For abbreviation, we set $\mathbb{J}^\sigma := \{1, 2, \dots, m\}^\sigma$, and $\mathbb{J}^\tau$ is defined alike. Given local coordinates $\varphi = \{x^1, \dots, x^m\}$, $(i) := (i_1, \dots, i_\sigma) \in \mathbb{J}^\sigma$ and $(j) := (j_1, \dots, j_\tau) \in \mathbb{J}^\tau$, we set

$$\frac{\partial}{\partial x^{(i)}} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}}, \quad \partial_{(i)} := \partial_{i_1} \circ \cdots \circ \partial_{i_\sigma} \quad \mathrm{d}x^{(j)} := \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_\tau}$$

with $\partial_i = \frac{\partial}{\partial x^i}$. The local representation of $a \in \Gamma(\mathsf{M}, T^\sigma_\tau \mathsf{M})$ with respect to these coordinates is given by

$$a = a^{(i)}_{(j)} \frac{\partial}{\partial x^{(i)}} \otimes \mathrm{d}x^{(j)} \tag{1.4}$$

with coefficients $a^{(i)}_{(j)}$ defined on O_κ.

For a topological set, U, $\mathring{U}$ denotes its interior. If U consists of only one point, we set $\mathring{U} := U$. For any two Banach spaces, X, Y, $X \doteq Y$ means that they are equal in the sense of equivalent norms. The notation $\mathcal{L}$ is (X, Y) stands for the set of all bounded linear isomorphisms from X to Y.

2. Function spaces on uniformly regular Riemannian manifolds

Most of the work in Sects. 2.1 and 2.2 is laid out in [3] and [4] for weighted functions and tensor fields defined on manifolds with "singular ends" characterized by a "singular function" $\rho \in C^\infty(\mathsf{M}, (0, \infty))$. Such manifolds are uniformly regular iff the singular datum satisfies $\rho \sim 1_{\mathsf{M}}$. Because of this, we will state some of the results therein without providing proofs below.

2.1. Tensor bundles

Let $\mathbb{A}$ be a countable index set. Suppose E_α is for each $\alpha \in \mathbb{A}$ a locally convex space. We endow $\prod_\alpha E_\alpha$ with the product topology, that is, the coarsest topology for which all projections $pr_\beta : \prod_\alpha E_\alpha \to E_\beta$, $(e_\alpha)_\alpha \mapsto e_\beta$ are continuous. By $\bigoplus_\alpha E_\alpha$, we mean the vector subspace of $\prod_\alpha E_\alpha$ consisting of all finitely supported elements, equipped with the inductive limit topology, that is, the finest locally convex topology for which all injections $E_\beta \to \bigoplus_\alpha E_\alpha$ are continuous.

We denote by $\nabla = \nabla_g$ the Levi–Civita connection on $T\mathsf{M}$. It has a unique extension over $\mathcal{T}^\sigma_\tau \mathsf{M}$ satisfying, for $X \in \mathcal{T}^1_0 \mathsf{M}$,

(i) $\nabla_X f = \langle df, X\rangle, \quad f \in C^\infty(\mathsf{M})$,
(ii) $\nabla_X(a \otimes b) = \nabla_X a \otimes b + a \otimes \nabla_X b, \quad a \in \mathcal{T}^{\sigma_1}_{\tau_1}\mathsf{M}, b \in \mathcal{T}^{\sigma_2}_{\tau_2}\mathsf{M}$,
(iii) $\nabla_X\langle a, b\rangle = \langle \nabla_X a, b\rangle + \langle a, \nabla_X b\rangle, \quad a \in \mathcal{T}^\sigma_\tau \mathsf{M}, b \in \mathcal{T}^\tau_\sigma \mathsf{M}$,

where $\langle\cdot,\cdot\rangle : \mathcal{T}^\sigma_\tau \mathsf{M} \times \mathcal{T}^\tau_\sigma \mathsf{M} \to C^\infty(\mathsf{M})$ is the extension of the fiberwise defined duality pairing on M, cf. [3, Section 3]. Then, the covariant (Levi–Civita) derivative is the linear map

$$\nabla : \mathcal{T}^\sigma_\tau \mathsf{M} \to \mathcal{T}^\sigma_{\tau+1}\mathsf{M}, a \mapsto \nabla a$$

defined by

$$\langle \nabla a, b \otimes X\rangle := \langle \nabla_X a, b\rangle, \quad b \in \mathcal{T}^\tau_\sigma \mathsf{M}, X \in \mathcal{T}^1_0 \mathsf{M}.$$

For $k \in \mathbb{N}_0$, we define

$$\nabla^k : \mathcal{T}^\sigma_\tau \mathsf{M} \to \mathcal{T}^\sigma_{\tau+k}\mathsf{M}, a \mapsto \nabla^k a$$

by letting $\nabla^0 a := a$ and $\nabla^{k+1} a := \nabla \circ \nabla^k a$. We can also extend the Riemannian metric $(\cdot|\cdot)_g$ from the tangent bundle to any (σ,τ)-tensor bundle $T^\sigma_\tau \mathsf{M}$ such that $(\cdot|\cdot)_g : T^\sigma_\tau \mathsf{M} \times T^\sigma_\tau \mathsf{M} \to \mathbb{K}$ by setting

$$(\cdot|\cdot)_g : (a_1,\dots,a_\sigma,b_1,\dots,b_\tau)\times(c_1,\dots,c_\sigma,d_1,\dots,d_\tau) \mapsto \prod_{i=1}^{\sigma}(a_i|c_i)_g \prod_{i=1}^{\tau}(b_i|d_i)_{g^*},$$

where $a_i, c_i \in T\mathsf{M}$, $b_i, d_i \in T^*\mathsf{M}$ and $(\cdot|\cdot)_{g^*}$ denotes the induced contravariant metric. In addition,

$$|\cdot|_g : \mathcal{T}^\sigma_\tau \mathsf{M} \to C^\infty(\mathsf{M}), a \mapsto \sqrt{(a|a)_g}$$

is called the (vector bundle) *norm* induced by g.

We assume that V is a $\mathbb{K}$-valued tensor bundle on M and E is a $\mathbb{K}$-valued vector space, i.e.,

$$V = V^\sigma_\tau := \left\{T^\sigma_\tau \mathsf{M}, (\cdot|\cdot)_g\right\}, \quad \text{and} \quad E = E^\sigma_\tau := \left\{\mathbb{K}^{m^\sigma \times m^\tau}, (\cdot|\cdot)\right\},$$

for some $\sigma,\tau \in \mathbb{N}_0$. Here, $(a|b) :=\operatorname{trace}(b^* a)$ with b^* being the conjugate matrix of b. By setting $N = m^{\sigma+\tau}$, we can identify $\mathfrak{F}^s(\mathsf{M}, E)$ with $\mathfrak{F}^s(\mathsf{M})^N$.

Throughout the rest of this paper, we always assume that

- (M, g) is a uniformly regular Riemannian manifold.
- $(\pi_\kappa, \zeta_\kappa)_{\kappa\in\mathfrak{K}}$ is a localization system subordinate to $\mathfrak{A}$.
- $\sigma,\tau \in \mathbb{N}_0$, $V = V^\sigma_\tau := \{T^\sigma_\tau \mathsf{M}, (\cdot|\cdot)_g\}$, $E = E^\sigma_\tau := \{\mathbb{K}^{m^\sigma\times m^\tau}, (\cdot|\cdot)\}$.

For $K \subset \mathsf{M}$, we put $\mathfrak{K}_K := \{\kappa \in \mathfrak{K} : \mathsf{O}_\kappa \cap K \neq \emptyset\}$. Then, given $\kappa \in \mathfrak{K}$,

$$\mathbb{X}_\kappa := \begin{cases} \mathbb{R}^m & \text{if } \kappa \in \mathfrak{K}\backslash\mathfrak{K}_{\partial\mathsf{M}}, \\ \mathbb{H}^m & \text{otherwise,}\end{cases}$$

endowed with the Euclidean metric g_m.

Given $a \in \Gamma(\mathsf{M}, V)$ with local representation (1.4), we define $\psi_\kappa^* a \in E$ by means of $\psi_\kappa^* a = [a_{(j)}^{(i)}]$, where $[a_{(j)}^{(i)}]$ stands for the $(m^\sigma \times m^\tau)$-matrix with entries $a_{(j)}^{(i)}$ in the $((i), (j))$ position, with (i), (j) arranged lexicographically.

For the sake of brevity, we set $\boldsymbol{L}_{1,\mathrm{loc}}(\mathbb{X}, E) := \prod_\kappa L_{1,\mathrm{loc}}(\mathbb{X}_\kappa, E)$. Then, we introduce two linear maps for $\kappa \in \mathfrak{K}$:

$$\mathcal{R}_\kappa^c : L_{1,\mathrm{loc}}(\mathsf{M}, V) \to L_{1,\mathrm{loc}}(\mathbb{X}_\kappa, E), u \mapsto \psi_\kappa^*(\pi_\kappa u),$$

and

$$\mathcal{R}_\kappa : L_{1,\mathrm{loc}}(\mathbb{X}_\kappa, E) \to L_{1,\mathrm{loc}}(\mathsf{M}, V), v_\kappa \mapsto \pi_\kappa \varphi_\kappa^* v_\kappa.$$

Here, and in the following, it is understood that a partially defined and compactly supported tensor field is automatically extended over the whole base manifold by identifying it to be zero outside its original domain. Moreover,

$$\mathcal{R}^c : L_{1,\mathrm{loc}}(\mathsf{M}, V) \to \boldsymbol{L}_{1,\mathrm{loc}}(\mathbb{X}, E), u \mapsto (\mathcal{R}_\kappa^c u)_\kappa,$$

and

$$\mathcal{R} : \boldsymbol{L}_{1,\mathrm{loc}}(\mathbb{X}, E) \to L_{1,\mathrm{loc}}(\mathsf{M}, V), (v_\kappa)_\kappa \mapsto \sum_\kappa \mathcal{R}_\kappa v_\kappa.$$

2.2. Hölder and little Hölder spaces

Before we study the Hölder and little Hölder spaces on uniformly regular Riemannian manifolds, we list some prerequisites for such spaces on $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$ from [4].

Throughout this subsection, we assume that $k \in \mathbb{N}_0$. For any given Banach space F, the Banach space $BC^k(\mathbb{X}, F)$ is defined by

$$BC^k(\mathbb{X}, F) := (\{u \in C^k(\mathbb{X}, F) : \|u\|_{k,\infty} < \infty\}, \|\cdot\|_{k,\infty}).$$

The closed linear subspace $BUC^k(\mathbb{X}, F)$ of $BC^k(\mathbb{X}, F)$ consists of all functions $u \in BC^k(\mathbb{X}, F)$ such that $\partial^\alpha u$ is uniformly continuous for all $|\alpha| \le k$. Moreover,

$$BC^\infty(\mathbb{X}, F) := \bigcap_k BC^k(\mathbb{X}, F) = \bigcap_k BUC^k(\mathbb{X}, F).$$

It is a Fréchet space equipped with the natural projective topology.

For $0 < s < 1$, $0 < \delta \le \infty$ and $u \in F^{\mathbb{X}}$, the seminorm $[\cdot]_{s,\infty}^\delta$ is defined by

$$[u]_{s,\infty}^\delta := \sup_{h \in (0,\delta)^m} \frac{\|u(\cdot + h) - u(\cdot)\|_\infty}{|h|^s}, \quad [\cdot]_{s,\infty} := [\cdot]_{s,\infty}^\infty.$$

Let $k < s < k+1$. The *Hölder* space $BC^s(\mathbb{X}, F)$ is defined as

$$BC^s(\mathbb{X}, F) := (\{u \in BC^k(\mathbb{X}, F) : \|u\|_{s,\infty} < \infty\}, \|\cdot\|_{s,\infty}),$$

where $\|u\|_{s,\infty} := \|u\|_{k,\infty} + \max_{|\alpha|=k}[\partial^\alpha u]_{s-k,\infty}$.

The *little Hölder* space of order $s \geq 0$ is defined by

$$bc^s(\mathbb{X}, F) := \text{the closure of } BC^\infty(\mathbb{X}, F) \text{ in } BC^s(\mathbb{X}, F).$$

By [4, formula (11.13), Corollary 11.2, Theorem 11.3], we have

$$bc^k(\mathbb{X}, F) = BUC^k(\mathbb{X}, F), \tag{2.1}$$

and for $k < s < k+1$

$$u \in BC^s(\mathbb{X}, F) \text{ belongs to } bc^s(\mathbb{X}, F) \text{ iff } \lim_{\delta\to 0}[\partial^\alpha u]^\delta_{s-[s],\infty} = 0, \quad |\alpha| = [s].$$

In the following context, let F_κ be Banach spaces. Then, we put $\boldsymbol{F} := \prod_\kappa F_\kappa$. We denote by $l_\infty(\boldsymbol{F})$ the linear subspace of $\boldsymbol{F}$ consisting of all $\boldsymbol{x} = (x_\kappa)$ such that

$$\|\boldsymbol{x}\|_{l_\infty(\boldsymbol{F})} := \sup_\kappa \|x_\kappa\|_{F_\kappa}$$

is finite. Then, $l_\infty(\boldsymbol{F})$ is a Banach space with norm $\|\cdot\|_{l_\infty(\boldsymbol{E})}$.

For $\mathfrak{F} \in \{bc, BC\}$, we put $\boldsymbol{\mathfrak{F}}^s := \prod_\kappa \mathfrak{F}^s_\kappa$, where $\mathfrak{F}^s_\kappa := \mathfrak{F}^s(\mathbb{X}_\kappa, E)$. Denote by

$$l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k)$$

the linear subspace of $l_\infty(\boldsymbol{BC}^k)$ of all $\boldsymbol{u} = (u_\kappa)_\kappa$ such that $\partial^\alpha u_\kappa$ is uniformly continuous on $\mathbb{X}_\kappa$ for $|\alpha| \leq k$, uniformly with respect to $\kappa \in \mathfrak{K}$. Similarly, for any $k < s < k+1$, we denote by

$$l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s)$$

the linear subspace of $l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k)$ of all $\boldsymbol{u} = (u_\kappa)_\kappa$ such that

$$\lim_{\delta\to 0}\max_{|\alpha|=k}[\partial^\alpha u_\kappa]^\delta_{s-k,\infty} = 0, \tag{2.2}$$

uniformly with respect to $\kappa \in \mathfrak{K}$.

$$\boldsymbol{f} : \boldsymbol{F}^{\mathbb{X}} \to \prod_\kappa F_\kappa^{\mathbb{X}}, \quad u \mapsto \boldsymbol{f}(u) := (\mathrm{pr}_\kappa \circ u)_\kappa$$

is a linear bijection. Set $E_\kappa := E^\sigma_\tau$ and $\boldsymbol{E} := \prod_\kappa E_\kappa$. Then, [4, Lemma 11.10, 11.11] tells us that for $s \geq 0$

$$\boldsymbol{f} \in \mathcal{L} \text{ is } \left(bc^s(\mathbb{X}, l_\infty(\boldsymbol{E})), l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s)\right), \tag{2.3}$$

and for $s > 0$, $s \notin \mathbb{N}$

$$\boldsymbol{f} \in \mathcal{L} \text{ is } \left(BC^s(\mathbb{X}, l_\infty(\boldsymbol{E})), l_\infty(\boldsymbol{BC}^s)\right). \tag{2.4}$$

Now, we are in a position to introduce the counterparts of these spaces on uniformly regular Riemannian manifolds. For $k \in \mathbb{N}_0$, we define

$$BC^k(\mathsf{M}, V) := \left(\{u \in C^k(\mathsf{M}, V) : \|u\|^{\mathsf{M}}_{k,\infty} < \infty\}, \|\cdot\|^{\mathsf{M}}_{k,\infty}\right),$$

where $\|u\|^{\mathsf{M}}_{k,\infty} := \max_{0\le i\le k} \||\nabla^i u|_g\|_\infty$.

