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J. Differential Equations 196 (2004) 418–447
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Quasilinear evolutionary equations and continuous interpolation spaces

Philippe Clément,[a] Stig-Olof Londen,[b,*] and Gieri Simonett[c]

[a] Department of Mathematics and Informatics, TU Delft 2600 GA Delft, The Netherlands
[b] Institute of Mathematics, Helsinki University of Technology, 02150 Espoo, Finland
[c] Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

Received August 26, 2002; revised June 17, 2003

Abstract

In this paper we analyze the abstract parabolic evolutionary equations

$$D_t^\alpha(u-x)+A(u)u=f(u)+h(t), \quad u(0)=x,$$

in continuous interpolation spaces allowing a singularity as $t\downarrow 0$. Here D_t^α denotes the time-derivative of order $\alpha\in(0,2)$. We first give a treatment of fractional derivatives in the spaces $L^p((0,T);X)$ and then consider these derivatives in spaces of continuous functions having (at most) a prescribed singularity as $t\downarrow 0$. The corresponding trace spaces are characterized and the dependence on α is demonstrated. Via maximal regularity results on the linear equation

$$D_t^\alpha(u-x)+Au=f, \quad u(0)=x,$$

we arrive at results on existence, uniqueness and continuation on the quasilinear equation. Finally, an example is presented.

Keywords: Abstract parabolic equations; Continuous interpolation spaces; Quasilinear evolutionary equations; Maximal regularity

*Corresponding author. Institute of Mathematics, Helsinki University of Technology, P.O. Box 1100, 02015 HUT, Finland. Fax: +35-89-45-13-016.
E-mail address: stig-olof.londen@hut.fi (S.-O. Londen).

0022-0396/$ - see front matter
doi:10.1016/j.jde.2003.07.014

1. Introduction

In a recent paper, [7], the quasilinear parabolic evolution equation

$$\frac{du}{dt} + A(u)u = f(u), \quad u(0) = x,$$

was considered in continuous interpolation spaces. The analysis was based on maximal regularity results concerning the linear equation

$$\frac{du}{dt} + Au = f, \quad u(0) = x.$$

In particular, the approach allowed for solutions having (at most) a prescribed singularity as $t \downarrow 0$. Thus the smoothing property of parabolic evolution equations could be incorporated.

In this paper we show that the approach and the principal results of [7] extend, in a very natural way, to the entire range of abstract parabolic evolutionary equations

$$D_t^\alpha(u - x) + A(u)u = f(u), \quad u(0) = x.$$

Here D_t^α denotes the time-derivative of arbitrary order $\alpha \in (0, 2)$.

As in [7], our basic setting is the following. Let E_0, E_1 be Banach spaces, with $E_1 \subset E_0$, and assume that, for each u, $A(u)$ is a linear bounded map of E_1 into E_0 which is positive and satisfies an appropriate spectral angle condition as a map in E_0. Moreover, $A(u)$ and $f(u)$ are to satisfy a specific local continuity assumption with respect to u.

Problems of fractional order occur in several applications, e.g., in viscoelasticity [10], and in the theory of heat conduction in materials with memory [17]. For an entire volume devoted to applications of fractional differential systems, see [16].

Our paper is structured as follows. We first (Section 2) define, and give a brief treatment of, fractional derivatives in the spaces $L^p((0, T); X)$ and then (Section 3) consider these derivatives in spaces of continuous functions having a prescribed singularity as $t \downarrow 0$. In Section 4 we characterize the corresponding trace spaces at $t = 0$ and show how these spaces depend on α.

In Section 5 we consider the maximal regularity of the linear equation

$$D_t^\alpha(u - x) + Au = f, \quad u(0) = x, \tag{1}$$

where again $\alpha \in (0, 2)$ and where the setting is the space of continuous functions having at most a prescribed singularity as $t \downarrow 0$. To obtain maximal regularity we make a further assumption on E_0, E_1.

In Section 6 we analyze the nonautonomous, $A = A(t)$, version of (1). Here we assume that for each fixed t the corresponding operator admits maximal regularity and deduce maximal regularity of the nonautonomous case.

In Sections 7 and 8 we combine our results of the previous sections with a contraction mapping technique to obtain existence, uniqueness, and continuation results on

$$D_t^\alpha(u-x)+A(u)u=f(u)+h(t), \quad u(0)=x.$$

Finally, in Section 9, we present an application of our results to the nonlinear equation

$$D_t^\alpha(u-u_0)-(\sigma(u_x))_x=h(t), \quad x\in(0,1), \quad t\geqslant 0,$$

with $u=u(t,x)$, $u(0,x)=u_0(x)$, $\alpha\in(0,2)$, Dirichlet boundary conditions, σ monotone increasing and sufficiently smooth.

This equation occurs in nonlinear viscoelasticity, and has been studied, e.g., in [10,12].

Parabolic evolution equations, linear and quasilinear, have been considered by several authors using different approaches. Of particular interest to our approach are the references, among others, [1,2,8,15]. The reader may consult [7] for more detailed comments on the relevant literature.

It should also be observed that we draw upon results of [4], where (1) is considered in spaces of continuous functions on $[0,T]$, i.e., without allowance for any singularity at the origin.

2. Fractional derivatives in L^p

We recall [20, II, pp. 134–136] the following definition and the ensuing properties. Let X be a Banach space and write

$$g_\beta(t)=\frac{1}{\Gamma(\beta)}t^{\beta-1}, \quad t>0, \quad \beta>0.$$

Definition 1. Let $u\in L^1((0,T);X)$ for some $T>0$. We say that u has a fractional derivative of order $\alpha>0$ provided $u=g_\alpha * f$ for some $f\in L^1((0,T);X)$. If this is the case, we write $D_t^\alpha u=f$.

Note that if $\alpha=1$, then the above condition is sufficient for u to be absolutely continuous and differentiable a.e. with $u'=f$ a.e.

Tradition has that the word fractional is used to characterize derivatives of noninteger order, although α may of course be any positive real number.

The fractional derivative (whenever existing) is essentially unique. Observe the consistency; if $u=g_\alpha * f$, and $\alpha\in(0,1)$, then $f=D_t^\alpha u=\frac{d}{dt}(g_{1-\alpha}*u)$. Thus, if u has a fractional derivative of order $\alpha\in(0,1)$, then $g_{1-\alpha}*u$ is differentiable a.e. and absolutely continuous. Also note a trivial consequence of the definition; i.e., $D_t^\alpha(g_\alpha * u)=u$.

Suppose $\alpha\in(0,1)$. By the Hausdorff–Young inequality one easily has that if the fractional derivative f of u satisfies $f\in L^p((0,T);X)$ with $p\in[1,\frac{1}{\alpha})$, then

$u \in L^q((0,T);X)$ for $1 \leqslant q < \frac{p}{1-\alpha p}$. Furthermore, if $f \in L^p((0,T);X)$; with $p = \alpha^{-1}$, then $u \in L^q((0,T);X)$ for $q \in [1,\infty)$. If $f \in L^p((0,T);X)$ with $\alpha^{-1} < p$, then $u \in h_{0\to 0}^{\alpha - \frac{1}{p}}([0,T];X)$ [20, II, p. 138]. In particular note that $u(0)$ is now well defined and that one has $u(0) = 0$. (By $h_{0\to 0}^{\theta}$ we denote the little-Hölder continuous functions having modulus of continuity θ and vanishing at the origin.)