We also set

$$BC^\infty(\mathsf{M}, V) := \bigcap_k BC^k(\mathsf{M}, V)$$

endowed with the conventional projective topology. Then,

$$bc^k(\mathsf{M}, V) := \text{the closure of } BC^\infty \text{ in } BC^k.$$

Let $k < s < k+1$. Now, the *Hölder* space $BC^s(\mathsf{M}, V)$ is defined by

$$BC^s(\mathsf{M}, V) := \left(bc^k(\mathsf{M}, V), bc^{k+1}(\mathsf{M}, V)\right)_{s-k,\infty}.$$

Here, $(\cdot,\cdot)_{\theta,\infty}$ is the real interpolation method, see [1, Example I.2.4.1] and [28, Definition 1.2.2]. It is a Banach space by interpolation theory. For $s \ge 0$, we define the *little Hölder* spaces by

$$bc^s(\mathsf{M}, V) := \text{the closure of } BC^\infty(\mathsf{M}, V) \text{ in } BC^s(\mathsf{M}, V).$$

THEOREM 2.1. *Suppose $s \ge 0$. Then, $\mathcal{R}$ is a retraction*

$$\textit{from } l_\infty(\boldsymbol{BC}^s) \textit{ onto } BC^s(\mathsf{M}, V),$$

and

$$\textit{from } l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s) \textit{ onto } bc^s(\mathsf{M}, V).$$

Moreover, $\mathcal{R}^c$ is a coretraction in both cases.

Proof. See [3, Theorem 6.3] and [4, Theorem 12.1, 12.3, formula (12.2)]. Note that for $s \notin \mathbb{N}_0$, $BC^s(\mathsf{M}, V)$ and $bc^s(\mathsf{M}, V)$ coincide with the spaces $B^s_\infty(\mathsf{M}, V)$ and $b^s_\infty(\mathsf{M}, V)$ defined in [4, Section 12]. □

In the following proposition, $(\cdot,\cdot)^0_{\theta,\infty}$ and $[\cdot,\cdot]_\theta$ are the continuous interpolation method and the complex interpolation method, respectively. See [1, Example I.2.4.2, I.2.4.4] for definitions.

PROPOSITION 2.2. *Suppose that $0 < \theta < 1$, $0 \le s_0 < s_1$ and $s = (1-\theta)s_0 + \theta s_1$ with $s \notin \mathbb{N}$. Then,*

(a) $(BC^{s_0}(\mathsf{M}, V), BC^{s_1}(\mathsf{M}, V))_{\theta,\infty} \doteq BC^s(\mathsf{M}, V)$
$\doteq [BC^{s_0}(\mathsf{M}, V), BC^{s_1}(\mathsf{M}, V)]_\theta$, *for* $s_0, s_1 \notin \mathbb{N}_0$.
(b) $(bc^{s_0}(\mathsf{M}, V), bc^{s_1}(\mathsf{M}, V))_{\theta,\infty} \doteq BC^s(\mathsf{M}, V)$, *for* $s_0, s_1 \in \mathbb{N}_0$.
(c) $(bc^{s_0}(\mathsf{M}, V), bc^{s_1}(\mathsf{M}, V))^0_{\theta,\infty} \doteq bc^s(\mathsf{M}, V) \doteq [bc^{s_0}(\mathsf{M}, V), bc^{s_1}(\mathsf{M}, V)]_\theta$, *for* $s_0, s_1 \notin \mathbb{N}_0$.
(d) $(bc^{s_0}(\mathsf{M}, V), bc^{s_1}(\mathsf{M}, V))^0_{\theta,\infty} \doteq bc^s(\mathsf{M}, V)$, *for* $s_0, s_1 \in \mathbb{N}_0$.

Proof. See [4, Corollary 12.2 (iii), (iv) and Corollary 12.4]. □

PROPOSITION 2.3. *Suppose that* $u \in \mathfrak{F}^s(\mathsf{M}, V)$ *for* $\mathfrak{F} \in \{bc, BC\}$. *Then,*

$$\|\cdot\|^{\mathsf{M}}_{\mathfrak{F}^s} \sim \|\mathcal{R}^c(\cdot)\|_{l_\infty(\mathfrak{F}^s)} = \sup_\kappa \|(\psi^*_\kappa(\pi_\kappa \cdot))_\kappa\|_{\mathfrak{F}^s_\kappa},$$

Proof. Since $\mathcal{R}$ and $\mathcal{R}^c$ are continuous, there exists a constant $C > 0$ such that for any $u \in \mathfrak{F}^s(\mathsf{M}, V)$, we have $\frac{1}{C}\|\mathcal{R}^c u\|_{l_\infty(\mathfrak{F}^s)} \le \|u\|^{\mathsf{M}}_{\mathfrak{F}^s} = \|\mathcal{R}\mathcal{R}^c u\|^{\mathsf{M}}_{\mathfrak{F}^s} \le C\|\mathcal{R}^c u\|_{l_\infty(\mathfrak{F}^s)}$. □

For a given $\kappa \in \mathfrak{K}$ and any $\eta \in \mathfrak{N}(\kappa)$, we define $S_{\eta\kappa} : E^{\mathbb{X}_\eta} \to E^{\mathbb{X}_\kappa}$ by

$$S_{\eta\kappa} : u \mapsto \varpi\, \psi^*_\kappa \pi_\eta \psi^*_\kappa \varphi^*_\eta u.$$

LEMMA 2.4. *Suppose that* $\mathfrak{F} \in \{bc, BC\}$. *Then,*

$$S_{\eta\kappa} \in \mathcal{L}(\mathfrak{F}^s_\eta, \mathfrak{F}^s_\kappa), \quad \textit{and} \quad \|S_{\eta\kappa}\| \le c,$$

for $\eta \in \mathfrak{N}(\kappa)$. *Here, the positive constant c is independent of* κ *and* $\eta \in \mathfrak{N}(\kappa)$. *The statement still holds true with* ϖ *replaced by* ζ *in the definition of* $S_{\eta\kappa}$ *or* π_η *being replaced by* ζ_η.

Proof. The case that $s \in \mathbb{N}_0$ follows from the pointwise multiplication result on $\mathbb{X}$, the chain rule, (R2) and (L3). The gaps left can be filled in by interpolation theory. □

An alternative of the spaces $\mathfrak{F}^s(\mathsf{M})$ can be defined as follows:

$$\breve{\mathfrak{F}}^s(\mathsf{M}) := \left(\{u \in L_{1,\mathrm{loc}}(\mathsf{M}) : \psi^*_\kappa(\pi^2_\kappa u) \in \mathfrak{F}^s\}, \|\cdot\|_{\breve{\mathfrak{F}}^s(\mathsf{M})}\right),$$

where $\mathfrak{F} \in \{bc, BC\}$ and $\|u\|_{\breve{\mathfrak{F}}^s(\mathsf{M})} := \left\|\psi^*_\kappa(\pi^2_\kappa u)\right\|_{l_\infty(\mathfrak{F}^s)}$.

PROPOSITION 2.5. $\breve{\mathfrak{F}}^s(\mathsf{M}) \doteq \mathfrak{F}^s(\mathsf{M})$.

Proof. The proof is straightforward. Indeed,

$$\|u\|_{\breve{\mathfrak{F}}^s(\mathsf{M})} = \|\psi^*_\kappa(\pi^2_\kappa u)\|_{l_\infty(\mathfrak{F}^s)} \le M\|\mathcal{R}^c_\kappa u\|_{l_\infty(\mathfrak{F}^s)} \le M\|u\|^{\mathsf{M}}_{\mathfrak{F}^s}.$$

To obtain the other direction, we adopt Lemma 2.4 to compute

$$\|\mathcal{R}^c_\kappa u\|_{\mathfrak{F}^s_\kappa} \le \sum_{\eta\in\mathfrak{N}(\kappa)} \|\psi^*_\kappa \pi_\kappa \varpi\, \psi^*_\kappa(\zeta_\eta \pi^2_\eta u)\|_{\mathfrak{F}^s_\kappa} \le M \sum_{\eta\in\mathfrak{N}(\kappa)} \|\psi^*_\eta(\pi^2_\eta u)\|_{\mathfrak{F}^s_\eta} \le M\|u\|_{\breve{\mathfrak{F}}^s(\mathsf{M})}.$$

□

2.3. Basic properties

In this subsection, we will list some basic properties of the function spaces and tensor fields introduced in the previous subsections. These properties are well known to be enjoyed by their counterparts in Euclidean spaces or on domains with smooth boundary. By using the interpolation and retraction properties setup above, we can verify them on a uniformly regular Riemannian manifold M.

PROPOSITION 2.6. *$\mathfrak{F}^t(\mathsf{M}, V) \overset{d}{\hookrightarrow} bc^s(\mathsf{M}, V)$, where $t > s \geq 0$ and $\mathfrak{F} \in \{bc, BC\}$.*

Proof. This result is a direct consequence of interpolation theory and the dense embedding $BC^\infty(\mathsf{M}, V) \overset{d}{\hookrightarrow} bc^s(\mathsf{M}, V)$. □

Let $V_j = V_{\tau_j}^{\sigma_j} := \{T_{\tau_j}^{\sigma_j}\mathsf{M}, (\cdot|\cdot)_g\}$ with $j = 1, 2, 3$ be $\mathbb{K}$-valued tensor bundles on M. By bundle multiplication from $V_1 \times V_2$ into V_3, denoted by

$$\mathsf{m} : V_1 \times V_2 \to V_3, \quad (v_1, v_2) \mapsto \mathsf{m}(v_1, v_2),$$

we mean a smooth bounded section $\mathfrak{m}$ of $\mathrm{Hom}(V_1 \otimes V_2, V_3)$, i.e.,

$$\mathfrak{m} \in BC^\infty(\mathsf{M}, \mathrm{Hom}(V_1 \otimes V_2, V_3)), \tag{2.5}$$

such that $\mathsf{m}(v_1, v_2) := \mathfrak{m}(v_1 \otimes v_2)$. Equation (2.5) implies that for some $c > 0$

$$|\mathsf{m}(v_1, v_2)|_g \leq c|v_1|_g|v_2|_g, \quad v_i \in \Gamma(\mathsf{M}, V_i) \text{ with } i = 1, 2.$$

Its pointwise extension from $\Gamma(\mathsf{M}, V_1 \oplus V_2)$ into $\Gamma(\mathsf{M}, V_3)$ is defined by:

$$\mathsf{m}(v_1, v_2)(p) := \mathsf{m}(p)(v_1(p), v_2(p))$$

for $v_i \in \Gamma(\mathsf{M}, V_i)$ and $p \in \mathsf{M}$. We still denote it by m. We can also prove the following pointwise multiplier theorem for function spaces over uniformly regular Riemannian manifolds.

PROPOSITION 2.7. *Let $k \in \mathbb{N}_0$, and $V_j = V_{\tau_j}^{\sigma_j} := \{T_{\tau_j}^{\sigma_j}\mathsf{M}, (\cdot|\cdot)_g\}$ with $j = 1, 2, 3$ be tensor bundles. Suppose that $\mathsf{m} : V_1 \times V_2 \to V_3$ is a bundle multiplication. Then, $[(v_1, v_2) \mapsto \mathsf{m}(v_1, v_2)]$ is a bilinear and continuous map for the following spaces:*

$$\mathfrak{F}^s(\mathsf{M}, V_1) \times \mathfrak{F}^s(\mathsf{M}, V_2) \to \mathfrak{F}^s(\mathsf{M}, V_3), \text{ where } s \geq 0 \text{ and } \mathfrak{F} \in \{bc, BC\}.$$

Proof. This assertion follows from [4, Theorem 13.5], wherein pointwise multiplication results for anisotropic function spaces are presented. For the reader's convenience, we will state herein a brief proof for the isotropic case.

(i) Let $(M, g) = (\mathbb{X}_\kappa, g_m)$. Set $E_j = E_{\tau_j}^{\sigma_j} := \{\mathbb{K}^{m^{\sigma_j} \times m^{\tau_j}}, (\cdot|\cdot)\}$ with $j = 1, 2, 3$. Suppose $\mathsf{b}_e \in \mathcal{L}(E_1, E_2; E_3)$, the space of all continuous bilinear maps: $E_1 \times E_2 \to E_3$, and denote its pointwise extension by m_e. The case $s \in \mathbb{N}$ follows

from the product rule. For the same reason, it suffices to prove the case $0 < s < 1$. One may check that

$$[\mathsf{m}_e(v_1, v_2)]^\delta_{s,\infty} \le c\left([v_1]^\delta_{s,\infty}\|v_2\|_\infty + [v_2]^\delta_{s,\infty}\|v_1\|_\infty\right) \tag{2.6}$$

for $v_i \in E_i^{\mathbb{X}_\kappa}$. Now it is an immediate consequence of the continuity of b_e that

$$\mathsf{m}_e \in \mathcal{L}\left(\mathfrak{F}^s(\mathbb{X}_\kappa, E_1), \mathfrak{F}^s(\mathbb{X}_\kappa, E_2); \mathfrak{F}^s(\mathbb{X}_\kappa, E_3)\right).$$

(ii) Suppose that $\mathsf{b}_e \in \mathfrak{F}^s(\mathbb{X}_\kappa, \mathcal{L}(E_1, E_2; E_3))$ and denote its pointwise extension by m_e, i.e.,

$$\mathsf{m}_e : E_1^{\mathbb{X}_\kappa} \times E_2^{\mathbb{X}_\kappa} \to E_3^{\mathbb{X}_\kappa} : \ (v_1, v_2) \mapsto \mathsf{b}_e(x)(v_1(x), v_2(x)).$$

Consider the multiplications:

$$\mathsf{b}_1 \in \mathcal{L}(\mathcal{L}(E_1, E_2; E_3), E_1; \mathcal{L}(E_2, E_3)), \ \ (f, v_1) \mapsto f(v_1, \cdot),$$

and

$$\mathsf{b}_2 \in \mathcal{L}(\mathcal{L}(E_2; E_3), E_2; E_3), \ \ (g, v_2) \mapsto g(v_2),$$

where $v_i \in E_i$. Denote by m_i the pointwise extension of b_i. Then by step (i), we deduce that

$$\mathsf{m}_1 \in \mathcal{L}\left(\mathfrak{F}^s(\mathbb{X}_\kappa, \mathcal{L}(E_1, E_2; E_3)), \mathfrak{F}^s(\mathbb{X}_\kappa, E_1); \mathfrak{F}^s(\mathbb{X}_\kappa, \mathcal{L}(E_2, E_3))\right),$$

and

$$\mathsf{m}_2 \in \mathcal{L}(\mathfrak{F}^s(\mathbb{X}_\kappa, \mathcal{L}(E_2, E_3)), \mathfrak{F}^s(\mathbb{X}_\kappa, E_2); \mathfrak{F}^s(\mathbb{X}_\kappa, E_3)).$$

Since $\mathsf{m}_e(v_1, v_2) = \mathsf{m}_2(\mathsf{m}_1(\mathsf{b}_e, v_1), v_2)$, it yields

$$\mathsf{m}_e \in \mathcal{L}(\mathfrak{F}^s(\mathbb{X}_\kappa, E_1), \mathfrak{F}^s(\mathbb{X}_\kappa, E_2); \mathfrak{F}^s(\mathbb{X}_\kappa, E_3)).$$

Moreover, the norm of m_e only relies on $\|\mathsf{b}_e\|_{s,\infty}$.

(iii) We define m_κ by

$$\mathsf{m}_\kappa(\xi_1, \xi_2) := \psi_\kappa^*(\zeta_\kappa \mathsf{m}(\varphi_\kappa^*\xi_1, \varphi_\kappa^*\xi_2))$$

for $\xi_i \in E_i^{\mathbb{X}_\kappa}$. Now it is a consequence of (L3), (2.5) and [3, Lemma 3.1(iv)] that

$$\mathsf{m}_\kappa \in BC^k(\mathbb{X}_\kappa, \mathcal{L}(E_1, E_2; E_3)), \quad \|\mathsf{m}_\kappa\|_{k,\infty} \le c(k)$$

for each $k \in \mathbb{N}_0$ and the constant $c(k)$ is independent of κ. Thus m_κ is a bundle multiplication. Now we conclude from (ii) that

$$\mathsf{m}_\kappa \in \mathcal{L}(\mathfrak{F}^s(\mathbb{X}_\kappa, E_1), \mathfrak{F}^s(\mathbb{X}_\kappa, E_2); \mathfrak{F}^s(\mathbb{X}_\kappa, E_3)).$$

Moreover, the norm of m_κ is independent of the choice of κ.

(iv) Given $v_1 \in \mathfrak{F}^s(\mathsf{M}, V_1)$ and $v_2 \in \mathfrak{F}^s(\mathsf{M}, V_2)$, we have

$$\begin{aligned}\mathcal{R}^c_\kappa(\mathsf{m}(v_1, v_2)) &= \psi^*_\kappa(\pi_\kappa \mathsf{m}(v_1, v_2)) \\ &= \sum_{\eta\in\mathfrak{N}(\kappa)} \mathsf{m}_\kappa(\mathcal{R}^c_\kappa v_1, \varpi\, \psi^*_\kappa(\pi^2_\eta v_2)).\end{aligned}$$

The discussion in (iii) shows that

$$\begin{aligned}\|\mathcal{R}^c_\kappa(\mathsf{m}(v_1, v_2))\|_{s,\infty,\mathbb{X}_\kappa} &\le \sum_{\eta\in\mathfrak{N}(\kappa)} \|\mathsf{m}_\kappa(\mathcal{R}^c_\kappa v_1, \varpi\, \psi^*_\kappa(\pi^2_\eta v_2))\|_{s,\infty,\mathbb{X}_\kappa} \\ &\le c \sum_{\eta\in\mathfrak{N}(\kappa)} \|\mathcal{R}^c_\kappa v_1\|_{s,\infty,\mathbb{X}_\kappa} \|\varpi\, \psi^*_\kappa(\pi^2_\eta v_2))\|_{s,\infty,\mathbb{X}_\kappa} \\ &\le c \sum_{\eta\in\mathfrak{N}(\kappa)} \|\mathcal{R}^c_\kappa v_1\|_{s,\infty,\mathbb{X}_\kappa} \|\mathcal{R}^c_\eta v_2\|_{s,\infty,\mathbb{X}_\eta} \\ &\le cK \|\mathcal{R}^c_\kappa v_1\|_{s,\infty,\mathbb{X}_\kappa} \|v_2\|^{\mathsf{M}}_{s,\infty}\end{aligned}$$

The penultimate line follows from the pointwise multiplication result in $\mathbb{X}_\kappa$ and Lemma 2.4. The last line is a straightforward consequence of (R1). Thus Theorem 2.1 implies that

$$\mathsf{m} \in \mathcal{L}(\mathfrak{F}^s(\mathsf{M}, V_1), \mathfrak{F}^s(\mathsf{M}, V_2); BC^s(\mathsf{M}, V_3)).$$

By adopting a density argument based on Proposition 2.6, we can show that in fact

$$\mathsf{m} \in \mathcal{L}(bc^s(\mathsf{M}, V_1), bc^s(\mathsf{M}, V_2); bc^s(\mathsf{M}, V_3)).$$

Indeed, pick an arbitrary $t > s$. For each $v_i \in bc^s(\mathsf{M}, V_i)$ with $i = 1, 2$, there exist $(u^i_j)_j \in BC^t(\mathsf{M}, V_i)$ converging to v_i in $BC^s(\mathsf{M}, V_i)$. Then by the above discussion and the triangle inequality, we deduce that $\mathsf{m}(u^1_j, u^2_j) \in BC^t(\mathsf{M}, V_3)$ and $\mathsf{m}(u^1_j, u^2_j) \to \mathsf{m}(v_1, v_2)$ in $BC^s(\mathsf{M}, V_3)$, which implies that $\mathsf{m}(v_1, v_2) \in bc^s(\mathsf{M}, V_3)$. □

Let $s \ge 0$ and $l \in \mathbb{N}_0$. A linear operator $\mathcal{A} : C^\infty(\mathsf{M}, V) \to \Gamma(\mathsf{M}, V)$ is called a linear differential operator of order l on M, if we can find that $\mathfrak{a} = (a^r)_r \in \prod_{r=0}^l \Gamma(\mathsf{M}, V^{\sigma+\tau+r}_{\tau+\sigma})$ such that

$$\mathcal{A} = \mathcal{A}(\mathfrak{a}) := \sum_{r=0}^{l} \mathsf{C}(a^r, \nabla^r \cdot). \tag{2.7}$$

Here, the complete contraction

$$\mathsf{C} : \Gamma\left(\mathsf{M}, V^{\sigma+\tau+r}_{\tau+\sigma} \times V^\sigma_{\tau+r}\right) \to \Gamma\left(\mathsf{M}, V^\sigma_\tau\right) : (a, b) \mapsto \mathsf{C}(a, b)$$

is defined as follows. Let $(i_1), (i_2), (i_3) \in \mathbb{J}^\sigma$, $(j_1), (j_2), (j_3) \in \mathbb{J}^\tau$ and $(r_1), (r_2) \in \mathbb{J}^r$.

$$\begin{aligned}\mathsf{C}(a,b)(p) := \mathsf{C}(&a^{(i_3;j_1;r_1)}_{(j_3;i_1)} \frac{\partial}{\partial x^{(i_3)}} \otimes \frac{\partial}{\partial x^{(j_1)}} \otimes \frac{\partial}{\partial x^{(r_1)}} \otimes dx^{(j_3)} \otimes dx^{(i_1)},\\ &b^{(i_2)}_{(j_2;r_2)} \frac{\partial}{\partial x^{(i_2)}} \otimes dx^{(j_2)} \otimes dx^{(r_2)})(p)\\ = \; &a^{(i_3;j_1;r_1)}_{(j_3;i_1)} b^{(i_1)}_{(j_1;r_1)} \frac{\partial}{\partial x^{(i_3)}} \otimes dx^{(j_3)}(p),\end{aligned}$$

in every local chart and for $p \in \mathsf{M}$. The index $(i_2; j_1; r_1)$ is defined by

$$(i_3; j_1; r_1) = (i_{3,1}, \ldots, i_{3,\sigma}; j_{1,1}, \ldots, j_{1,\tau}; r_{1,1}, \ldots, r_{1,r}).$$

The other indices are defined in a similar way. Amann [4, Lemma 14.2] implies that C is a bundle multiplication. Making use of [3, formula (3.18)], one can check that for any lth order linear differential operator so defined, in every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$, there exists some linear differential operator

$$\mathcal{A}_\kappa(x, \partial) := \sum_{|\alpha| \le l} a^\kappa_\alpha(x)\partial^\alpha, \quad \text{with } a^\kappa_\alpha \in \mathcal{L}(E)^{\mathbb{B}^m_\kappa}, \tag{2.8}$$

called the local representation of $\mathcal{A}$ in $(\mathsf{O}_\kappa, \varphi_\kappa)$, such that for any $u \in C^\infty(\mathsf{M}, V)$

$$\psi^*_\kappa(\mathcal{A}u) = \mathcal{A}_\kappa(\psi^*_\kappa u). \tag{2.9}$$

What is more, the $\mathcal{A}_\kappa$'s satisfy

$$\psi^*_\eta \varphi^*_\kappa \mathcal{A}_\kappa \psi^*_\kappa \varphi^*_\eta = \mathcal{A}_\eta, \quad \eta \in \mathfrak{N}(\kappa). \tag{2.10}$$

Conversely, suppose that $\mathcal{A}$ is a linear map acting on $C^\infty(\mathsf{M}, V)$, and its localizations satisfy (2.8)–(2.10). Then, $\mathcal{A}$ is a well-defined linear operator from $C^\infty(\mathsf{M}, V)$ to $\Gamma(\mathsf{M}, V)$. Moreover, one can show that $\mathcal{A}$ is actually a linear differential operator of order l with expression (2.7). Indeed, we can construct $\mathfrak{a} = (a^r)_r$ in a recursive way as follows. For notational brevity, we express $\mathcal{A}_\kappa$ as

$$\mathcal{A}_\kappa = a^{(r)}_\kappa \partial_{(r)}, \quad (r) \in \mathbb{J}^s \text{ with } s \le l, \text{ and } a^{(r)}_\kappa \in \mathcal{L}(E)^{\mathbb{B}^m_\kappa}.$$

Then, we write the coefficients $a^{(r)}_\kappa$ of $\mathcal{A}^\pi_\kappa$, the principal part of $\mathcal{A}_\kappa$, in the matrix form $(a^{(i_2;j_1;r)}_{\kappa,(j_2;i_1)})$, where $a^{(i_2;j_1;r)}_{\kappa,(j_2;i_1)}$ is the entry of $a^{(r)}_\kappa$ in the $(i_1; j_1)$th column and $(i_2; j_2)$th row with indices ordered lexicographically. Here, $(i_1), (i_2) \in \mathbb{J}^\sigma$, $(j_1), (j_2) \in \mathbb{J}^\tau$ and $(r) \in \mathbb{J}^l$. Then, we define

$$a^l|_{\mathsf{O}_\kappa} := a^{(i_2;j_1;r)}_{\kappa,(j_2;i_1)} \tfrac{\partial}{\partial x^{(i_2)}} \otimes \tfrac{\partial}{\partial x^{(j_1)}} \otimes \tfrac{\partial}{\partial x^{(r)}} \otimes dx^{(j_2)} \otimes dx^{(i_1)} \tag{2.11}$$

on the local patch O_κ. It follows from [3, formula (3.18)] that

$$\psi^*_\kappa \mathsf{C}(a^l, \nabla^l u) = \mathcal{A}^\pi_\kappa \psi^*_\kappa u + \mathcal{B}_\kappa \psi^*_\kappa u,$$

where $\mathcal{A}_\kappa^\pi$ is the principal part of $\mathcal{A}_\kappa$, and $\mathcal{B}_\kappa$ is a linear differential operator of order at most $l-1$ defined on $\mathbb{B}_\kappa^m$. By the above argument, $\mathcal{A}$ is well defined only if for all local coordinates $\varphi_\kappa = \{x_1, \dots, x_m\}$ and $\varphi_\eta = \{y_1, \dots, y_m\}$ with $\eta \in \mathfrak{N}(\kappa)$

$$\psi_\eta^* \varphi_\kappa^* \left(\sum_{(r)\in\mathbb{J}^l} a_\kappa^{(r)} \otimes \psi_\kappa^* \frac{\partial}{\partial x^{(r)}} \right) = \sum_{(r)\in\mathbb{J}^l} a_\eta^{(r)} \otimes \psi_\eta^* \frac{\partial}{\partial y^{(r)}}$$

only if a^l is invariant. Hence, a^l is globally well defined. Therefore,

$$\tilde{\mathcal{A}}u := \mathcal{A}u - \mathsf{C}(a^l, \nabla^l u)$$

is a well-defined linear differential operator of order at most $l-1$. Then, we repeat this process to lower the order of $\mathcal{A}$ till it reduces to zero. We find

$$\mathfrak{a} = (a^r)_r \in \prod_{r=0}^{l} \Gamma\left(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r}\right) \tag{2.12}$$

such that $\mathcal{A}$ can be formulated in the form of (2.7).

In the rest of this section, we assume that $\mathcal{A}$ is a linear differential operator of order l over M with local expressions $\mathcal{A}_\kappa(x,\partial) := \sum_{|\alpha|\le l} a_\alpha^\kappa(x)\partial^\alpha$. We first state a useful proposition concerning the equivalence of regularity of local versus global coefficients.

PROPOSITION 2.8. *If $(a_\alpha^\kappa)_\kappa \in l_b(\mathfrak{F}^s(\mathbb{B}_\kappa^m, \mathcal{L}(E))$ for every $|\alpha| \le l$ with $b =$ "∞" for $\mathfrak{F} = BC$, or $b =$ "∞, unif" for $\mathfrak{F} = bc$, then $\mathfrak{a}$ in (2.12) belongs to the class $\prod_{r=0}^l \mathfrak{F}^s(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r})$, and vice versa.*

Proof. $(a_\alpha^\kappa)_\kappa \in l_b(\mathfrak{F}^s(\mathbb{B}_\kappa^m, \mathcal{L}(E)))$ implies that $(a_{\kappa,(j_2;i_1)}^{(i_2;j_1;r)})_\kappa \in l_b(\mathfrak{F}^s(\mathbb{B}_\kappa^m))$. By (2.11), we attain $\mathcal{R}_\kappa^c a^l = \psi_\kappa^* \pi_\kappa a_\kappa^{(r)} \otimes \psi_\kappa^* \frac{\partial}{\partial x^{(r)}}$ with $(r) \in \mathbb{J}^l$. The pointwise multiplication results on $\mathbb{B}_\kappa^m$ yield

$$\mathcal{R}^c a^l \in l_b\left(\mathfrak{F}^s\left(\mathbb{B}_\kappa^m, E_{\tau+\sigma}^{\sigma+\tau+r}\right)\right).$$

By Theorem 2.1, we have that $a^l \in \mathfrak{F}^s(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r})$. The rest of the proof, including the converse statement, follows from the recursive construction of a^r above and [3, formula (3.18)]. □

PROPOSITION 2.9. *Suppose that $\mathfrak{a} = (a^r)_r \in \prod_{r=0}^l bc^s(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r})$. Then,*

$$\mathcal{A} \in \mathcal{L}\left(bc^{s+l}(\mathsf{M}, V), bc^s(\mathsf{M}, V)\right).$$

Proof. By [4, Theorem 16.1], we have

$$\nabla \in \mathcal{L}(bc^{s+1}(\mathsf{M}, V), bc^s(\mathsf{M}, V_{\tau+1}^\sigma)), \quad s \notin \mathbb{N}_0. \tag{2.13}$$

The case that $s \in \mathbb{N}_0$ follows by the definition of Hölder spaces and a density argument. Since C is a bundle multiplication, the statement is a straightforward conclusion of Proposition 2.7. □

The following corollary is a special case of Proposition 2.9.

COROLLARY 2.10. *Suppose that* $\mathfrak{a} = (a^r)_r \in \prod_{r=0}^{l} BC^t(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r})$. *Then,*

$$\mathcal{A} \in \mathcal{L}\left(\mathfrak{F}^{s+l}(\mathsf{M}, V), \mathfrak{F}^{s}(\mathsf{M}, V)\right).$$

Here, we choose $t > s$ *for* $\mathfrak{F} = bc$, *or* $t \geq s$ *for* $\mathfrak{F} = BC$.