The extension of the last statement to higher order fractional derivatives is obvious. Thus, if u has a fractional derivative f of order $\alpha \in (1,2)$ and $f \in L^p$ with $(\alpha - 1)^{-1} < p$, then $u_t \in h_{0\to 0}^{\alpha - 1 - p^{-1}}$.

We also note that if $u \in L^1((0,T);X)$ with $D_t^\alpha u \in L^\infty((0,T);X)$, $\alpha \in (0,1)$, then $u \in C_{0\to 0}^\alpha([0,T];X)$. The converse is not true, for $u \in C_{0\to 0}^\alpha([0,T];X)$ the fractional derivative of order α of u does not necessarily even exist. To see this, take $v \in \Lambda_*$ [20, I, p. 43], then [20, II, Theorem 8.14(ii), p. 136] $D_t^{1-\alpha} v \in C^\alpha([0,T];X)$. Without loss of generality, assume $D_t^{1-\alpha} v$ vanishes at $t = 0$. Assume that there exists $f \in L^1((0,T);X)$ such that

$$D_t^{1-\alpha} v = t^{-1+\alpha} * f.$$

But this implies (convolve by $t^{-\alpha}$) $v = 1 * f$, which does not in general hold for $v \in \Lambda_*$ [20, I, p. 433].

The following proposition shows that the L^p-fractional derivative is the fractional power of the realization of the derivative in L^p.

Proposition 2. *Let* $1 \leqslant p < \infty$ *and define*

$$\mathscr{D}(L) \stackrel{\text{def}}{=} W_0^{1,p}((0,T);X),$$

and

$$Lu \stackrel{\text{def}}{=} u', \quad u \in \mathscr{D}(L).$$

Then L is m-accretive in $L^p((0,T);X)$ *with spectral angle* $\frac{\pi}{2}$. *With* $\alpha \in (0,1)$ *we have*

$$L^\alpha u = D_t^\alpha u, \quad u \in \mathscr{D}(L^\alpha),$$

where in fact $\mathscr{D}(L^\alpha)$ *coincides with the set of functions u having a fractional derivative in* L^p, *i.e.*,

$$\mathscr{D}(L^\alpha) = \left\{ u \in L^p((0,T);X) \mid g_{1-\alpha} * u \in W_0^{1,p}((0,T);X) \right\}.$$

Moreover, L^α *has spectral angle* $\frac{\alpha\pi}{2}$.

We only briefly indicate the proof of this known result. (Cf. the proof of Proposition 5 below.)

The fact that L is m-accretive and has spectral angle $\frac{\pi}{2}$ is well known. See, e.g., [3, Theorem 3.1]. The representation formula given in the proof of Proposition 5 and the arguments following give the equality of L^α and D_t^α. The reasoning used to prove [4, Lemma 11(b)] can be adapted to give that L^α has spectral angle $\frac{\alpha\pi}{2}$.

We remark that if X has the UMD-property then (in $L^p((0,T);X)$ with $1<p<\infty$) we have

$$\mathscr{D}(L^\alpha) = \mathscr{D}(D_t^\alpha) = [L^p((0,T);X); W_0^{1,p}((0,T);X)]_\alpha.$$

See [9, p. 20] or [19, pp. 103–104], and observe that $\frac{d}{dt}$ admits bounded imaginary powers in $L^p((0,T);X)$.

3. Fractional derivatives in $\mathbf{BUC}_{1-\mu}$

Let X be a Banach space and $T>0$. We consider functions defined on $J_0 = (0,T]$ having (at most) a singularity of prescribed order at $t=0$.

Let $J=[0,T]$, $\mu\in(0,1)$, and define

$$\begin{aligned} &BUC_{1-\mu}(J,X) \\ &\quad = \{u\in C(J_0;X) | t^{1-\mu}u(t)\in BUC(J_0;X), \lim_{t\downarrow 0} t^{1-\mu}||u(t)||_X = 0\}, \end{aligned}$$

with

$$||u||_{BUC_{1-\mu}(J,X)} \stackrel{\text{def}}{=} \sup_{t\in J_0} t^{1-\mu}||u(t)||_X. \tag{2}$$

(In this paper, we restrict ourselves to the case $\mu\in(0,1)$. The case $\mu=1$ was considered in [4].) It is not difficult to verify that $BUC_{1-\mu}(J;X)$, with the norm given in (2), is a Banach space. Note the obvious fact that for $T_1>T_2$ we may view $BUC_{1-\mu}([0,T_1];X)$ as a subset of $BUC_{1-\mu}([0,T_2];X)$, and also that if $u\in BUC_{1-\mu}([0,T];X)$ for some $T>0$, then (for this same u) one has

$$\lim_{\tau\downarrow 0} ||u||_{BUC_{1-\mu}([0,\tau];X)} = 0. \tag{3}$$

Moreover, one easily deduces the inequality

$$||u||_{L^p(J;X)} \leqslant c||u||_{BUC_{1-\mu}(J;X)}, \quad \mu\in(0,1), \quad 1\leqslant p<(1-\mu)^{-1},$$

and so, for these (μ,p)-values,

$$BUC_{1-\mu}(J;X)\subset L^p(J;X),$$

with dense imbedding. To see that this last fact holds, recall that $C(J,X)$ is dense in $L^p(J;X)$ and that obviously $C(J,X)\subset BUC_{1-\mu}(J;X)$.

We make the following fundamental assumption:

$$\alpha + \mu > 1. \tag{4}$$

To motivate this assumption, suppose we require (as we will do) that both u and $D_t^\alpha u$ lie in $BUC_{1-\mu}$ and that $u(0)$ $(= 0)$ is well defined. The requirement $D_t^\alpha u \in BUC_{1-\mu}$ implies, by the above, $D_t^\alpha u \in L^p((0,T);X)$; for $1 \leqslant p < \frac{1}{1-\mu}$. On the other hand, if $D_t^\alpha u \in L^p$ with $\alpha^{-1} < p$ then $u \in h_{0\to 0}^{\alpha - \frac{1}{p}}$ and $u(0)$ $(= 0)$ is well defined. Thus our requirements motivate the assumption that the interval $(\alpha^{-1}, (1-\mu)^{-1})$ be nonempty. But this is (4).

Therefore, under the assumption (4), the following definition makes sense.

$$BUC_{1-\mu}^\alpha(J;X) \stackrel{\text{def}}{=} \{u \in BUC_{1-\mu}(J;X)| \text{ there exist } x \in X \text{ and } f \in BUC_{1-\mu}(J;X) \text{ such that } u = x + g_\alpha * f\}. \tag{5}$$

We keep in mind that if $u \in BUC_{1-\mu}^\alpha(J;X)$, then (assuming (4)) $u(0) = x$ and u is Hölder-continuous.