REMARK 2.11. Let $\mathfrak{a} = (a^r)_r \in \prod_{r=0}^{l} \mathfrak{F}^{s}(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r})$ with $\mathfrak{F} \in \{bc, BC\}$. Then, due to (2.13) and Proposition 2.7, we deduce that

$$[\mathfrak{a} \mapsto \mathcal{A}(\mathfrak{a})] \in \mathcal{L}\left(\prod_{r=0}^{l} \mathfrak{F}^{s}\left(\mathsf{M}, V_{\tau+\sigma}^{\sigma+\tau+r}\right), \mathcal{L}\left(\mathfrak{F}^{s+l}(\mathsf{M}, V), \mathfrak{F}^{s}(\mathsf{M}, V)\right)\right).$$

3. Analytic semigroups and continuous maximal regularity on uniformly regular Riemannian manifolds without boundary

Let X be a Banach space. Given some $\mathcal{E} \geq 1$ and $\vartheta \in [0, \pi)$, a linear differential operator of order l,

$$\mathcal{A} := \mathcal{A}(x, \partial) := \sum_{|\alpha| \leq l} a_\alpha(x)\partial^\alpha,$$

defined on an open subset $U \subset \mathbb{H}^m$ with $a_\alpha : U \to \mathcal{L}(X)$ is said to be $(\mathcal{E}, \vartheta; l)$-*elliptic* if its principal symbol

$$\hat{\sigma}\mathcal{A}^\pi(x, \xi) : U \times \mathbb{R}^m \to \mathcal{L}(X), \quad (x, \xi) \mapsto \sum_{|\alpha|=l} a_\alpha(x)(-i\xi)^\alpha$$

satisfies

$$\begin{cases} \Sigma_\vartheta := \{z \in \mathbb{C} : |\arg z| \leq \vartheta\} \cup \{0\} \subset \rho(-\hat{\sigma}\mathcal{A}^\pi(x, \xi)) \\ (1 + |\lambda|)\|R(\lambda, -\hat{\sigma}\mathcal{A}^\pi(x, \xi))\|_{\mathcal{L}(X)} \leq \mathcal{E}, \quad \lambda \in \Sigma_\vartheta, \end{cases} \tag{3.1}$$

and all $(x, \xi) \in U \times \mathbb{R}^m$ with $|\xi| = 1$. The constant $\mathcal{E}$ is called the *ellipticity constant* of $\mathcal{A}$. $\hat{\sigma}\mathcal{A}^\pi(x, \xi)$ is considered as an element of $\mathcal{L}(X)$. Here, i is the complex identity, if necessary, we consider the complexification of X. In particular, $\mathcal{A}$ is called *normally elliptic of order* l if it is $(\mathcal{E}, \frac{\pi}{2}; l)$-elliptic for some constant $\mathcal{E} \geq 1$. We readily check that a normally elliptic operator must be of even order. This concept was introduced by H. Amann in [2].

If X is of finite dimension, then $\mathcal{A}$ is $(\mathcal{E}, \vartheta; l)$-elliptic on U iff there exist some $0 < r(\mathcal{E}, \vartheta) < R(\mathcal{E}, \vartheta)$ such that the spectrum of $\hat{\sigma}\mathcal{A}(x, \xi)$ is contained in

$$\{z \in \mathbb{C} : r < |z| < R\} \cap \mathring{\Sigma}_{\pi-\vartheta}$$

for all $(x,\xi) \in U \times \mathbb{R}^m$ with $|\xi| = 1$. In particular, in the case that $X = \mathbb{K}$, $\mathcal{A}$ is normally elliptic of order l on U if there exist $0 < r(\mathcal{E},\vartheta) < R(\mathcal{E},\vartheta)$ such that

$$R \geq Re(\hat{\sigma}\mathcal{A}(x,\xi)) \geq r, \quad (x,\xi) \in U \times \mathbb{R}^m, \text{ with } |\xi| = 1.$$

It is worthwhile mentioning that the concept of normal ellipticity is usually referred to as uniformly strong ellipticity in the scalar-valued case.

Let $\mathsf{C} : \Gamma(\mathsf{M}, V^{\sigma+\tau+l}_{\tau+\sigma} \times V^0_l) \to \mathrm{Hom}(V)$ be the complete contraction. A linear operator $\mathcal{A} := \mathcal{A}(\mathfrak{a}) : C^\infty(\mathsf{M}, V) \to \Gamma(\mathsf{M}, V)$ of order l is said to be $(\mathcal{E},\vartheta;l)$-*elliptic* with $\mathcal{E} \geq 1$ if there exists a $\mathcal{E} \geq 1$ such that its principal symbol

$$\hat{\sigma}\mathcal{A}^\pi(p,\xi(p)) := \mathsf{C}(a^l, (-i\xi)^{\otimes l})(p) \in \mathcal{L}\left(T_p\mathsf{M}^{\otimes\sigma} \otimes T^*_p\mathsf{M}^{\otimes\tau}\right)$$

satisfies that $S := \Sigma_\vartheta \subset \rho(-\hat{\sigma}\mathcal{A}^\pi(p,\xi(p)))$ and

$$(1+|\lambda|)\|R(\lambda, -\hat{\sigma}\mathcal{A}^\pi(p,\xi(p)))\|_{\mathcal{L}(T_p\mathsf{M}^{\otimes\sigma}\otimes T^*_p\mathsf{M}^{\otimes\tau})} \leq \mathcal{E}, \quad \lambda \in S,$$

for all $(p,\xi) \in \mathsf{M}\times\Gamma(\mathsf{M}, T^*M)$ with $|\xi|_{g^*} = 1_{\mathsf{M}}$. This definition is a natural extension of its Euclidean version. In fact, the following proposition holds true.

PROPOSITION 3.1. *A linear differential operator $\mathcal{A}$ of order l is $(\mathcal{E},\vartheta;l)$-elliptic iff all its local realizations*

$$\mathcal{A}_\kappa(x,\partial) = \sum_{|\alpha|\leq l} a^\kappa_\alpha(x)\partial^\alpha$$

are $(\mathcal{E}',\vartheta;l)$-elliptic on $\mathbb{B}^m_\kappa$ with uniform constants $\mathcal{E}'$, ϑ in condition (3.1).

Proof. We first show the "only if" part. In every local patch $(\mathsf{O}_\kappa, \varphi_\kappa)$, by definition we have

$$\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi) = \sum_{|\alpha|=l} a^\kappa_\alpha(x)(-i\xi)^\alpha = \sum_{(r)\in\mathbb{J}^l} a^{(r)}_\kappa(x)(-i\xi)^{(r)} = \psi^*_\kappa \mathsf{C}\left(a^l, \left(-i\xi^{\mathsf{M}}\right)^{\otimes l}\right)(p)$$

with $x \in \mathbb{B}^m_\kappa$, $\xi \in \mathbb{R}^m$ and $p = \psi_\kappa(x)$. Here, a^l is defined in the same way as in (2.11), and $\xi^{\mathsf{M}}|_{\mathsf{O}_\kappa} = \xi_j dx^j$ can be considered as the restriction of a 1-form onto O_κ. In light of the equality $\mathsf{C}(\varphi^*_\kappa T, \varphi^*_\kappa v) = \varphi^*_\kappa T v$ for $T \in \mathcal{L}(E)^{\mathbb{B}^m_\kappa}$ and $v \in E^{\mathbb{B}^m_\kappa}$, one readily verifies that, letting $S := \Sigma_\vartheta$,

$$S \subset \rho(-\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi)) \text{ iff } S \subset \rho\left(-\mathsf{C}\left(a^l, (-i\xi^{\mathsf{M}})^{\otimes l}\right)(p)\right). \tag{3.2}$$

The resolvent estimates of $-\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi)$ can be obtained as follows. Taking $\xi \in \mathbb{S}^{m-1}$ and letting $c = |\xi^{\mathsf{M}}(p)|^l_{g(p)}$, for every $\lambda \in S$, $\eta, \gamma \in E$ with $\gamma = (\lambda + \hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi))\eta$

$$\begin{aligned}(1+|\lambda|)\|R(\lambda, -\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi))\gamma\|_{g_m} &= (1+|\lambda|)\|\eta\|_{g_m} \leq M(1+|\lambda|)\|\mathrm{d}\psi_\kappa(x)\eta\|_{g(p)} \\ &\leq M\mathcal{E}\frac{1+|\lambda|}{1+|\lambda|/c}\left\|\left[\frac{\lambda}{c} + \mathsf{C}\left(a^l, \frac{(-i\xi^{\mathsf{M}})^{\otimes l}}{c}\right)(p)\right]\mathrm{d}\psi_\kappa(x)\eta\right\|_{g(p)}\end{aligned}$$

$$\leq M\mathcal{E}\frac{1+|\lambda|}{c+|\lambda|}\left\|\left[\lambda+\mathsf{C}\left(a^l,(-i\xi^{\mathsf{M}})^{\otimes l}\right)(p)\right]\mathrm{d}\psi_\kappa(x)\eta\right\|_{g(p)}$$
$$\leq \mathcal{E}'\|\left(\lambda+\hat{\sigma}\mathcal{A}^\pi_\kappa(x,\xi)\right)\eta\|_{g_m}$$

by virtue of (R3) for some $\mathcal{E}'$. The "if" part follows by a similar argument. □

$\mathcal{A}$ is called *normally elliptic of order* l, if $\mathcal{A}$ is $(\mathcal{E},\frac{\pi}{2};l)$-elliptic for some constant $\mathcal{E}\geq 1$. Analogously, the constant $\mathcal{E}$ is called the *ellipticity constant* of $\mathcal{A}$ defined on M.

For the proof of the main theorem, we first quote a result of Amann.

PROPOSITION 3.2. [2, Theorem 4.2, Remark 4.6] *Let $s>0$, $s\notin\mathbb{N}$ and X be a Banach space. Suppose that $\mathcal{A}=\sum_{|\alpha|\leq l}a_\alpha\partial^\alpha$ is a $(\mathcal{E},\vartheta;l)$-elliptic operator on $\mathbb{R}^m$ with*

$$a_\alpha\in bc^s(\mathbb{R}^m,\mathcal{L}(X)),\ \text{and}\ \|a_\alpha\|_{s,\infty}\leq\mathcal{K}\ \text{for all}\ |\alpha|\leq l.$$

Then, there exist constants $\omega(\mathcal{E},\mathcal{K}),\mathcal{N}(\mathcal{E},\mathcal{K})$ such that $S:=\omega+\Sigma_\vartheta\subset\rho(-\mathcal{A})$ and

$$|\lambda|^{1-j}\|R(\lambda,-\mathcal{A})\|_{\mathcal{L}(\mathfrak{F}^s(\mathbb{R}^m,X),\mathfrak{F}^{s+lj}(\mathbb{R}^m,X))}\leq\mathcal{N},\quad\text{for}\ \lambda\in S,\ \ j=0,1,$$

with $\mathfrak{F}\in\{bc,BC\}$. Here, $\mathcal{E}$ is the ellipticity constant of $\mathcal{A}$.

THEOREM 3.3. *Let $s>0$ and $s\notin\mathbb{N}$. Suppose that M is a uniformly regular Riemannian manifold without boundary, and $\mathcal{A}$ is a $(\mathcal{E},\vartheta;2l)$-elliptic operator with $\mathfrak{a}=(a^r)_r\in\prod_{r=0}^{2l}bc^s(\mathsf{M},V^{\sigma+\tau+r}_{\tau+\sigma})$. Then, there exist constant $\omega(\mathcal{E},\mathcal{K}),\mathcal{N}(\mathcal{E},\mathcal{K})$ such that $S:=\omega+\Sigma_\vartheta\subset\rho(-\mathcal{A})$ and*

$$|\lambda|^{1-j}\|R(\lambda,-\mathcal{A})\|_{\mathcal{L}(\mathfrak{F}^s(\mathsf{M},V),\mathfrak{F}^{s+2lj}(\mathsf{M},V))}\leq\mathcal{N},\quad\text{for}\ \lambda\in S,\ \ j=0,1$$

with $\mathfrak{F}\in\{bc,BC\}$. Here, $\mathcal{E}$ is the ellipticity constant of $\mathcal{A}$ and $\mathcal{K}:=\max_r\|a^r\|^{\mathsf{M}}_{s,\infty}$.

Proof. To economize notations, we set

$$E_0=\mathfrak{F}^s,\ \ E_\theta=\mathfrak{F}^{s+2l-1},\ \ \text{and}\ E_1=\mathfrak{F}^{s+2l}.$$

In virtue of Proposition 2.8 and the discussions at the beginning of this section, we know that there exist constants $\mathcal{E}'$ and $\mathcal{K}'$ such that all localizations $\mathcal{A}_\kappa$'s are $(\mathcal{E}',\vartheta;2l)$-elliptic and their coefficients satisfy

$$(a^\kappa_\alpha)_\kappa\in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s(\mathbb{B}^m_\kappa,\mathcal{L}(E)))\quad\text{with}\quad\max_{|\alpha|\leq 2l}\sup_\kappa\|a^\kappa_\alpha\|_{s,\infty}\leq\mathcal{K}'.$$

Without loss of generality, we may assume that $\mathcal{E}=\mathcal{E}'$ and $\mathcal{K}=\mathcal{K}'$.

(i) For simplicity, we first assume that $s\in(0,1)$. It can be easily seen through step (ii) of [2, Theorem 4.1] that this assumption will not harm our proof. Define $h:\mathbb{R}^m\to\mathbb{B}^m\colon x\mapsto\zeta(x)x$. It is easy to see that $h\in BC^\infty(\mathbb{R}^m,\mathbb{B}^m)$. Let

$$\bar{\mathcal{A}}_\kappa(x,\partial):=\sum_{|\alpha|\leq 2l}\bar{a}^\kappa_\alpha(x)\partial^\alpha:=\sum_{|\alpha|\leq 2l}(a^\kappa_\alpha\circ h)(x)\partial^\alpha.$$

It is not hard to check that $(\bar{a}^\kappa_\alpha)_\kappa\in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s(\mathbb{R}^m,\mathcal{L}(E)))$ and $\bar{\mathcal{A}}_\kappa$ is $(\mathcal{E},\vartheta;2l)$-elliptic on $\mathbb{R}^m$.

(ii) For any $\lambda \in \mathbb{C}$ and $u \in E_1$, consider

$$\begin{aligned}\mathcal{R}^c_\kappa(\lambda+\mathcal{A})u - (\lambda+\bar{\mathcal{A}}_\kappa)\mathcal{R}^c_\kappa u &= \psi^*_\kappa(\pi_\kappa(\lambda+\mathcal{A})u) - (\lambda+\bar{\mathcal{A}}_\kappa)\psi^*_\kappa(\pi_\kappa u)\\ &= \psi^*_\kappa\pi_\kappa(\lambda+\bar{\mathcal{A}}_\kappa)\psi^*_\kappa u - (\lambda+\bar{\mathcal{A}}_\kappa)\psi^*_\kappa(\pi_\kappa u)\\ &= \psi^*_\kappa\pi_\kappa\bar{\mathcal{A}}_\kappa\psi^*_\kappa u - \bar{\mathcal{A}}_\kappa\psi^*_\kappa(\pi_\kappa u)\\ &= -\sum_{|\alpha|\le 2l}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\bar{a}^\kappa_\alpha\partial^{\alpha-\beta}(\zeta\psi^*_\kappa u)\partial^\beta(\psi^*_\kappa\pi_\kappa).\end{aligned}$$

In the rest of the proof, we always conventionally put

$$b = \text{“}\infty\text{”} \quad \text{for } \mathfrak{F} = BC, \quad \text{or } b = \text{“}\infty, \text{unif”} \quad \text{for } \mathfrak{F} = bc.$$

Using the proof of [4, Theorem 12.1] and interpolation theory, one readily checks that for $t \ge 0$

$$[u \mapsto (\zeta\psi^*_\kappa u)_\kappa] \in \mathcal{L}(\mathfrak{F}^t(\mathsf{M}, V), l_b(\mathfrak{F}^t)). \tag{3.3}$$

By Proposition 2.7, the following map is bilinear and continuous for $\mathfrak{F} \in \{bc, BC\}$

$$\begin{aligned}&(\mathfrak{F}^t(\mathbb{R}^m, \mathcal{L}(E)), \mathfrak{F}^t(\mathbb{R}^m, E)) \to \mathfrak{F}^t(\mathbb{R}^m, E)\\ &\quad [(F, v) \mapsto [x \mapsto F(x)(v(x))]]\end{aligned} \tag{3.4}$$

for $t \ge 0$. In view of (3.3) and (3.4), we conclude that

$$\mathcal{R}^c_\kappa((\lambda+\mathcal{A})\cdot) - (\lambda+\bar{\mathcal{A}}_\kappa)\mathcal{R}^c_\kappa(\cdot) =: B_\kappa(\cdot) \in \mathcal{L}(E_\theta(\mathsf{M}, V), E_0(\mathbb{R}^m, E)).$$

Analogously, we can verify that

$$(\lambda+\mathcal{A})\mathcal{R}_\kappa(\cdot) - \mathcal{R}_\kappa((\lambda+\bar{\mathcal{A}}_\kappa)\cdot) =: C_\kappa(\cdot) \in \mathcal{L}(E_\theta(\mathbb{R}^m, E), E_0(\mathsf{M}, V)).$$

(iii) Set $\bar{\mathcal{A}} : l_b(\boldsymbol{E}_1) \to \prod_\kappa \mathfrak{F}^s_\kappa : (u_\kappa)_\kappa \mapsto (\bar{\mathcal{A}}_\kappa u_\kappa)_\kappa$. By Proposition 2.7 and (2.6), one attains

$$\bar{\mathcal{A}} \in \mathcal{L}(l_b(\boldsymbol{E}_1), l_b(\boldsymbol{E}_0)).$$

Put $\boldsymbol{u} = (u_\kappa)_\kappa$ and $\boldsymbol{a}_\alpha = (\bar{a}^\kappa_\alpha)_\kappa$. Define $\bar{\boldsymbol{\mathcal{A}}} : E_1(\mathbb{R}^m, l_\infty(\boldsymbol{E})) \to E_0(\mathbb{R}^m, l_\infty(\boldsymbol{E}))$ by

$$\bar{\boldsymbol{\mathcal{A}}} := \sum_{|\alpha|\le 2l} \boldsymbol{f}^{-1}(\boldsymbol{a}_\alpha)\partial^\alpha.$$

Then by [4, formula (11.29)], we obtain

$$\boldsymbol{f}^{-1}(\bar{\mathcal{A}}\boldsymbol{u}) = \bar{\boldsymbol{\mathcal{A}}}\boldsymbol{f}^{-1}(\boldsymbol{u}) = \sum_{|\alpha|\le 2l} \boldsymbol{f}^{-1}(\boldsymbol{a}_\alpha)\partial^\alpha \boldsymbol{f}^{-1}(\boldsymbol{u}). \tag{3.5}$$

Theorem 2.1 implies that $\boldsymbol{f}^{-1}(\boldsymbol{a}_\alpha) \in bc^s(\mathbb{R}^m, l_\infty(\mathcal{L}(E))) \hookrightarrow bc^s(\mathbb{R}^m, \mathcal{L}(l_\infty(\boldsymbol{E})))$. By the $(\mathcal{E}, \vartheta; 2l)$-ellipticity of $\bar{\mathcal{A}}_\kappa$ and the uniform ellipticity constant $\mathcal{E}$

in (3.1), we conclude that $\bar{\boldsymbol{\mathcal{A}}}$ is $(\mathcal{E}', \vartheta; 2l)$-elliptic, i.e., for all $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$ with $|\xi| = 1$

$$(1 + |\lambda|)\|R(\lambda, -\hat{\sigma}\bar{\boldsymbol{\mathcal{A}}}^{\pi}(x, \xi))\|_{\mathcal{L}(l_\infty(\boldsymbol{E}))} \leq \mathcal{E}', \quad \lambda \in \Sigma_\vartheta,$$

for some $\mathcal{E}' \geq 1$. Its proof follows in a similar manner to that of Proposition 3.1 Without loss of generality, we assume again that $\mathcal{E}' = \mathcal{E}$. Set $\hat{E}_j := E_j(\mathbb{R}^m, l_\infty(\boldsymbol{E}))$ for $j = 0, 1$. Proposition 3.2 now yields the existence of constants ω_0 and M_0 such that $S_0 := \omega_0 + \Sigma_\vartheta \subset \rho(-\bar{\boldsymbol{\mathcal{A}}})$ and

$$|\lambda|^{1-j}\|R(\lambda, -\bar{\boldsymbol{\mathcal{A}}})\|_{\mathcal{L}(\hat{E}_0, \hat{E}_j)} \leq M_0, \quad \lambda \in S_0, \ \ j = 0, 1.$$

Pick $\boldsymbol{u} \in l_b(\boldsymbol{E}_1)$. By (2.3) and (3.5) we have for every $\lambda \in S_0$

$$\boldsymbol{f}(\lambda + \bar{\boldsymbol{\mathcal{A}}})^{-1}\boldsymbol{f}^{-1}(\lambda + \bar{\mathcal{A}})\boldsymbol{u} = \boldsymbol{f}(\lambda + \bar{\boldsymbol{\mathcal{A}}})^{-1}(\lambda + \bar{\boldsymbol{\mathcal{A}}})\boldsymbol{f}^{-1}\boldsymbol{u} = \boldsymbol{u}.$$

Similarly, by [4, formula (11.29)]

$$(\lambda + \bar{\mathcal{A}})\boldsymbol{f}(\lambda + \bar{\boldsymbol{\mathcal{A}}})^{-1}\boldsymbol{f}^{-1}\boldsymbol{u} = \boldsymbol{f}(\lambda + \bar{\boldsymbol{\mathcal{A}}})(\lambda + \bar{\boldsymbol{\mathcal{A}}})^{-1}\boldsymbol{f}^{-1}\boldsymbol{u} = \boldsymbol{u}.$$

Thus $S_0 \subset \rho(-\bar{\mathcal{A}})$. Furthermore, $(\lambda + \bar{\mathcal{A}})^{-1} = \boldsymbol{f}(\lambda + \bar{\boldsymbol{\mathcal{A}}})^{-1}\boldsymbol{f}^{-1}$ and

$$|\lambda|^{1-j}\|R(\lambda, -\bar{\mathcal{A}})\|_{\mathcal{L}(l_b(\boldsymbol{E}_0), l_b(\boldsymbol{E}_j))} \leq K_0, \quad \lambda \in S_0, \ \ j = 0, 1 \tag{3.6}$$

for some $K_0 \geq 1$.

(iv) Define $B : E_\theta(\mathsf{M}, V) \to \prod_\kappa \mathfrak{B}^s_\kappa : u \mapsto (B_\kappa u)_\kappa$. By (3.3), we have

$$B \in \mathcal{L}(E_\theta(\mathsf{M}, V), l_b(\boldsymbol{E}_0)).$$

By Theorem 2.1, we have that $B\mathcal{R} \in \mathcal{L}(l_b(\boldsymbol{E}_\theta), l_b(\boldsymbol{E}_0))$. It follows from (2.3), (2.4) and [1, Proposition I.2.3.2] that

$$l_b(\boldsymbol{E}_\theta) \doteq (l_b(\boldsymbol{E}_0), l_b(\boldsymbol{E}_1))_\theta,$$

where either $(\cdot, \cdot)_\theta = (\cdot, \cdot)^0_{\theta,\infty}$ for $\mathfrak{F} = bc$, or $(\cdot, \cdot)_\theta = (\cdot, \cdot)_{\theta,\infty}$ for $\mathfrak{F} = BC$, and $\theta = 1 - 1/(2l)$. Thus $B\mathcal{R}$ is a lower order perturbation. For each $\boldsymbol{u} \in l_b(\boldsymbol{E}_0)$ and $\lambda \in S_0$, one computes

$$\begin{aligned}
&\|B\mathcal{R}(\lambda + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l_b(\boldsymbol{E}_0)} \\
&\quad \leq M\varepsilon\|(\lambda + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l_b(\boldsymbol{E}_1)} + MC(\varepsilon)\|(\lambda + \bar{\mathcal{A}})^{-1}\boldsymbol{u}\|_{l_b(\boldsymbol{E}_0)} \\
&\quad \leq MK_0\varepsilon\|\boldsymbol{u}\|_{l_b(\boldsymbol{E}_0)} + MC(\varepsilon)\frac{K_0}{|\lambda|}\|\boldsymbol{u}\|_{l_b(\boldsymbol{E}_0)}.
\end{aligned}$$

The penultimate line is a consequence of interpolation theory [1, formula (I.2.2.2)]. The last inequality follows from (3.6). By choosing ε small enough and a sufficiently large $\omega_1 \geq \omega_0$, we can conclude that $\|B\mathcal{R}(\lambda + \bar{\mathcal{A}})^{-1}\| \leq \frac{1}{2}$. By a Neumann series argument, we infer that $S_1 := \omega_1 + \Sigma_\vartheta \subset \rho(-\bar{\mathcal{A}} - B\mathcal{R})$ and

$$|\lambda|^{1-j}\|R(\lambda, -\bar{\mathcal{A}} - B\mathcal{R})\|_{\mathcal{L}(l_b(\boldsymbol{E}_0), l_b(\boldsymbol{E}_j))} \leq M_1, \quad \lambda \in S_1, \ \ j = 0, 1,$$

for some M_1. For any $\lambda \in S_1$, $(\lambda + \bar{\mathcal{A}} + B\mathcal{R})^{-1}$ exists. Moreover,

$$\mathcal{R}(\lambda + \bar{\mathcal{A}} + B\mathcal{R})^{-1}\mathcal{R}^c(\lambda + \mathcal{A}) = \mathcal{R}(\lambda + \bar{\mathcal{A}} + B\mathcal{R})^{-1} \times(\lambda + \bar{\mathcal{A}} + B\mathcal{R})\mathcal{R}^c = \mathrm{id}_{E_1(\mathsf{M},V)}.$$

Therefore, $\lambda + \mathcal{A}$ is injective.

(v) Define $C : l_b(\boldsymbol{E}_\theta) \to E_0(\mathsf{M}, V)$: $(u_\kappa)_\kappa \mapsto \sum_\kappa C_\kappa(u_\kappa)$. Then analogously we can verify that $C \in \mathcal{L}(l_b(\boldsymbol{E}_\theta), E_0(\mathsf{M}, V))$ and in turn $\mathcal{R}^c C \in \mathcal{L}(l_b(\boldsymbol{E}_\theta), l_b(\boldsymbol{E}_0))$.

For any $\boldsymbol{u} \in l_b(\boldsymbol{E}_1)$, we have

$$(\lambda + \mathcal{A})\mathcal{R}\boldsymbol{u} = \mathcal{R}(\lambda + \bar{\mathcal{A}})\boldsymbol{u} + \mathcal{R}\mathcal{R}^c C\boldsymbol{u} = \mathcal{R}(\lambda + \bar{\mathcal{A}} + \mathcal{R}^c C)\boldsymbol{u}.$$

An analogous argument to (iv) reveals that there exist $\omega \geq \omega_1$ and $\mathcal{N} > M_1$ such that $S := \omega + \Sigma_\vartheta \subset \rho(-\bar{\mathcal{A}} - \mathcal{R}^c C)$ and

$$|\lambda|^{1-j}\|R(\lambda, -\bar{\mathcal{A}} - \mathcal{R}^c C)\|_{\mathcal{L}(l_b(\boldsymbol{E}_0), l_b(\boldsymbol{E}_j))} \leq \mathcal{N}, \quad \lambda \in S, \quad j = 0, 1.$$

Moreover, these constants depend only on $\mathcal{E}$ and $\mathcal{K}$. For any $\lambda \in S$, $(\lambda + \bar{\mathcal{A}} + \mathcal{R}^c C)^{-1}$ exists and $(\lambda + \mathcal{A})\mathcal{R}(\lambda + \bar{\mathcal{A}} + \mathcal{R}^c C)^{-1}\mathcal{R}^c = \mathrm{id}_{E_0(\mathsf{M},V)}$. Thus, $(\lambda + \mathcal{A})$ is surjective. Now, we conclude that $S \subset \rho(-\mathcal{A})$, and for every $\lambda \in S$, $u \in E_0(\mathsf{M}, V)$

$$\begin{aligned}|\lambda|^{1-j}\|R(\lambda, -\mathcal{A})u\|^{\mathsf{M}}_{E_j} &= |\lambda|^{1-j}\|\mathcal{R}R(\lambda, -\bar{\mathcal{A}} - \mathcal{R}^c C)\mathcal{R}^c u\|^{\mathsf{M}}_{E_j} \\ &\leq C_0|\lambda|^{1-j}\|R(\lambda, -\bar{\mathcal{A}} - \mathcal{R}^c C)\mathcal{R}^c u\|_{l_b(\boldsymbol{E}_j)} \leq C_0^2\mathcal{N}\|u\|^{\mathsf{M}}_{E_0}.\end{aligned}$$

□

REMARK 3.4. With the necessary modification, the above proof also works for Banach-valued functions defined on a uniformly regular manifold without boundary, provided the counterparts of these spaces, and the elliptic conditions are properly defined.

Recall an operator A is said to belong to the class $\mathcal{H}(E_1, E_0)$ for some densely embedded Banach couple $E_1 \overset{d}{\hookrightarrow} E_0$, if $-A$ generates a strongly continuous analytic semigroup on E_0 with $\mathrm{dom}(-A) = E_1$. By the well-known semigroup theory, Theorem 3.3 immediately implies

THEOREM 3.5. *Suppose $\mathcal{A}$ is normally elliptic of order $2l$ and the coefficients $\mathfrak{a} = (a^r)_r$ satisfy the conditions in Theorem 3.3. Then*

$$\mathcal{A} \in \mathcal{H}(bc^{s+2l}(\mathsf{M}, V), bc^s(\mathsf{M}, V)).$$

The reader should be aware that Theorem 3.5 does not apply to Hölder spaces, since $BC^{s+2l}(\mathsf{M}, V)$ is not densely embedded into $BC^s(\mathsf{M}, V)$.

For some fixed interval $I = [0, T]$, $\gamma \in (0, 1)$ and some Banach space X, we define

$$BUC_{1-\gamma}(I, X) := \{u \in C(\dot{I}, X); [t \mapsto t^{1-\gamma}u] \in C(\dot{I}, X), \lim_{t\to 0^+} t^{1-\gamma}\|u(t)\|_X = 0\},$$

$$\|u\|_{C_{1-\gamma}} := \sup_{t\in\dot{I}} t^{1-\gamma}\|u(t)\|_X,$$

and

$$BUC^1_{1-\gamma}(I,X) := \{u \in C^1(\dot{I},X) : u, \dot{u} \in BUC_{1-\gamma}(I,X)\}.$$

Recall that in the above definition $\dot{I} = I\backslash\{0\}$. Moreover, we put

$$BUC_0(I,X) := BUC(I,X) \quad \text{and} \quad BUC^1_0(I,X) := BUC^1(I,X).$$

In addition, if $I = [0,T)$ is a half open interval, then

$$C_{1-\gamma}(I,X) := \{v \in C(\dot{I},X) : v \in BUC_{1-\gamma}([0,t],X),\ t < T\},$$
$$C^1_{1-\gamma}(I,X) := \{v \in C^1(\dot{I},X) : v, \dot{v} \in C_{1-\gamma}(I,X)\}.$$

We equip these two spaces with the natural Fréchet topology induced by the topology of $BUC_{1-\gamma}([0,t],X)$ and $BUC^1_{1-\gamma}([0,t],X)$, respectively.

Assume that $E_1 \overset{d}{\hookrightarrow} E_0$ is a densely embedded Banach couple. Define

$$\mathbb{E}_0(I) := BUC_{1-\gamma}(I,E_0), \quad \mathbb{E}_1(I) := BUC_{1-\gamma}(I,E_1) \cap BUC^1_{1-\gamma}(I,E_0). \tag{3.7}$$

For $A \in \mathcal{H}(E_1,E_0)$, we say $(\mathbb{E}_0(I),\mathbb{E}_1(I))$ is a pair of maximal regularity of A, if

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + A, \gamma_0\right) \in \mathcal{L} \text{ is } (\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\gamma),$$

where γ_0 is the evaluation map at 0, i.e., $\gamma_0(u) = u(0)$, and $E_\gamma := (E_0,E_1)^0_{\gamma,\infty}$. Symbolically, we denote it by

$$A \in \mathcal{M}_\gamma(E_1,E_0).$$

Let $E_2 := (\mathrm{dom}(A^2), \|\cdot\|_{E_2})$, where $\|\cdot\|_{E_2} := \|A\cdot\|_{E_1} + \|\cdot\|_{E_1}$. Put

- $E_{1+\theta} := (E_1,E_2)^0_{\theta,\infty}$, $\theta \in (0,1)$,
- $A_\theta :=$ the maximal E_θ-realization of A.