We equip $BUC_{1-\mu}^\alpha(J;X)$ with the following norm:

$$||u||_{BUC_{1-\mu}^\alpha(J;X)} \stackrel{\text{def}}{=} ||u||_{BUC_{1-\mu}(J;X)} + ||D_t^\alpha(u-x)||_{BUC_{1-\mu}(J;X)}. \tag{6}$$

Lemma 3. *Let* $\alpha > 0$, $\mu \in (0,1)$, *and let* (4) *hold. Space* (5), *equipped with norm* (6), *is a Banach space. In particular*, $BUC_{1-\mu}^\alpha(J,X) \subset BUC(J,X)$.

Proof. Take $\{w_n\}_{n=1}^\infty$ to be a Cauchy-sequence in $BUC_{1-\mu}^\alpha(J;X)$. Then, by (6), and as $BUC_{1-\mu}(J;X)$ is a Banach space, there exists $w \in BUC_{1-\mu}(J;X)$ such that $||w_n - w||_{BUC_{1-\mu}(J;X)} \to 0$. Moreover, $f_n \stackrel{\text{def}}{=} D_t^\alpha(w_n - w_n(0))$ converges in $BUC_{1-\mu}(J;X)$ to some function z.

We claim that $w(0)$ is well defined and that $z = D_t^\alpha(w - w(0))$. To this end, note that

$$w_n(t) - w_n(0) = g_\alpha * f_n = g_\alpha * z + g_\alpha * [f_n - z]. \tag{7}$$

We have $\lim_{n\to\infty} ||t^{1-\mu}[f_n(t) - z(t)]||_X = 0$, uniformly on J. Thus, by (4), $\lim_{n\to\infty} ||g_\alpha * [f_n - z]||_X = 0$, uniformly on J. So, uniformly on J,

$$\lim_{n\to\infty} [w_n(t) - w_n(0)] = g_\alpha * z.$$

For each fixed $t>0$, $\{w_n(t)\}_{n=1}^{\infty}$ converges to $w(t)$ in X. Thus $\{w_n(0)\}_{n=1}^{\infty}$ must converge in X and by (4) and (7) we must have $w_n(0)\to w(0)$ as $n\to\infty$. For the proof of the last statement, use the considerations preceding the theorem. □

Our next purpose is to consider in more detail differentiation on $\tilde{X}\overset{\text{def}}{=}BUC_{1-\mu}(J;X)$ and to connect the fractional powers of this operation with that of taking fractional derivatives. First consider the derivative of integer order.

Take $\alpha=1$ in (5), (6), (thus $\alpha+\mu>1$) and define

$$\mathscr{D}(\tilde{L})\overset{\text{def}}{=}\left\{u\in BUC^1_{1-\mu}(J;X)\mid u(0)=0\right\},$$

and

$$\tilde{L}u=u'(t),\quad u\in\mathscr{D}(\tilde{L}).$$

We have

Lemma 4.

(i) *$\mathscr{D}(\tilde{L})$ is dense in $\tilde{X}$,*
(ii) *$\tilde{L}$ is a positive operator in $\tilde{X}$, with spectral angle $\frac{\pi}{2}$.*

Proof. (i) Clearly, $\tilde{Y}\overset{\text{def}}{=}\{u\in C^1(J;X)\mid u(0)=0\}\subset\mathscr{D}(\tilde{L})$. It is therefore sufficient to prove that $\tilde{Y}$ is dense in $\tilde{X}$. Observe that $\tilde{Y}\subset C_{0\to 0}(J;X)\subset\tilde{X}$. It is well known that $\tilde{Y}$ is dense in $C_{0\to 0}(J;X)$ with respect to the sup-norm (which is stronger than the norm in $\tilde{X}$). So it suffices to prove that $C_{0\to 0}(J;X)$ is dense in $\tilde{X}$.

Let $u\in\tilde{X}$. There exists $v\in C_{0\to 0}(J;X)$ such that $u(t)=t^{\mu-1}v(t)$, $t\in(0,T]$. Set, for n large enough,

$$v_n(t)=\begin{cases}0, & t\in[0,\frac{1}{n}],\\ v(t-\frac{1}{n}), & t\in(\frac{1}{n},T],\end{cases}$$
$$u_n(t)=t^{\mu-1}v_n(t),\quad t\in(0,T],\quad u_n(0)=0.$$

Then $u_n(t)\in C_{0\to 0}(J;X)$, and

$$\begin{aligned}\sup_{t\in(0,T]}\|t^{1-\mu}[u(t)-u_n(t)]\|_X &= \sup_{t\in(0,T]}\|v(t)-v_n(t)\|_X\\ &\leqslant \sup_{0\leqslant t\leqslant\frac{1}{n}}\|v(t)\|_X+\sup_{\frac{1}{n}<t\leqslant T}\left\|v(t)-v\left(t-\frac{1}{n}\right)\right\|_X\to 0,\end{aligned}$$

as $n\to\infty$. It follows that $C_{0\to 0}(J;X)$ is dense in $\tilde{X}$ and (i) holds.

(ii). First, note that $\tilde{X} \subset L^1(J;X)$ and that for every $\lambda \in \mathbb{C}$ and every $f \in L^1(J;X)$, the problem

$$\lambda u + u' = f, \quad u(0) = 0,$$

has a unique solution $u \in W_0^{1,1}((0,T);X) \subset C_{0\to 0}([0,T];X)$, given by

$$u(t) = \int_0^t \exp[-\lambda(t-s)] f(s)\, ds, \quad t \in J.$$

We use this expression to estimate

$$\sup_{|arg\ \lambda| \leqslant \theta} \sup_{t \in (0,T]} |\lambda| t^{1-\mu} ||u(t)||_X,$$

in case $f \in \tilde{X}$ and $\theta \in [0, \frac{\pi}{2})$. Thus

$$\begin{aligned} ||\lambda t^{1-\mu} u(t)||_X &\leqslant t^{1-\mu} \int_0^t |\lambda| \exp[-\Re\lambda(t-s)] s^{\mu-1}\, ds\, ||f||_{\tilde{X}} \\ &\leqslant \frac{1}{\cos\theta} t^{1-\mu} \int_0^t (\Re\lambda) \exp[-\Re\lambda(t-s)] s^{\mu-1}\, ds ||f||_{\tilde{X}}. \end{aligned}$$

We write $\eta \overset{\text{def}}{=} \Re\lambda > 0$, $\tau \overset{\text{def}}{=} \eta s$, to obtain

$$\begin{aligned} &(\cos\theta)^{-1} t^{1-\mu} \int_0^t (\Re\lambda) \exp[-\Re\lambda(t-s)] s^{\mu-1}\, ds \\ &\quad = (\cos\theta)^{-1} (\eta t)^{1-\mu} \int_0^{\eta t} \exp[-\eta t + \tau] \tau^{\mu-1}\, d\tau \leqslant c_\theta, \end{aligned}$$

where c_θ is independent of $\eta > 0$, $t > 0$. To see that the last inequality holds, first observe that the expression to be estimated only depends on the product ηt (and on μ, θ). Then split the integral into two parts, over $(0, \frac{\eta t}{2})$, and over $(\frac{\eta t}{2}, \eta t)$, respectively (cf. [2, p. 106]).