The next step is to conclude a maximal regularity result for normally elliptic operators. To this end, we quote a famous theorem, which was first proved by Da Prato and Grisvard [17], and then generalized later by Angenent.

THEOREM 3.6. [8, Theorem 2.14] *Suppose $A \in \mathcal{H}(E_1,E_0)$ and let $\gamma \in (0,1]$. Then,*

$$(\mathbb{E}_\theta(I), \mathbb{E}_{1+\theta}(I)) := \left(BUC_{1-\gamma}(I,E_\theta), BUC_{1-\gamma}(I,E_{1+\theta}) \cap BUC^1_{1-\gamma}(I,E_\theta)\right)$$

is a pair of maximal regularity for A_θ, that is,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + A_\theta, \gamma_0\right) \in \mathcal{L} \text{ is } \left(\mathbb{E}_{1+\theta}(I), \mathbb{E}_\theta(I) \times (E_\theta, E_{1+\theta})^0_{\gamma,\infty}\right).$$

THEOREM 3.7. *Let $\gamma \in (0, 1]$ and $s \notin \mathbb{N}_0$. Suppose that $\mathcal{A}$ satisfies the conditions in Theorem 3.5. Then,*

$$\mathcal{A} \in \mathcal{M}_\gamma(E_1, E_0)$$

with $E_0 := bc^s(\mathsf{M}, V)$ and $E_1 := bc^{s+2l}(\mathsf{M}, V)$.

Proof. Pick $\max\{0, s - 2l\} < \alpha < s$ with $\alpha \notin \mathbb{N}_0$. Let $F_0 = bc^\alpha(\mathsf{M}, V)$, $F_1 = bc^{\alpha+2l}(\mathsf{M}, V)$. Set $\theta = \frac{s-\alpha}{2l}$. By Proposition 2.2(c), we have $F_\theta = E_0$ and $\mathcal{A} \in \mathcal{H}(F_1, F_0)$.

By the preceding theorem, we infer that $\mathcal{A}_\theta \in \mathcal{M}_\gamma(F_{1+\theta}, E_0)$. In particular, $\mathcal{A}_\theta \in \mathcal{H}(F_{1+\theta}, E_0)$. Note that $\mathrm{dom}(\mathcal{A}) = \mathrm{dom}(\mathcal{A}_\theta)$.

By analytic semigroup theory, there exists a $\lambda > 0$ such that

$$\lambda + \mathcal{A} \in \mathcal{L}\,\mathrm{is}\,(F_{1+\theta}, E_0) \cap \mathcal{L}\,\mathrm{is}\,(E_1, E_0).$$

Consequently, it implies that $F_{1+\theta} \doteq E_1$. This completes the proof. □

PROPOSITION 3.8. *Suppose that all the local representations $\mathcal{A}_\kappa = \sum_{|\alpha|\le 2l} a_\alpha^\kappa \partial^\alpha$ of $\mathcal{A}$ are normally elliptic (or $(\mathcal{E}, \vartheta; 2l)$-elliptic with uniform constants $\mathcal{E}$, ϑ, respectively), and their coefficients satisfy*

$$\left(a_\alpha^\kappa\right)_\kappa \in l_\infty\left(BC^t\left(\mathbb{B}_\kappa^m, \mathcal{L}(E)\right)\right)$$

for some $t > s$. Then, the assertions in Theorem 3.5 and 3.7 (or Theorem 3.3, respectively) hold true.

4. Parameter-dependent diffeomorphisms on uniformly regular Riemannian manifolds

The main purpose of this section is to introduce a family of parameter-dependent diffeomorphisms induced by a truncated translation technique. As will be shown later, this technique combined with the implicit function theorem serves as a crucial tool to study regularity of solutions to parabolic equations on uniformly regular Riemannian manifolds. The idea of a family of truncated translations was introduced by Escher et al. [22] to establish regularity for solutions to parabolic and elliptic equations in Euclidean spaces. The major obstruction of bringing in the localized translations for uniformly regular Riemannian manifolds lies in how to introduce parameters so that the transformed functions and differential operators depend ideally on the parameters as long as they are smooth enough around the "center" of the localized translations. Thanks to the discussions in the previous section, we are able to set up these properties based on their counterparts in [22].

Suppose that (M, g) is a uniformly regular Riemannian manifold, with a uniformly regular atlas $\mathfrak{A}$. Given any point $p \in \mathring{\mathsf{M}}$, there is a local chart $(\mathsf{O}_{\kappa_\mathsf{p}}, \varphi_{\kappa_\mathsf{p}}) \in \mathfrak{A}$ containing

p. Let $x_{\mathsf{p}} := \varphi_{\kappa_{\mathsf{p}}}(p)$ and $d := \operatorname{dist}(x_{\mathsf{p}}, \partial\mathbb{B}^m_{\kappa_{\mathsf{p}}})$. We define a new local patch $(\mathsf{O}_\iota, \varphi_\iota)$ around p by means of

$$\mathsf{O}_\iota := \psi_{\kappa_{\mathsf{p}}}(\mathbb{B}(x_{\mathsf{p}}, d)), \quad \varphi_\iota(q) := \frac{\varphi_{\kappa_{\mathsf{p}}}(q) - x_{\mathsf{p}}}{d}, \quad q \in \mathsf{O}_{\kappa_{\mathsf{p}}}.$$

Then, $\varphi_\iota(p) = 0 \in \mathbb{R}^m$, $\varphi_\iota(\mathsf{O}_\iota) = \mathbb{B}^m$, and the transition maps between $(\mathsf{O}_\iota, \varphi_\iota)$ and $(\mathsf{O}_{\kappa_{\mathsf{p}}}, \varphi_{\kappa_{\mathsf{p}}})$ are given by

$$\varphi_\iota \circ \psi_{\kappa_{\mathsf{p}}}(y) = \frac{y - x_{\mathsf{p}}}{d}, \quad \varphi_{\kappa_{\mathsf{p}}} \circ \psi_\iota(x) = d\,x + x_{\mathsf{p}}, \quad x \in \mathbb{B}^m, \ y \in \mathbb{B}(x_{\mathsf{p}}, d).$$

Choose $\varepsilon_0 > 0$ small such that $5\varepsilon_0 < 1$ and set

$$B_i := \mathbb{B}^m(0, i\varepsilon_0), \quad \text{for } i = 1, 2, ..., 5.$$

As part of the preparations for introducing a family of parameter-dependent diffeomorphisms on M, we pick two cut-off functions on $\mathbb{B}^m$:

- $\chi \in \mathcal{D}(B_2, [0, 1])$ such that $\chi|_{\bar{B}_1} \equiv 1$.
- $\tilde{\varsigma} \in \mathcal{D}(B_5, [0, 1])$ such that $\tilde{\varsigma}|_{\bar{B}_4} \equiv 1$. We write $\varsigma = \varphi_\iota^* \tilde{\varsigma}$.

We define a rescaled translation on $\mathbb{B}^m$ for any $\mu \in \mathbb{B}(0, r) \subset \mathbb{R}^m$ with $r > 0$ sufficiently small:

$$\theta_\mu(x) := x + \chi(x)\mu, \quad x \in \mathbb{B}^m.$$

This localization technique in Euclidean spaces was first introduced by Escher et al. [22]. θ_μ induces a transformation Θ_μ on M by:

$$\Theta_\mu(q) = \begin{cases} \psi_\iota(\theta_\mu(\varphi_\iota(q))) & q \in \mathsf{O}_\iota, \\ q & q \notin \mathsf{O}_\iota. \end{cases}$$

In particular, we have $\Theta_\mu \in \operatorname{Diff}^\infty(\mathsf{M})$ for $\mu \in \mathbb{B}(0, r)$ with sufficiently small $r > 0$.

We may find an explicit global expression for Θ_μ^*, i.e., the pull-back of Θ_μ. Given $u \in \Gamma(\mathsf{M}, V)$,

$$\Theta_\mu^* u = \varphi_\iota^* \theta_\mu^* \psi_\iota^*(\varsigma u) + (1 - \varsigma)u.$$

Likewise, Θ_*^μ can be expressed by

$$\Theta_*^\mu u = \varphi_\iota^* \theta_*^\mu \psi_\iota^*(\varsigma u) + (1 - \varsigma)u.$$

Let $I = [0, T]$, $T > 0$. Assuming that $t_0 \in \mathring{I}$ is a fixed point, we choose ε_0 so small that $\mathbb{B}(t_0, 3\varepsilon_0) \subset \mathring{I}$. Next pick another auxiliary function

$$\xi \in \mathcal{D}(\mathbb{B}(t_0, 2\varepsilon_0), [0, 1]) \quad \text{with} \quad \xi|_{\mathbb{B}(t_0, \varepsilon_0)} \equiv 1.$$

The above construction now engenders a parameter-dependent transformation in terms of the time variable:

$$\varrho_\lambda(t) := t + \xi(t)\lambda, \quad \text{for any} \quad t \in I \text{ and } \lambda \in \mathbb{R}.$$

Now, we are in a situation to define a family of parameter-dependent transformations on $I \times \mathsf{M}$. Given a function $u : I \times \mathsf{M} \to V$, we set

$$u_{\lambda,\mu}(t,\cdot) := \Theta^*_{\lambda,\mu} u(t,\cdot) := T_\mu(t) \varrho^*_\lambda u(t,\cdot),$$

where $T_\mu(t) := \Theta^*_{\xi(t)\mu}$ and $(\lambda,\mu) \in \mathbb{B}(0,r)$.

It is important to note that $u_{\lambda,\mu}(0,\cdot) = u(0,\cdot)$ for any $(\lambda,\mu) \in \mathbb{B}(0,r)$ and any function u. Here and in the following, I will not distinguish between $\mathbb{B}(0,r)$, $\mathbb{B}^m(0,r)$ and $\mathbb{B}^{m+1}(0,r)$. As long as the dimension of the ball is clear from the context, we always simply write them as $\mathbb{B}(0,r)$.

The importance of the family of parameter-dependent diffeomorphisms $\{\Theta^*_{\lambda,\mu} : (\lambda,\mu) \in \mathbb{B}(0,r)\}$ lies in the following results. Their proofs and the additional properties of this technique can be found in [36]. Let ω be the symbol for real analyticity. It is understood that under the condition that $k = \omega$ in the following theorems, M is assumed to be a C^ω-uniformly regular Riemannian manifold.

THEOREM 4.1. *Let $k \in \mathbb{N} \cup \{\infty, \omega\}$. Suppose that $u \in BC(I \times \mathsf{M}, V)$. Then, $u \in C^k(\mathring{I} \times \mathring{\mathsf{M}}, V)$ iff for any $(t_0, p) \in \mathring{I} \times \mathring{\mathsf{M}}$, there exists $r = r(t_0,p) > 0$ and a corresponding family of parameter-dependent diffeomorphisms $\Theta^*_{\lambda,\mu}$ such that*

$$\left[(\lambda,\mu) \mapsto \Theta^*_{\lambda,\mu} u\right] \in C^k(\mathbb{B}(0,r), BC(I \times \mathsf{M}, V)).$$

Henceforth, assume that $\mathfrak{F} \in \{bc, BC\}$. Let $E_0 := \mathfrak{F}^s(M,V)$ and $E_1 := \mathfrak{F}^{s+l}(M,V)$ with $l \in \mathbb{N}$. Define $\mathbb{E}_1(I)$ as in (3.7) by fixing $\gamma = 1$.

PROPOSITION 4.2. *Suppose that $u \in \mathbb{E}_1(I)$. Then, $u_{\lambda,\mu} \in \mathbb{E}_1(I)$, and*

$$\partial_t [u_{\lambda,\mu}] = (1 + \xi'\lambda)\Theta^*_{\lambda,\mu} u_t + B_{\lambda,\mu}(u_{\lambda,\mu}), \quad (\lambda,\mu) \in \mathbb{B}(0,r),$$

where $[(\lambda,\mu) \mapsto B_{\lambda,\mu}] \in C^\omega(\mathbb{B}(0,r), C(I, \mathcal{L}(\mathfrak{F}^{s+l}(M,V), \mathfrak{F}^s(M,V))))$. Furthermore, $B_{\lambda,0} = 0$.

PROPOSITION 4.3. *Let $n \geq 0$, $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$, and $p \in \mathring{\mathsf{M}}$. Suppose that $\mathcal{A} := \mathcal{A}(\mathfrak{a})$ is a linear differential operator on M of order l satisfying*

$$\mathfrak{a} = (a^r)_r \in \prod_{r=0}^{l} bc^s\left(\mathsf{M}, V^{\sigma+\tau+r}_{\tau+\sigma}\right) \cap C^{n+k}\left(\mathsf{O}, V^{\sigma+\tau+r}_{\tau+\sigma}\right), \quad \text{for } s \in [0,n],$$

where O is an open subset containing p. Here, $n + \infty := \infty$, $n + \omega := \omega$. Then, for sufficiently small ε_0 and r

$$\left[\mu \mapsto T_\mu \mathcal{A} T_\mu^{-1}\right] \in C^k\left(\mathbb{B}(0,r), C(I, \mathcal{L}(bc^{s+l}(\mathsf{M},V), bc^s(\mathsf{M},V)))\right).$$

REMARK 4.4. The conditions in Proposition 4.3 can be equivalently stated as

$$a^{\kappa_\mathsf{p}}_\alpha \in C^{n+k}(\varphi_{\kappa_\mathsf{p}}(\mathsf{O} \cap \mathsf{O}_{\kappa_\mathsf{p}}), \mathcal{L}(E)), \quad a^\kappa_\alpha \in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s(\mathbb{B}, \mathcal{L}(E))), \quad |\alpha| \leq l,$$

for the local expressions $\mathcal{A}_\kappa(x,\partial) := \sum_{|\alpha| \leq l} a^\kappa_\alpha(x)\partial^\alpha$.

PROPOSITION 4.5. *Let $l \in \mathbb{N}_0$ and $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$. Suppose that*

$$u \in C^{l+k}(\mathsf{O}, V) \cap \mathfrak{F}^s(\mathsf{M}, V),$$

where either $s \in [0, l]$ if $\mathfrak{F} = bc$, or $s = l$ if $\mathfrak{F} = BC$. O has the same meaning as in Proposition 4.3. Then, for sufficiently small ε_0 and r, we have

$$[\mu \mapsto T_\mu u] \in C^k(\mathbb{B}(0, r), C(I, \mathfrak{F}^s(\mathsf{M}, V))).$$

REMARK 4.6. As a first application of the parameter-dependent diffeomorphism method and our main theorem, following the proof in Sect. 5, on a given closed C^ω-uniformly regular Riemannian manifold (M, g), we can show that the solution to the heat equation

$$\begin{cases} \partial_t u = \Delta_g u, \\ u(0) = u_0, \end{cases}$$

with Δ_g the Laplace–Beltrami operator with respect to the metric g, immediately becomes analytic jointly in time and space for any initial value $u_0 \in bc^s(\mathsf{M})$, $s > 0$.