We conclude that the spectral angle of $\tilde{L}$ is not strictly greater that $\frac{\pi}{2}$.

Finally, assume that the spectral angle is less than $\frac{\pi}{2}$. Then $-\tilde{L}$ would generate an analytic semigroup. To obtain a contradiction, observe that $\tilde{L}$ is the restriction to $\tilde{X}$ of $\tilde{L}_1$ considered on $L^1((0,T);X)$, where $\mathscr{D}(\tilde{L}_1) = W^{1,1}_{0\to 0}((0,T);X)$; $\tilde{L}_1 u \overset{\text{def}}{=} u'$; $u \in \mathscr{D}(\tilde{L}_1)$. Thus the analytic semigroup $T(t)$ generated by $-\tilde{L}$ would be the restriction to $\tilde{X}$ of right translation, i.e.,

$$(T(t)f)(s) = \begin{cases} f(s-t), & 0 \leqslant t \leqslant s, \\ 0, & s < t. \end{cases}$$

But $\tilde{X}$ is not invariant under right translation. By this contradiction, (ii) follows and Lemma 4 is proved. □

Proceeding next to the fractional powers and fractional derivatives we have:

Proposition 5. *Let* $\alpha, \mu \in (0,1)$. *Then*

$$\mathscr{D}(\tilde{L}^{\alpha}) = \mathscr{D}(D_t^{\alpha}) \stackrel{\text{def}}{=} \{u \in \tilde{X} \mid u = g_\alpha * f \text{ for some } f \in \tilde{X}\},$$

and $\tilde{L}^{\alpha}u = D_t^{\alpha}u$, *for* $u \in \mathscr{D}(\tilde{L}^{\alpha})$. *Moreover,*

$$D_t^{\alpha} \text{ is positive, densely defined on } \tilde{X}, \text{ and has spectral angle } \frac{\alpha\pi}{2}. \tag{8}$$

Proof. We first show that

$$(\tilde{L}^{-1})^{\alpha} f = g_\alpha * f, \quad \text{for } f \in \tilde{X}. \tag{9}$$

Observe that $0 \in \rho(\tilde{L})$, and that $\tilde{L}$ is positive. Thus

$$(\tilde{L}^{-1})^{\alpha} f = \tilde{L}^{-\alpha} f = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} s^{-\alpha}(sI + \tilde{L})^{-1} f \, ds,$$

where the integral converges absolutely. But

$$(sI + \tilde{L})^{-1} f = \int_0^t \exp[-s(t-\sigma)] f(\sigma) \, d\sigma, \quad 0 \leqslant t \leqslant T,$$

and so, after a use of Fubini's theorem,

$$(\tilde{L}^{-1})^{\alpha} f = \int_0^t \left(\int_0^{\infty} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} s^{-\alpha} \exp[-s\sigma] ds \right) f(t-\sigma) \, d\sigma.$$

To obtain (9), note that the inner integral equals $g_\alpha(\sigma)$.

Let $u \in \mathscr{D}(D_t^{\alpha})$. Then $u = g_\alpha * f$, with $D_t^{\alpha} u = f \in \tilde{X}$. So, by (9), $u = (\tilde{L}^{-1})^{\alpha} f$, which implies $u \in \mathscr{D}(\tilde{L}^{\alpha})$ and $\tilde{L}^{\alpha} u = f$.

Conversely, let $u \in \mathscr{D}(\tilde{L}^{\alpha})$. Then, for some $f \in \tilde{X}$, $\tilde{L}^{\alpha} u = f$, and so $u = (\tilde{L}^{\alpha})^{-1} f$. By (9), this gives $u = g_\alpha * f$ and so $u \in \mathscr{D}(D_t^{\alpha})$.

We conclude that $\mathscr{D}(\tilde{L}^{\alpha}) = \mathscr{D}(D_t^{\alpha})$ and that $\tilde{L}^{\alpha} u = D_t^{\alpha} u$, $u \in \mathscr{D}(\tilde{L}^{\alpha})$.

To get that D_t^{α} is densely defined, use (i) of Lemma 4 and apply, e.g., [18, Proposition 2.3.1]. The fact that the spectral angle is $\frac{\alpha\pi}{2}$ follows, e.g., by the same arguments as those used to prove [4, Lemma 11(b)]. □

Analogously, higher order fractional derivatives may be connected to fractional powers. We have, e.g., the following statement.

Proposition 6. *Let* $\alpha, \mu \in (0,1)$. *Define*

$$\mathscr{D}(D_t^{1+\alpha}) \stackrel{\text{def}}{=} \Big\{u \in BUC_{1-\mu}^1([0,T];X) \mid u(0) = 0,\ u_t \in \mathscr{D}(D_t^\alpha)\Big\},$$

and $D_t^{1+\alpha}u = D_t^\alpha u_t$, *for* $u \in \mathscr{D}(D_t^{1+\alpha})$. *Then*

$$\tilde{L}^{1+\alpha}u = D_t^\alpha u_t, \quad u \in \mathscr{D}(D_t^{1+\alpha}).$$

Moreover, $\tilde{L}^{1+\alpha}$ *is positive, densely defined on* $\tilde{X}$ *with spectral angle* $\frac{(1+\alpha)\pi}{2}$ *and with* (*cf*. (9)),

$$(\tilde{L}^{1+\alpha})^{-1}f = g_{1+\alpha} * f, \quad \text{for } f \in \tilde{X}.$$

For the proof of Proposition 6, first use Proposition 5 and the definition $D_t^{1+\alpha}u = D_t^\alpha u_t$, $u \in \mathscr{D}(D_t^{1+\alpha})$. To obtain the size of the spectral angle one may argue as in the proof of [5, Lemma 8(a)].

4. Trace spaces

Let E_1, E_0 be Banach spaces with $E_1 \subset E_0$ and dense imbedding and let A be an isomorphism mapping E_1 into E_0. Take $\alpha \in (0,2)$, $\mu \in (0,1)$. Further, let A as an operator in E_0 be nonnegative with spectral angle ϕ_A satisfying

$$\phi_A < \pi\Big(1 - \frac{\alpha}{2}\Big).$$

Assume (4) holds and write $J = [0,T]$.

We consider the spaces

$$\tilde{E}_0(J) \stackrel{\text{def}}{=} BUC_{1-\mu}(J;E_0), \tag{10}$$

$$\tilde{E}_1(J) \stackrel{\text{def}}{=} BUC_{1-\mu}(J;E_1) \cap BUC_{1-\mu}^\alpha(J;E_0), \tag{11}$$

and equip $\tilde{E}_1(J)$ with the norm

$$||u||_{\tilde{E}_1(J)} \stackrel{\text{def}}{=} \sup_{t \in (0,T]} t^{1-\mu}\Big[||f(t)||_{E_0} + ||u(t)||_{E_1}\Big],$$

where f is defined through the fact that $u \in \tilde{E}_1(J)$ implies $u = x + g_\alpha * f$, for some $f \in \tilde{E}_0(J)$.

Without loss of generality, we take $||y||_{E_1} = ||Ay||_{E_0}$, for $y \in E_1$, and note that by Lemma 3, $\tilde{E}_1(J)$ is a Banach space. We write

$$E_\theta \stackrel{\text{def}}{=} (E_0, E_1)_\theta \stackrel{\text{def}}{=} (E_0, E_1)_{\theta,\infty}^0, \quad \theta \in (0,1),$$

for the continuous interpolation spaces between E_0 and E_1. Recall that if η is some number such that $0 \leqslant \eta < \pi - \phi_A$, then

$$x \in E_\theta \quad \text{iff} \lim_{|\lambda| \to \infty, |arg\ \lambda| \leqslant \eta} ||\lambda^\theta A(\lambda I + A)^{-1} x||_{E_0} = 0, \tag{12}$$

and that we may take

$$||x||_\theta \overset{\text{def}}{=} \sup_{|arg\ \lambda| \leqslant \eta, \lambda \neq 0} ||\lambda^\theta A(\lambda I + A)^{-1} x||_{E_0}$$

as norm on E_θ (see [13, Theorem 3.1, p. 159] and [14, p. 314]).

Our purpose is to investigate the trace space of $\tilde{E}_1(J)$.

We define

$$\gamma : \tilde{E}_1(J) \to E_0 \quad \text{by } \gamma(u) = u(0),$$

and the trace space $\gamma(\tilde{E}_1(J)) \overset{\text{def}}{=} Im(\gamma)$, with

$$||x||_{\gamma(\tilde{E}_1(J))} \overset{\text{def}}{=} \inf\{||v||_{\tilde{E}_1(J)} \mid v \in \tilde{E}_1(J),\ \gamma(v) = x\}.$$

It is straightforward to show that this norm makes $\gamma(\tilde{E}_1(J))$ a Banach space.

Define

$$\hat{\mu} = 1 - \frac{1-\mu}{\alpha}$$

for $\mu \in (0,1)$, $\alpha \in (0,2)$ with $\alpha + \mu > 1$. Observe that this very last condition is equivalent to $\hat{\mu} > 0$ and that $\alpha < 1$ implies $\hat{\mu} < \mu$, whereas $\alpha \in (1,2)$ gives $\mu < \hat{\mu}$. Thus

$$0 < \hat{\mu} < \mu < 1, \quad \alpha \in (0,1); \quad 0 < \mu < \hat{\mu} < 1, \quad \alpha \in (1,2).$$

Obviously, if $\alpha = 1$, then $\hat{\mu} = \mu$.

We claim

Theorem 7. *For* $\mu \in (0,1)$, $\alpha \in (0,2)$, $\alpha + \mu > 1$, *one has*

$$\gamma(\tilde{E}_1(J)) = E_{\hat{\mu}}.$$

Proof. The case $\alpha = 1$ is treated in [7]. Thus let $\alpha \neq 1$ and first consider the case $\alpha \in (0,1)$.

Let $x \in E_{\hat{\mu}}$. We define u as the solution of

$$u - x + g_\alpha * Au = 0, \quad t \in J, \tag{13}$$

or, equivalently, as the solution of

$$D_t^\alpha (u - x) + Au = 0, \quad t \in J. \tag{14}$$

By Clément et al. [4, Lemma 7], u is well defined and given by

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_{1,\psi}} \exp[\lambda t](\lambda^\alpha I + A)^{-1}\lambda^{\alpha-1}x\,d\lambda, \quad t>0, \tag{15}$$

Here $\psi \in (\frac{\pi}{2}, \min(\pi, \frac{\pi-\phi_A}{\alpha}))$ and

$$\Gamma_{r,\psi} \stackrel{\text{def}}{=} \{re^{it} \mid |t| \leqslant \psi\} \cup \{\rho e^{i\psi} \mid r<\rho<\infty\} \cup \{\rho e^{-i\psi} \mid r<\rho<\infty\}.$$

Note that $\lim_{t\downarrow 0} ||u(t) - x||_{E_0} = 0$. We assert that $\lim_{t\downarrow 0} ||t^{1-\mu}D_t^\alpha(u-x)||_{E_0} = 0$, i.e., that

$$\lim_{t\to 0} t^{1-\mu}\int_{\Gamma_{1,\psi}} \exp[\lambda t]A(\lambda^\alpha I + A)^{-1}\lambda^{\alpha-1}x\,d\lambda = 0 \tag{16}$$

in E_0. To show this assertion, we take $t>0$ arbitrary and rewrite the expression in (16) ($\stackrel{\text{def}}{=} I$) as follows:

$$\begin{aligned} I &= t^{1-\mu}\int_{\Gamma_{\frac{1}{t},\psi}} \exp[\lambda t]A(\lambda^\alpha I + A)^{-1}\lambda^{\alpha-1}x\,d\lambda \\ &= \int_{\Gamma_{1,\psi}} \exp[s]\left(\left(\frac{s}{t}\right)^{\alpha\hat{\mu}} A\left\{\left(\frac{s}{t}\right)^\alpha I + A\right\}^{-1} x\right)s^{-\mu}\,ds. \end{aligned} \tag{17}$$

The first equality followed by analyticity; to obtain the second we made the variable transform $s \stackrel{\text{def}}{=} \lambda t$ and used the definition of $\hat{\mu}$.

Now recall that $x \in E_{\hat{\mu}}$ and use (12) in (17) to get (16). Observe also that by the above one has

$$\sup_{t\in J_0} ||t^{1-\mu}D_t^\alpha(u-x)||_{E_0} \leqslant c||x||_{E_{\hat{\mu}}}, \tag{18}$$

where $c = c(\mu, \psi)$ but where c does not depend on T.

By (14), (16), (18),

$$\sup_{t\in J_0} ||t^{1-\mu}Au(t)||_{E_0} \leqslant c||x||_{E_{\hat{\mu}}}, \quad \lim_{t\downarrow 0} ||t^{1-\mu}Au(t)||_{E_0} = 0. \tag{19}$$

Continuity of $Au(t)$ and $D_t^\alpha(u-x)$ in E_0 for $t\in(0,T]$ follows from (15). One concludes that

$$E_{\hat{\mu}} \subset \gamma(\tilde{E}_1(J)). \tag{20}$$

Observe that we also have:

$$\text{If} \quad x \in E_{\hat{\mu}}, \text{ and } u \text{ solves (13), then } u \in \tilde{E}_1(J). \tag{21}$$

Conversely, take $x \in \gamma(\tilde{E}_1(J))$ and take $v \in \tilde{E}_1(J)$ such that $v(0) = x$. Then

$$H_0(t) \stackrel{\text{def}}{=} t^{1-\mu} D_t^\alpha (v - x) \in BUC_{0\to 0}(J; E_0),$$

$$H_1(t) \stackrel{\text{def}}{=} t^{1-\mu} Av(t) \in BUC_{0\to 0}(J; E_0).$$

It follows that, with $H \stackrel{\text{def}}{=} H_0 + H_1$,

$$D_t^\alpha (v - x) + Av(t) = t^{\mu-1} H(t). \tag{22}$$

We take the Laplace transform ($\lambda > 0$) of $t^{\mu-1}H(t)$ (take $H(t) = 0$, $t > T$), to obtain, in E_0,

$$\int_0^T \exp[-\lambda t] t^{\mu-1} H(t)\, dt = \lambda^{-\mu} \int_0^{\lambda T} \exp[-s] s^{\mu-1} H\left(\frac{s}{\lambda}\right) ds = o(\lambda^{-\mu}) \tag{23}$$

for $\lambda \to \infty$. For the last equality, use $H \in C_{0\to 0}(J; E_0)$.