5. The Yamabe flow

A well-known problem in differential geometry is the Yamabe problem. In 1960, Yamabe [39] conjectured the following:

Yamabe Conjecture *Let (M, g) be a compact Riemannian manifold of dimension $m \geq 3$. Then, every conformal class of metrics contains a representative with constant scalar curvature.*

The proof for Yamabe's conjecture was completed by Trudinger [38], Aubin [9], Schoen [34] using the calculus of variations and elliptic partial differential equations, see [27] for a survey. The normalized Yamabe flow can be considered as another approach to this problem, which asks whether a metric converges conformally to one with constant scalar curvature on a compact manifold under the following flow:

$$\begin{cases} \partial_t g = (s_g - R_g)g, \\ g(0) = g_0, \end{cases} \tag{5.1}$$

where R_g is the scalar curvature with respect to the metric g, and s_g is the average of the scalar curvature. It was introduced by R. Hamilton shortly after the Ricci flow, and studied extensively by many authors afterward, among them Hamilton [25], Chow [15], Ye [40], Schwetlick and Struwe [35], Brendle [11–13]. A global existence and regularity result was presented by Ye in [40]. The author asserts that the unique solution to (5.1) exists globally and smoothly for any smooth initial metric. R. Hamilton conjectured that on a compact Riemannian manifold, the solution of (5.1) converges to a metric of constant scalar curvature as $t \to \infty$. Chow [15] commenced the study

of Hamilton's conjecture and proved convergence in the case when (M, g_0) is locally conformally flat and has positive Ricci curvature. Later, this result was improved by Ye [40], wherein the author removed the restriction on the positivity of Ricci curvature by lifting the flow to a sphere and deriving a Harnack inequality. In the case that $3 \le m \le 5$, Schwetlick and Struwe [35] showed that the normalized Yamabe flow evolves any initial metric to one with constant scalar curvature as long as the initial Yamabe energy is small enough. In [12], S. Brendle was able to remove the smallness assumption on the initial Yamabe energy. A convergence result is stated in [13] by the same author for higher dimension cases. Finally, it is worthwhile to point out that for two-dimensional manifolds, the Yamabe flow agrees with the Ricci flow.

As claimed in the introductory section, we will establish a regularity result for (5.1) to show that for any initial metric g_0 belonging to the class C^s with $s > 0$ in a conformal class containing at least one real analytic metric, the solution to (5.1) immediately evolves into an analytic metric. We shall point out here that not every conformal class contains a real analytic metric, but these conformal classes are in no sense trivial nevertheless. For instance, if g is a real analytic metric and $f \in C^\omega(\mathsf{M})$ has sufficiently small gradient, then we can also construct another real analytic metric by $g_f = g + \nabla f \otimes \nabla f$. In general, $g_f \notin [g]$.

In Sect. 5.1, we first study a generalization of the (unnormalized) Yamabe flow on noncompact manifolds. In recent years, these problems have been studied by several mathematicians, including An and Ma [7], Burchard et al. [14]. In comparison with the existing results, we do not ask for a uniform bound on the curvatures of the initial metric.

5.1. The Yamabe flow on uniformly regular manifolds without boundary

Let (M, g_0) be a C^ω-uniformly regular Riemannian manifold without boundary of dimension $m \ge 3$. The Yamabe flow reads as

$$\begin{cases} \partial_t g = -R_g g, \\ g(0) = g^0, \end{cases} \tag{5.2}$$

where R_g is the scalar curvature with respect to the metric g, and $g^0 \in [g_0]$, the conformal class of the real analytic background metric g_0 of M. Note that on a compact manifold M, (5.2) is equivalent to (5.1) in the sense that any solution to (5.2) can be transformed into a solution to (5.1) by a rescaling procedure.

We seek solutions to the Yamabe flow (5.2) in $[g_0]$. Define $c(m) := \frac{m-2}{4(m-1)}$ and the conformal Laplacian operator L_g with respect to the metric g as:

$$L_g u := \Delta_g u - c(m) R_g u.$$

Let $g = u^{\frac{4}{m-2}} g_0$ for some $u > 0$. It is well known that

$$R_g = -\frac{1}{c(m)} u^{-\frac{m+2}{m-2}} L_0 u.$$

where $L_0 := L_{g_0}$, see [40, formula (1.5)]. By rescaling the time variable, Eq. (5.2) is equivalent to

$$\begin{cases} \partial_t u = u^{-\frac{4}{m-2}} L_0 u, \\ u(0) = u_0 \end{cases} \tag{5.3}$$

with a positive function u_0, see also [29, formula (7)]. In the following, we will use the continuous maximal regularity theory in Sect. 3 to establish existence and uniqueness of solutions to (5.3). (R4), Theorem 2.1 and Remark 6.5(b) imply that

$$g_0 \in BC^\infty(\mathsf{M}, V_2^0) \cap C^\omega\left(\mathsf{M}, V_2^0\right), \quad g_0^* \in BC^\infty\left(\mathsf{M}, V_0^2\right) \cap C^\omega\left(\mathsf{M}, V_0^2\right). \tag{5.4}$$

By the well-known formula of scalar curvature in local coordinates,

$$R_g = \frac{1}{2} g^{ki} g^{lj} (g_{jk,li} + g_{il,kj} - g_{jl,ki} - g_{ik,lj}),$$

see [26, formula (3.3.15), Definition 3.3.3]. This, combined with (5.4), thus yields

$$R_{g_0} \in BC^\infty(\mathsf{M}) \cap C^\omega(\mathsf{M}). \tag{5.5}$$

Put

$$P(u)h := -u^{-\frac{4}{m-2}} \Delta_{g_0} h, \quad \text{and} \quad Q(u) := -c(m) u^{\frac{m-6}{m-2}} R_{g_0}.$$

Given any $0 < s < 1$, we choose $0 < \alpha < s$, $\gamma = \frac{s-\alpha}{2}$, and $b > 0$. Let $V = \mathbb{K}$ and

$$E_0 := bc^\alpha(\mathsf{M}), \quad E_1 := bc^{2+\alpha}(\mathsf{M}), \quad \text{and} \quad E_\gamma := (E_0, E_1)^0_{\gamma,\infty}.$$

Proposition 2.2 implies that $E_\gamma = bc^s(\mathsf{M})$. We put $W_b^s := \{u \in bc^s(\mathsf{M}) : \inf u > b\}$. By Proposition 6.3, for each $\beta \in \mathbb{R}$, we have

$$\mathsf{P}_\beta : [u \mapsto u^\beta] \in C^\omega\left(W_b^s, bc^s(\mathsf{M})\right), \tag{5.6}$$

which together with (5.5) and Proposition 2.7 now shows that

$$Q(u) = -c(m) u^{\frac{m-6}{m-2}} R_{g_0} \in E_0, \quad u \in W_b^s. \tag{5.7}$$

In every local chart, the Laplace–Beltrami operator Δ_{g_0} reads as

$$\Delta_{g_0} = g_0^{jk} \left(\partial_j \partial_k - \Gamma^i_{jk} \partial_i\right),$$

where Γ^i_{jk} are the Christoffel symbols of g_0. By (5.4) and Corollary 2.10, we obtain

$$\Delta_{g_0} \in \mathcal{L}\left(bc^{2+\alpha}(\mathsf{M}, V), bc^\alpha(\mathsf{M}, V)\right). \tag{5.8}$$

Then, it follows from Proposition 2.7, (5.6)–(5.8) that

$$(P, Q) \in C^\omega\left(W_b^s, \mathcal{L}(E_1, E_0) \times E_0\right).$$

One verifies that the symbols of the principal parts for the local expressions $(P(u))_\kappa$ in every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$ satisfy

$$\hat{\sigma}(P(u))^\pi_\kappa(x,\xi) = (\psi_\kappa^* u)^{-\frac{4}{m-2}}(x) g^{ij}(x)\xi_i\xi_j \geq C, \quad u \in W^s_b$$

for any $\xi \in \mathbb{R}^m$ with $|\xi| = 1$, and some $C > 0$ independent of κ. It follows from Theorem 3.7 that

$$P(u) \in \mathcal{M}_\gamma(E_1, E_0), \quad u \in W^s_b. \tag{5.9}$$

Now, [16, Theorem 4.1] leads to

THEOREM 5.1. *If $u_0 \in W^s_b := \{u \in bc^s(\mathsf{M}) : \inf u > b\}$ with $0 < s < 1$ and $b > 0$, then for every fixed $\alpha \in (0, s)$ Eq. (5.3) has a unique local positive solution*

$$\hat{u} \in C^1_{1-\gamma}(J(u_0), bc^\alpha(\mathsf{M})) \cap C_{1-\gamma}(J(u_0), bc^{2+\alpha}(\mathsf{M})) \cap C(J(u_0), W^s_b)$$

existing on $J(u_0) := [0, T(u_0))$ for some $T(u_0) > 0$ with $\gamma = \frac{s-\alpha}{2}$.

From now on, we use the notation $\hat{u}$ exclusively for the solution in Theorem 5.1. Next, set $G(u) := P(u)u - Q(u)$, $u \in W^s_b \cap E_1$. For any $(t_0, p) \in \dot{J}(u_0) \times \mathsf{M}$, we find a closed interval $I := [\varepsilon, T] \subset \dot{J}(u_0)$ and $t_0 \in \mathring{I}$. We define two function spaces as follows:

$$\mathbb{E}_0(I) := C(I, E_0), \quad \mathbb{E}_1(I) := C(I, E_1) \cap C^1(I, E_0),$$

and an open subset of $\mathbb{E}_1(I)$ by

$$\mathbb{U}(I) := \{u \in \mathbb{E}_1(I) : \inf u > b\}.$$

Let $u := \Theta^*_{\lambda,\mu}\hat{u} = \hat{u}_{\lambda,\mu}$. By Proposition 4.2, one computes that

$$\begin{aligned} u_t = \partial_t[\hat{u}_{\lambda,\mu}] &= (1 + \xi'\lambda)\Theta^*_{\lambda,\mu}\hat{u}_t + B_{\lambda,\mu}(u) \\ &= -(1 + \xi'\lambda)\Theta^*_{\lambda,\mu} G(\hat{u}) + B_{\lambda,\mu}(u) \\ &= -(1 + \xi'\lambda)T_\mu G(\varrho^*_\lambda \hat{u}) + B_{\lambda,\mu}(u) \\ &= -(1 + \xi'\lambda)T_\mu G(T_\mu^{-1}u) + B_{\lambda,\mu}(u). \end{aligned}$$

Thus, we define a map $\Phi : \mathbb{U}(I) \times \mathbb{B}(0, r) \to \mathbb{E}_0(I) \times E_1$ by

$$\Phi(v, (\lambda, \mu)) = \begin{pmatrix} \partial_t v + (1 + \xi'\lambda)T_\mu G(T_\mu^{-1}v) - B_{\lambda,\mu}(v) \\ \gamma_\varepsilon v - \hat{u}(\varepsilon) \end{pmatrix},$$

where γ_ε is the evaluation map at ε, i.e., $\gamma_\varepsilon(v) = v(\varepsilon)$. Now, the subsequent step is to verify the conditions of the implicit function theorem.

(i) Clearly, $\Phi(\hat{u}_{\lambda,\mu}, (\lambda, \mu)) = \binom{0}{0}$ for any $(\lambda, \mu) \in \mathbb{B}(0, r)$. On the other hand, we have

$$D_1\Phi(\hat{u}, (0, 0))w = \begin{pmatrix} \partial_t w + DG(\hat{u})w \\ \gamma_\varepsilon w \end{pmatrix}.$$

One verifies that the symbol of the principal part for $DG(\hat{u})$ agrees with that of $P(\hat{u})$. By Theorem 3.7, in (5.9) we have the freedom to choose γ. Thus choose $\gamma = 1$. It yields

$$D_1\Phi(\hat{u}(t,\cdot),(0,0)) \in \mathcal{L} \text{ is } (\mathbb{E}_1(I),\mathbb{E}_0(I)\times E_1), \quad \text{for every } t\in I.$$

Set $\mathcal{A}(t) := DG(\hat{u}(t))$. The above formula shows that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\mathcal{A}(s),\gamma_\varepsilon\right) \in \mathcal{L} \text{ is } (\mathbb{E}_1(I),\mathbb{E}_0(I)\times E_1), \quad \text{for every } s\in I.$$

It follows from [16, Lemma 2.8(a)] that

$$D_1\Phi(\hat{u},(0,0)) = \left(\frac{\mathrm{d}}{\mathrm{d}t}+\mathcal{A}(\cdot),\gamma_\varepsilon\right) \in \mathcal{L} \text{ is } (\mathbb{E}_1(I),\mathbb{E}_0(I)\times E_1).$$

(ii) The next goal is to show that

$$\Phi \in C^\omega(\mathbb{U}(I)\times\mathbb{B}(0,r),\mathbb{E}_0(I)\times E_1). \tag{5.10}$$

By Proposition 4.2, $B_{\lambda,\mu}\in C^\omega(\mathbb{B}(0,r),C(I,\mathcal{L}(E_1,E_0)))$. We define a bilinear and continuous map $f : C(I,\mathcal{L}(E_1,E_0))\times\mathbb{E}_1(I)\to\mathbb{E}_0(I)$ by:

$$f : (T(t),u(t))\mapsto T(t)u(t).$$

Hence $[(v,(\lambda,\mu))\mapsto f(B_{\lambda,\mu},v)=B_{\lambda,\mu}(v)]\in C^\omega(\mathbb{U}(I)\times\mathbb{B}(0,r),\mathbb{E}_0(I))$.

By Proposition 6.4, the pointwise extension of P_β on I, i.e., $\mathsf{P}_\beta(u)(t) = \mathsf{P}_\beta(u(t))$, fulfills

$$\mathsf{P}_\beta \in C^\omega(C(I,W_b^s),C(I,E_0)). \tag{5.11}$$

In virtue of Proposition 4.5, we get

$$[\mu\mapsto T_\mu R_{g_0}]\in C^\omega(\mathbb{B}(0,r),\mathbb{E}_0(I)).$$

By noticing $T_\mu Q(T_\mu^{-1}u) = -c(m)u^{\frac{m-6}{m-2}}T_\mu R_{g_0}$, Proposition 2.7 and (5.11) hence imply that

$$[(u,\mu)\mapsto T_\mu Q(T_\mu^{-1}u)]\in C^\omega(\mathbb{U}(I)\times\mathbb{B}(0,r),\mathbb{E}_0(I)).$$

On the other hand, in light of Proposition 4.3, we infer that

$$[\mu\mapsto T_\mu\Delta_{g_0}T_\mu^{-1}]\in C^\omega(\mathbb{B}(0,r),C(I,\mathcal{L}(E_1,E_0))).$$

It follows again from Proposition 2.7 and (5.11) that

$$[(u,\mu)\mapsto T_\mu P(T_\mu^{-1}u)T_\mu^{-1}]\in C^\omega(\mathbb{U}(I)\times\mathbb{B}(0,r),\mathcal{L}(\mathbb{E}_1(I),\mathbb{E}_0(I))).$$

The rest of proof for (5.10) follows straight away. Consequently, we have proved the desired assertion. Now we are in a position to apply the implicit function

theorem. Hence there exists an open neighborhood, say $\mathbb{B}(0, r_0) \subset \mathbb{B}(0, r)$, such that

$$[(\lambda, \mu) \mapsto \hat{u}_{\lambda,\mu}] \in C^\omega(\mathbb{B}(0, r_0), \mathbb{E}_1(I)).$$

As a consequence of Theorem 4.1, we deduce that

THEOREM 5.2. *For each* $u_0 \in W_b^s := \{u \in bc^s(\mathsf{M}) : \inf u > b\}$ *with* $0 < s < 1$ *and* $b > 0$, *Eq.* (5.3) *has a unique local positive solution*

$$\hat{u} \in C^\omega(\dot{J}(u_0) \times \mathsf{M})$$

for some interval $J(u_0) := [0, T(u_0))$. *In particular,*

$$g = \hat{u}^{\frac{4}{m-2}} g_0 \in C^\omega\left(\dot{J}(u_0) \times \mathsf{M}, V_2^0\right).$$

REMARKS 5.3. (a) In general, the presence of a metric $g_0 \in BC^\infty(\mathsf{M}, V_2^0) \cap C^\omega(\mathsf{M}, V_2^0)$ is unnecessary. As long as there is a metric $g_0 \in bc^{2+\alpha}(\mathsf{M}, V_2^0) \cap C^k(\mathsf{M}, V_2^0)$ with $k \in \{\infty, \omega\}$, the solution to (5.2) on M immediately evolves conformally in $[g_0]$ into the class $C^k(\dot{J}(u_0) \times \mathsf{M}, V_2^0)$.