Obviously, (23) holds with H replaced by H_0. Hence, by the way H_0 was defined and after some straightforward calculations,

$$\tilde{v} - \lambda^{-1} x = \lambda^{-\alpha} o(\lambda^{-\mu}) \quad \text{for } \lambda \to \infty. \tag{24}$$

Take transforms in (22), use (23), (24) to obtain

$$A(\lambda^\alpha I + A)^{-1} x = \lambda^{1-\alpha} o(\lambda^{-\mu}),$$

and so, in E_0,

$$\lambda^{\alpha\hat{\mu}} A(\lambda^\alpha I + A)^{-1} x \to 0, \quad \lambda \to \infty.$$

Hence $x \in E_{\hat{\mu}}$.

The case $\alpha \in (1, 2)$ follows in the same way. Again, define u by (13) (or (14)) but now use [5, Lemma 3] instead of [4, Lemma 7]. Note that one in fact takes $u_t(0) = 0$. Relations (15)–(19) remain valid and (20) follows. The proof of the converse part also carries over from the case where $\alpha \in (0, 1)$. □

We next show that $u \in \tilde{E}_1(J)$ implies that the values of u remain in $E_{\hat{\mu}}$. In particular, we have:

Theorem 8. *Let* $\mu \in (0, 1)$, $\alpha \in (0, 2)$ *and let* (4) *hold. Then*

$$\tilde{E}_1(J) \subset BUC(J; E_{\hat{\mu}}).$$

Proof. Take $u \in \tilde{E}_1(J)$. By Theorem 7, $u(0) \in E_{\hat{\mu}}$. We split u into two parts, writing $u = v + w$ where v, w satisfy

$$D_t^\alpha (v - u(0)) + Av(t) = 0, \quad v(0) = u(0) \in E_{\hat{\mu}}, \tag{25}$$

$$D_t^\alpha w + Aw(t) = t^{\mu-1} h(t), \quad w(0) = 0. \tag{26}$$

The function $h \in BUC_{0\to 0}(J; E_0)$ is defined through Eqs. (25), (26).

We consider the equations separately, beginning with the former. The claim is then that $v \in \tilde{E}_1(J) \cap BUC(J; E_{\hat{\mu}})$.

Take transforms in (25), use analyticity and invert to get, for $t > 0$,

$$v(t) - u(0) = -\frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{t},\psi}} \exp[\lambda t] \lambda^{-1} A(\lambda^\alpha I + A)^{-1} u(0)\, d\lambda,$$

and so

$$\begin{aligned} &\eta^{\hat{\mu}} A(\eta I + A)^{-1} (v(t) - u(0)) \\ &\quad = -\frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{t},\psi}} \exp[\lambda t] \lambda^{-1} A(\lambda^\alpha I + A)^{-1} \eta^{\hat{\mu}} A(\eta I + A)^{-1} u(0)\, d\lambda. \end{aligned}$$

Thus, using $u(0) \in E_{\hat{\mu}}$,

$$\begin{aligned} ||\eta^{\hat{\mu}} A(\eta I + A)^{-1} (v(t) - u(0))||_{E_0} &\leqslant \varepsilon \int_{\Gamma_{\frac{1}{t},\psi}} |\exp[\lambda t] \lambda^{-1}|\, d|\lambda| \\ &= \varepsilon \int_{\Gamma_{1,\psi}} |\exp[\tau]| |\tau|^{-1}\, d|\tau| \leqslant c\varepsilon, \end{aligned}$$

where $\varepsilon > 0$ arbitrary, and $\eta \geqslant \eta(\varepsilon)$ sufficiently large.

The conclusion is that $[v(t) - u(0)] \in E_{\hat{\mu}}$, for all $t > 0$. Moreover, $||v(t) - u(0)||_{E_{\hat{\mu}}} \leqslant c||u(0)||_{E_{\hat{\mu}}}$, and so

$$||v(t)||_{E_{\hat{\mu}}} \leqslant ||v(t) - u(0)||_{E_{\hat{\mu}}} + ||u(0)||_{E_{\hat{\mu}}} \leqslant [c+1] ||u(0)||_{E_{\hat{\mu}}}.$$

Continuity in $E_{\hat{\mu}}$ follows as in the proof of [4, Lemma 12f]. We infer that $v \in BUC(J; E_{\hat{\mu}})$.

The fact that $v \in \tilde{E}_1(J)$ is stated in (21).

We proceed to (26).

By assumption, $u \in \tilde{E}_1(J)$. Hence, $w = u - v \in \tilde{E}_1(J)$. We claim that $w \in BUC(J; E_{\hat{\mu}})$. To show this, first note that $w \in \tilde{E}_1(J)$, $w(0) = 0$, implies that

$$D_t^\alpha w = t^{\mu-1} h(t), \quad \text{where } h \in BUC_{0\to 0}(J; E_0), \tag{27}$$

and where $\sup_{t\in J} ||h(t)||_{E_0} \leqslant ||w||_{\tilde{E}_1(J)}$. So, after convolving (27) by $t^{-1+\alpha}$ and estimating in E_0,

$$||w(t)||_{E_0} \leqslant (\Gamma(\alpha))^{-1} ||w||_{\tilde{E}_1(J)} \int_0^t (t-s)^{-1+\alpha} s^{\mu-1}\, ds \leqslant \Gamma(1-\alpha) t^{\alpha+\mu-1} ||w||_{\tilde{E}_1(J)}. \quad (28)$$

Moreover,

$$||w(t)||_{E_1} = ||Aw(t)||_{E_0} \leqslant t^{\mu-1} ||w||_{\tilde{E}_1(J)}. \quad (29)$$

We interpolate between the two estimates (28),(29). To this end, recall that

$$K(\tau, w(t), E_0, E_1) \overset{\text{def}}{=} \inf_{w(t)=a+b} \Big(||a||_{E_0} + \tau ||b||_{E_1} \Big),$$

fix t, and choose $a = \frac{\tau}{\tau+t^\alpha} w(t)$, $b = \frac{t^\alpha w(t)}{\tau+t^\alpha}$. Then, by (28), (29),

$$K(\tau, w(t), E_0, E_1) \leqslant \frac{2\Gamma(1-\alpha)\tau t^{\alpha+\mu-1}}{\tau + t^\alpha} ||w||_{\tilde{E}_1(J)}.$$

So, without loss of generality,

$$\begin{aligned} ||w(t)||_{E_{\hat{\mu}}} &= \sup_{\tau\in(0,1]} \tau^{-\hat{\mu}} K(\tau, w(t), E_0, E_1) \\ &\leqslant \sup_{\tau\in(0,1]} \frac{2\Gamma(1-\alpha)\tau^{1-\hat{\mu}} t^{\alpha+\mu-1}}{\tau+t^\alpha} ||w||_{\tilde{E}_1(J)}. \end{aligned}$$

It is not hard to show that from this follows:

$$||w(t)||_{E_{\hat{\mu}}} \leqslant 2\Gamma(1-\alpha) ||w||_{\tilde{E}_1(J)}, \quad t\in J. \quad (30)$$

Finally observe that the same estimate holds with $J = [0, T]$ replaced by $J_1 = [0, T_1]$ for any $0 < T_1 < T$, and recall (3). Thus $w(t)$ is continuous in $E_{\hat{\mu}}$ at $t = 0$.

To have continuity for $t > 0$ it suffices to observe that since $w \in \tilde{E}_1(J)$, then $w \in BUC_{1-\mu}(J; \mathscr{D}(A))$, and so, (with $\mathscr{D}(A) = E_1$) a fortiori, $w \in C((0, T], E_{\hat{\mu}})$. Thus $w \in BUC([0, T]; E_{\hat{\mu}})$.

Adding up, we have $u = v + w \in BUC(J; E_{\hat{\mu}})$. Theorem 8 is proved. □

Corollary 9. *For $u \in \tilde{E}_1(J)$ with $\gamma(u) = 0$ one has*

$$||u||_{BUC(J,E_{\hat{\mu}})} \leqslant 2\Gamma(1-\alpha) ||u||_{\tilde{E}_1(J)}. \quad (31)$$

Proof. It suffices to note that if $u \in \tilde{E}_1(J)$, with $\gamma(u) = 0$, then v in (25) vanishes identically and $u = w$, (w as in (26)) and to recall (30). □

Next, we consider Hölder continuity.

Theorem 10. *Let* $\mu \in (0,1)$, $\alpha \in (0,2)$, $\alpha + \mu > 1$. *Then*

$$\tilde{E}_1(J) \subset BUC^{\alpha[1-\sigma]-[1-\mu]}(J, E_\sigma), \quad 0 \leqslant \sigma \leqslant \hat{\mu}.$$

Note that if $\alpha + \mu > 2$, then the Hölder exponent exceeds 1, provided $\sigma > 0$ is sufficiently small.

Proof. The case $\alpha = 1$ was in fact covered in [7]. The case $\sigma = \hat{\mu}$ was already considered above. In case $\sigma = 0$, the claim is

$$\tilde{E}_1(J) \subset BUC^{\alpha+\mu-1}(J, E_0).$$

To see that this claim is true, note that if $u \in \tilde{E}_1(J)$, then $D_t^\alpha(u - u(0)) = t^{\mu-1}h(t)$, where $h \in BUC_{0\to 0}(J, E_0)$ and $\sup_{t\in J} ||h(t)||_{E_0} \leqslant ||u(t)||_{\tilde{E}_1(J)}$. Then

$$||u(t) - u(0)||_{E_0} \leqslant \Gamma(1-\alpha) t^{\alpha+\mu-1} ||u||_{\tilde{E}_1((0,t))}. \tag{32}$$

So we have the desired Hölder continuity at $t = 0$ for $\sigma = 0$. The case $t > 0$ is straightforward and left to the reader.

There remains the case $\sigma \in (0, \hat{\mu})$. By the Reiteration theorem, $E_\sigma = (E_0, E_{\hat{\mu}})_{\frac{\sigma}{\hat{\mu}}}$, and by the interpolation inequality,

$$||u(t) - u(s)||_{E_\sigma} \leqslant c||u(t) - u(s)||_{E_0}^{1-\frac{\sigma}{\hat{\mu}}} ||u(t) - u(s)||_{E_{\hat{\mu}}}^{\frac{\sigma}{\hat{\mu}}},$$

Hence, for $s = 0$, using (32) and the fact that $||u(t)||_{E_{\hat{\mu}}}$ is bounded,

$$||u(t) - u(0)||_{E_\sigma} \leqslant ct^{[\alpha+\mu-1][1-\frac{\sigma}{\hat{\mu}}]} = ct^{\alpha[1-\sigma]-[1-\mu]}.$$

We leave the case $0 < s < t$ to the reader. □

5. Maximal regularity

Let E_1, E_0, A be as in Section 4. Let $\mu \in (0,1)$, $\alpha \in (0,2)$, $\alpha + \mu > 1$. We have shown that given $u \in \tilde{E}_1(J)$ we have $u(0) \in E_{\hat{\mu}}$. Also, by definition, if $u \in \tilde{E}_1(J)$, then

$$f \stackrel{\text{def}}{=} D_t^\alpha(u - u(0)) + Au \in \tilde{E}_0(J).$$

We now consider the converse question, i.e., the maximal regularity. We ask whether there exists $c>0$ such that

$$||u||_{\tilde{E}_1(J)} \leqslant c\Big[||f||_{\tilde{E}_0(J)} + ||x||_{E_{\hat{\mu}}}\Big],$$

where u solves $D_t^\alpha(u-x)+Au=f$.

By (21) and linearity we may obviously take $x=0$. Thus we let u solve

$$D_t^\alpha u + Au = f, \quad u(0)=0, \tag{33}$$

with $f \in \tilde{E}_0(J)$, and claim that $u \in \tilde{E}_1(J)$. This will follow only under a particular additional assumption on E_0, E_1.

We first need to formulate some definitions. We write, for $\omega \geqslant 0$,

$$\mathscr{H}_\alpha(E_1,E_0,\omega) \overset{\text{def}}{=} \Big\{A \in L(E_1,E_0) \mid A_\omega \overset{\text{def}}{=} \omega I + A$$

is a nonnegative closed operator in E_0 with spectral angle $<\pi(1-\frac{\alpha}{2})\big\}$ and

$$\mathscr{H}_\alpha(E_1,E_0) \overset{\text{def}}{=} \bigcup_{\omega \geqslant 0} \mathscr{H}_\alpha(E_1,E_0,\omega).$$

Note that as $\mathscr{H}_\alpha(E_1,E_0,\omega_1) \subset \mathscr{H}_\alpha(E_1,E_0,\omega_2)$, for $\omega_1<\omega_2$, we may as well take the union over, e.g., $\omega>0$. Also note that $\mathscr{H}_\alpha(E_1,E_0)$ is open in $L(E_1,E_0)$.

Furthermore, we let

$$\mathscr{M}_{\alpha\mu}(E_1,E_0) \overset{\text{def}}{=} \{A \in \mathscr{H}_\alpha(E_1,E_0) |\ D_t^\alpha u + Au = f,$$

$$u(0)=0, \text{ has maximal regularity in } \tilde{E}_0(J)\}.$$

Observe that using the assumption $\alpha+\mu>1$ one can show that if $D_t^\alpha u + Au = f$ has maximal regularity in $\tilde{E}_0(J)$, then $D_t^\alpha u + (\omega I + A)u = f$ has maximal regularity in $\tilde{E}_0(J)$ for any $\omega \in R$.

We equip $\mathscr{M}_{\alpha\mu}(E_1,E_0)$ with the topology of $L(E_1,E_0)$ and make the following assumptions on E_0, E_1.