(b) For any Riemannian manifold M without boundary, by [32, Theorem 1.4, Corollary 1.5], in every conformal class containing a C^k-metric with $k \in \{\infty, \omega\}$ there exists a C^k-metric g_0 with bounded geometry, which makes (M, g_0) complete. It then follows that (M, g_0) is uniformly regular. Choosing g_0 as the background metric, we can infer from the proof above that the solution to (5.2) belongs to $C^k(\dot{J}(u_0) \times \mathsf{M}, V_2^0)$ for any initial metric $g^0 = u_0^{\frac{4}{m-2}} g_0$ with $u_0 \in W_b^s := \{bc^s(\mathsf{M}) : \inf u > b\}$.

5.2. The normalized Yamabe flow on compact manifolds

Next, we consider regularity of the normalized Yamabe flow on compact manifolds. Let (M, g_0) be a connected compact closed m-dimensional C^∞-Riemannian manifold with $m \geq 3$. By a well-known result of H. Whitney, M admits a compatible real analytic structure. The existence of a real analytic metric on M follows from [31]. Without loss of generality, we may assume that the atlas $\mathfrak{A}$ and the metric g_0 are real analytic.

The normalized Yamabe flow reads as

$$\begin{cases} \partial_t g = (s_g - R_g)g, \\ g(0) = g^0, \end{cases} \tag{5.12}$$

where $g^0 \in [g_0]$, $V(g) = \int_{\mathsf{M}} \mathrm{d}V_g$ and

$$s_g = \frac{1}{V(g)} \int_{\mathsf{M}} R_g \, \mathrm{d}V_g.$$

The normalized Yamabe flow preserves the volume, that is, $V(g) \equiv V(g^0)$. This can be easily verified by checking the time derivative of $V(g)$.

As in the above subsection, we can reduce Eq. (5.12) to the following form.

$$\begin{cases} \partial_t u = u^{-\frac{4}{m-2}}[L_0 u + c(m) s_g u^{\frac{m+2}{m-2}}], \\ u(0) = u_0. \end{cases} \tag{5.13}$$

Put

$$P(u)h := -u^{-\frac{4}{m-2}} \left[\Delta_{g_0} h - \frac{u^{\frac{m+2}{m-2}}}{V(g)} \int_{\mathsf{M}} u^{-\frac{m+2}{m-2}} L_0 h \, \mathrm{d}V_g \right],$$

and

$$Q(u) = -c(m) u^{\frac{m-6}{m-2}} R_{g_0}.$$

Given $0 < s < 1$, we choose $0 < \alpha < s$ and $\gamma = \frac{s-\alpha}{2}$. Set E_0, E_1 and E_γ as in Sect. 5.1. Additionally, we put $W^s := \{u \in bc^s(\mathsf{M}) : u > 0\}$. Let

$$A(u)h := \frac{1}{V(g)} \int_{\mathsf{M}} u^{-\frac{m+2}{m-2}} L_0 h \, \mathrm{d}V_g = \frac{1}{\int_{\mathsf{M}} u^{\frac{2m}{m-2}} \, \mathrm{d}V_{g_0}} \int_{\mathsf{M}} u L_0 h \, \mathrm{d}V_{g_0}. \tag{5.14}$$

By Corollary 2.10 and Proposition 6.3, for each $\beta \in \mathbb{R}$

$$\Delta_{g_0} \in \mathcal{L}(BC^2(\mathsf{M}), BC(\mathsf{M})), \quad \text{and} \quad \mathsf{P}_\beta \in C^\omega(W^s, bc^s(\mathsf{M})). \tag{5.15}$$

One readily checks that (5.14), (5.15) and Remark 6.5(a) imply that

$$A \in C^\omega(W^s, \mathcal{L}(BC^2(\mathsf{M}), \mathbb{R})).$$

It yields

$$(P, Q) \in C^\omega(W^s, \mathcal{L}(E_1, E_0) \times E_0).$$

We verify that the symbol of the principal part for the local expression of $u^{-\frac{4}{m-2}} \Delta_{g_0}$ in every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$ satisfies

$$\hat{\sigma}(u^{-\frac{4}{m-2}} \Delta_{g_0})^\pi_\kappa (x, \xi) = u^{-\frac{4}{m-2}}(x) g^{ij}(x) \xi^i \xi^j \geq C, \quad u \in W^s$$

for any $\xi \in \mathbb{R}^m$ with $|\xi| = 1$, and some $C > 0$ by the compactness of M. Thus, $u^{-\frac{4}{m-2}} \Delta_{g_0}$ is normally elliptic for any $u \in W^s$. Since $uA(u)$ is a lower order perturbation compared to $u^{-\frac{4}{m-2}} \Delta_{g_0}$ for every $u \in W^s$, it follows from Theorem 3.7 and [16, Lemma 2.7] that

$$P(u) \in \mathcal{M}_\gamma(E_1, E_0), \quad u \in W^s. \tag{5.16}$$

Then, [16, Theorem 4.1] implies that

THEOREM 5.4. *If* $u_0 \in W^s := \{u \in bc^s(\mathsf{M}) : \inf u > 0\}$ *with* $0 < s < 1$, *then for each fixed* $\alpha \in (0, s)$ *Eq.* (5.13) *has a unique local positive solution*

$$\hat{u} \in C^1_{1-\gamma}(J(u_0), bc^\alpha(\mathsf{M})) \cap C_{1-\gamma}(J(u_0), bc^{2+\alpha}(\mathsf{M})) \cap C(J(u_0), W^s)$$

existing on $J(u_0) := [0, T(u_0))$ *for some* $T(u_0) > 0$ *with* $\gamma = \frac{s-\alpha}{2}$.

Next, set $G(u) = P(u)u - Q(u)$, $u \in W^s \cap E_1$. For any closed interval $I := [\varepsilon, T] \subset \dot{J}(u_0)$, the function spaces $\mathbb{E}_0(I)$, $\mathbb{E}_1(I)$ and $\mathbb{U}(I)$ have the same definitions as in Sect. 5.1 with $b = 0$. The map $\Phi : \mathbb{U}(I) \times \mathbb{B}(0, r) \to \mathbb{E}_0(I) \times E_1$ is defined by

$$\Phi(v, (\lambda, \mu)) = \begin{pmatrix} \partial_t v + (1 + \xi'\lambda) T_\mu G(T_\mu^{-1} v) - B_{\lambda,\mu}(v) \\ \gamma_\varepsilon v - \hat{u}(\varepsilon) \end{pmatrix}.$$

Clearly, $\Phi(\hat{u}_{\lambda,\mu}, (\lambda, \mu)) = \binom{0}{0}$ for any $(\lambda, \mu) \in \mathbb{B}(0, r)$. On the other hand, we have

$$D_1\Phi(\hat{u}, (0, 0))w = \begin{pmatrix} \partial_t w + DG(\hat{u})w \\ \gamma_\varepsilon w \end{pmatrix}.$$

Since $DA(\hat{u}) \in \mathcal{L}(BC^2(\mathsf{M}), \mathbb{R})$, it follows from a similar argument to the previous problem that

$$D_1\Phi(\hat{u}, (0, 0)) \in \mathcal{L}\,\text{is}\,(\mathbb{U}(I), \mathbb{E}_0(I) \times E_1).$$

Now, it remains to show that $\Phi \in C^\omega(\mathbb{U}(I) \times \mathbb{B}(0, r), \mathbb{E}_0(I) \times E_1)$. Following the previous proof, we only need to consider $\bar{A}(u) := A(u)u$. Note that

$$T_\mu \bar{A}(T_\mu^{-1} u) = \frac{1}{\int_{\mathsf{M}} (T_\mu^{-1} u)^{\frac{2m}{m-2}} \, \mathrm{d}V_{g_0}} \int_{\mathsf{M}} T_\mu^{-1} u L_0 T_\mu^{-1} u \, \mathrm{d}V_{g_0}.$$

An analogous argument as in [36, Proposition 6.2] yields

$$[(u, \mu) \mapsto T_\mu \bar{A}(T_\mu^{-1} u)] \in C^\omega(\mathbb{U}(I) \times \mathbb{B}(0, r), \mathbb{E}_0(I)).$$

The rest follows by a slight change of step (ii) in Sect. 5.1 and [22, Lemma 5.1]. Then, we adopt the implicit function theorem and get

THEOREM 5.5. *If* $u_0 \in W^s := \{u \in bc^s(\mathsf{M}) : \inf u > 0\}$ *with* $0 < s < 1$, *then Eq.* (5.13) *has a unique local positive solution*

$$\hat{u} \in C^\omega(\dot{J}(u_0) \times \mathsf{M})$$

for some interval $J(u_0) := [0, T(u_0))$. *In particular,*

$$g = \hat{u}^{\frac{4}{m-2}} g_0 \in C^\omega(\dot{J}(u_0) \times \mathsf{M}, V_2^0).$$

REMARKS 5.6. (a) We shall point out the global existence of the real analytic solution obtained above. By [16, Theorem 4.1(d)], it suffices to show that $\operatorname{dist}(u(t, \cdot), \partial W^s) > 0$ and there exists a $\theta \in (\gamma, 1]$ such that $\|\hat{u}\|_{E_\theta}$ is bounded in finite time. Actually, this is guaranteed by [40, Theorem 3, Lemma 4].

(b) It follows from Theorem 5.5 that the equilibria in the conformal class $[g_0]$, that is to say, those of constant scalar curvature, must be analytic. This implies that the solution to the Yamabe problem in any conformal class possessing at least one real analytic metric is analytic.

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6. Appendix

LEMMA 6.1. *Let $k \in \mathbb{N}$ and $s \in [0, k]$. Suppose that $f \in C^k(U, \mathbb{R})$ for some open interval $U \subset \mathbb{R}$ and $u \in W := \{u \in bc^s(\mathsf{M}) : \mathrm{im}(u) \subset\subset U\}$. Then $f(u) := f \circ u \in bc^s(\mathsf{M})$.*

Proof. It suffices to prove the assertion for $0 < s < 1$, since the other cases are direct consequences of the chain rule. Clearly, $f(u) \in BC(\mathsf{M})$. On the other hand, for $|x - y| < \delta$ for some sufficiently small δ

$$\begin{aligned}
&\frac{|\psi_\kappa^*\pi_\kappa(x)\psi_\kappa^* f(u)(x) - \psi_\kappa^*\pi_\kappa(y)\psi_\kappa^* f(u)(y)|}{|x-y|^s} \\
&= \psi_\kappa^*\pi_\kappa(y)\frac{|\psi_\kappa^* f(u)(x) - \psi_\kappa^* f(u)(y)|}{|x-y|^s} + |\psi_\kappa^* f(u)(x)|\frac{|\psi_\kappa^*\pi_\kappa(x) - \psi_\kappa^*\pi_\kappa(y)|}{|x-y|^s} \\
&\le \psi_\kappa^*\pi_\kappa(y)\int_0^1 |f'(\psi_\kappa^* u(y) + t(\psi_\kappa^* u(x) - \psi_\kappa^* u(y)))| \\
&\quad \times \frac{|\zeta(x)\psi_\kappa^* u(x) - \zeta(y)\psi_\kappa^* u(y)|}{|x-y|^s}\,\mathrm{d}t + \delta^{1-s}\|\psi_\kappa^*\pi_\kappa\|_{1,\infty,\mathbb{X}_\kappa}\|f(u)\|_\infty \\
&\le M(u)[\zeta\psi_\kappa^* u]_{s,\infty}^\delta + \delta^{1-s}\|\psi_\kappa^*\pi_\kappa\|_{1,\infty,\mathbb{X}_\kappa}\|f(u)\|_\infty.
\end{aligned}$$

The validity of the last step is supported by (L2). By (2.2) and (3.3), we attain

$$\mathcal{R}^c f(u) \in l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s).$$

Now, the assertion follows from Theorem 2.1. □

Henceforth, we assume that $\alpha \in \mathbb{R}$, $s \ge 0$ and $b \ge 0$.

LEMMA 6.2. *Suppose that $W_b^s := \{u \in bc^s(U) : \inf u > b\}$ for some open subset $U \subset \mathbb{R}^m$. Then,*

$$[u \mapsto u^\alpha] \in C^\omega(W_b^s, bc^s(U)).$$

Proof. We only need to show the case $0 < s < 1$, and the rest follows from knowledge of calculus. Put $W := \{u \in bc^s(U) : \|u\|_\infty < 1\}$. It is well known that

$$(1+u)^\alpha = \sum_n \binom{\alpha}{n} u^n, \quad \text{for } u \in W, \tag{6.1}$$

converges in $BC(U)$. Equation (2.6) and induction imply $[u^n]^\delta_{s,\infty} \le n\|u\|^{n-1}_\infty [u]^\delta_{s,\infty}$. Hence,

$$\sum_n \binom{\alpha}{n} \|u^n\|_{s,\infty} \le \sum_n \binom{\alpha}{n} \|u\|^n_\infty + \alpha [u]^\delta_{s,\infty} \sum_n \binom{\alpha-1}{n} \|u\|^n_\infty < \infty.$$

By Lemma 6.1, (6.1) converges in $bc^s(U)$.

For each $u \in W^s_b$, choose $\varepsilon \le \inf(u-b)$. Then, for any $v \in \mathbb{B}_{bc^s(U)}(u,\varepsilon) \subset W^s_b$,

$$v^\alpha = u^\alpha (1 + \frac{v-u}{u})^\alpha = u^\alpha \sum_n \binom{\alpha}{n} \left(\frac{v-u}{u}\right)^n = \sum_n \binom{\alpha}{n} u^{(\alpha-n)} (v-u)^n.$$

By Proposition 2.7 and Lemma 6.1, the series converges in $bc^s(U)$. □

PROPOSITION 6.3. *Suppose that* $W^s_b := \{u \in bc^s(\mathsf{M}) : \inf u > b\}$. *Then,* $\mathsf{P}_\alpha := [u \mapsto u^\alpha] \in C^\omega(W^s_b, bc^s(\mathsf{M}))$.

Proof. For each $u \in W^s_b$, choose $\varepsilon \le \inf(u-b)$ sufficiently small. By Lemma 6.2, (3.3) and (3.4), for any $v \in \mathbb{B}_\varepsilon(u) \subset W^s_b$

$$\begin{aligned} \mathcal{R}^c_\kappa v^\alpha &= \psi^*_\kappa \pi_\kappa \sum_n \binom{\alpha}{n} (\zeta \psi^*_\kappa u)^{(\alpha-n)} (\zeta \psi^*_\kappa v - \zeta \psi^*_\kappa u)^n \\ &= \sum_n \binom{\alpha}{n} \psi^*_\kappa \pi_\kappa (\psi^*_\kappa u)^{(\alpha-n)} (\psi^*_\kappa v - \psi^*_\kappa u)^n \end{aligned}$$

converges in bc^s_κ uniformly with respect to κ. One can check that

$$\mathcal{R}^c \circ \mathsf{P}_\alpha \in C^\omega(W^s_b, l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s)).$$

The assertion follows from Theorem 2.1, and the fact that $\mathcal{R}$ is real analytic. □

A slight modification of the above proofs now implies the pointwise extension of Proposition 6.3.

PROPOSITION 6.4. *Let* $X := bc^s(\mathsf{M})$ *and* $W^s_b(I) := \{u \in C(I,X) : \inf u > b\}$. *Then* $\mathsf{P}_\alpha \in C^\omega(W^s_b(I), C(I,X))$.

REMARK 6.5. The estimates in Lemmas 6.1 and 6.2 show that the case bc replaced by BC is admissible. So Propositions 6.3 and 6.4 still hold true for $BC^s(\mathsf{M})$ and $C(I, BC^s(\mathsf{M}))$, respectively.

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