Let F_1, F_0 be Banach spaces such that

$$E_1 \subset F_1 \subset E_0 \subset F_0, \tag{34}$$

and assume that there is an isomorphism $\tilde{A}: F_1 \to F_0$ such that $\tilde{A}$ (as an operator in F_0) is nonnegative with spectral angle $\phi_{\tilde{A}}$ satisfying

$$\phi_{\tilde{A}} < \pi\Big(1-\frac{\alpha}{2}\Big), \tag{35}$$

and such that for some $\theta \in (0,1)$,

$$E_0 = F_\theta \overset{\text{def}}{=} (F_0, F_1)_\theta^{0,\infty} \tag{36}$$

and such that

$$Ax = \tilde{A}x \quad \text{for } x \in E_1. \tag{37}$$

Our claim is that if $f \in \tilde{E}_0(J) = BUC_{1-\mu}(J, F_\theta)$, then Aw lies in the same space and we have a norm estimate. Specifically:

Theorem 11. *Let* $\mu \in (0,1)$, $\alpha \in (0,2)$, $\alpha + \mu > 1$. *Assume* (34), *let* $\tilde{A}$ *be as in* (35) *and suppose* (36), (37) *hold. Then* $A \in \mathcal{M}_{\alpha\mu}(E_1, E_0)$.

Proof. We define

$$\tilde{F}_0 = BUC_{1-\mu}(J; F_0); \quad \tilde{F}_1 = BUC_{1-\mu}(J; F_1).$$

Then

$$(\tilde{F}_0, \tilde{F}_1)_\theta = BUC_{1-\mu}(J; (F_0, F_1)_\theta) = BUC_{1-\mu}(J; E_0) = \tilde{E}_0(J).$$

To get the first equality above one recalls the characterization of F_0, F_1, and that by Clément et al. [4, Lemma 9(c)] the statement holds for $\mu = 1$. The cases $\mu \in (0,1)$ follow by an easy adaptation of the proof of [4, Lemma 9(c)]. The second equality above is (36), the third is the definition of $\tilde{E}_0(J)$.

Write, for $\alpha \in (0,2)$,

$$(\tilde{\mathscr{A}}u)(t) \overset{\text{def}}{=} \tilde{A}u(t); \quad u \in \mathscr{D}(\tilde{\mathscr{A}}) \overset{\text{def}}{=} \tilde{F}_1,$$

$$(\tilde{\mathscr{B}}u)(t) \overset{\text{def}}{=} D_t^\alpha u(t); \quad u \in \mathscr{D}(\tilde{\mathscr{B}}) \overset{\text{def}}{=} \left\{ u \mid u \in BUC_{1-\mu}^\alpha([0,T]; F_0); u(0) = 0 \right\}.$$

One then has, using (8), (35), and Proposition 6,

$$\tilde{\mathscr{A}} \text{ is positive, densely defined in } \tilde{F}_0, \text{ with spectral angle } < \pi\left(1 - \frac{\alpha}{2}\right),$$

$$\tilde{\mathscr{B}} \text{ is positive densely defined in } \tilde{F}_0 \text{ with spectral angle} = \frac{\pi\alpha}{2}.$$

Moreover, the operators $\tilde{\mathscr{A}}, \tilde{\mathscr{B}}$ are resolvent commuting and $0 \in \rho(\tilde{\mathscr{A}}) \cap \rho(\tilde{\mathscr{B}})$.

Consider the equation

$$\tilde{\mathscr{B}}u + \tilde{\mathscr{A}}u = f, \tag{38}$$

where $f \in \tilde{E}_0(J)$. By the Da Prato–Grisvard Method of Sums (in particular see [6, Theorem 4]) there exists a unique $u \in \mathscr{D}(\tilde{\mathscr{A}}) \cap \mathscr{D}(\tilde{\mathscr{B}})$ such that (38) holds, and such that $\tilde{\mathscr{A}}u, \tilde{\mathscr{B}}u \in \tilde{E}_0$ with

$$||\tilde{\mathscr{A}}u||_{\tilde{E}_0} \leqslant c||f||_{\tilde{E}_0},$$

where c is independent of f. Thus, recall (37), the function u satisfies (33), $u \in \tilde{E}_1(J)$, and there exists c such that

$$||u||_{\tilde{E}_1(J)} \leqslant c||f||_{\tilde{E}_0(J)}.$$

Observe that $c = c(T)$ but can be taken the same for all intervals $[0, T_1]$, with $T_1 \leqslant T$. $\square$

6. Linear nonautonomous equations

As earlier, we take $\mu \in (0,1)$, $\alpha \in (0,2)$, $\alpha + \mu > 1$, and define $\hat{\mu} = 1 - \frac{1-\mu}{\alpha}$. Consider the equation

$$u + g_\alpha * B(t)u = u_0 + g_\alpha * h. \tag{39}$$

We prove

Theorem 12. *Let E_0, E_1 be as in Section* 4, *let $T \in (0, \infty)$, $J = [0, T]$ and assume that*

$$B \in C(J; \mathscr{M}_{\alpha\mu}(E_1, E_0) \cap \mathscr{H}_\alpha(E_1, E_0, 0)),$$

$$u_0 \in E_{\hat{\mu}}, \quad h \in \tilde{E}_0(J). \tag{40}$$

Then there exists a unique $u \in \tilde{E}_1(J)$ solving (39) *such that $B(t)u(t) \in \tilde{E}_0(J)$ and there exists $c > 0$ such that*

$$||u||_{BUC_{1-\mu}(J;E_1)} + ||D_t^\alpha(u - u_0)||_{\tilde{E}_0(J)} \leqslant c\Big(||u_0||_{E_{\hat{\mu}}} + ||h||_{\tilde{E}_0(J)}\Big). \tag{41}$$

Proof. From (40) it follows that the norms

$$||x||_{E_{\hat{\mu}}} \stackrel{\text{def}}{=} \sup_{\lambda > 0} ||\lambda^{\hat{\mu}} B(s)(\lambda I + B(s))^{-1} x||_{E_0}$$

are all uniformly equivalent for $s \in [0, T]$.

Fix $s\in[0,T]$, $T'\in(0,T]$, and write $J'=[0,T']$. Let $u^{(s)}=u^{(s)}(t)$ be the solution of

$$D_t^\alpha(u^{(s)}-u_0)+B(s)u^{(s)}=h, \quad \text{on } J'.$$

We claim that there exists $c_1>0$, independent of s, T', such that

$$||D_t^\alpha(u^{(s)}-u_0)||_{\tilde{E}_0(J')}+||B(s)u^{(s)}(t)||_{\tilde{E}_0(J')}\leqslant c_1\Big(||u_0||_{E_{\hat{\mu}}}+||h||_{\tilde{E}_0(J')}\Big). \tag{42}$$

To prove (42), write $u^{(s)}=u_1^{(s)}+u_2^{(s)}$, where

$$D_t^\alpha(u_1^{(s)}-u_0)+B(s)u_1^{(s)}=0; \quad u_1^{(s)}(0)=u_0,$$

$$D_t^\alpha u_2^{(s)}+B(s)u_2^{(s)}=h; \quad u_2^{(s)}(0)=0.$$

By (18),

$$||D_t^\alpha(u_1^{(s)}-u_0)||_{\tilde{E}_0(J')}\leqslant c||u_0||_{E_{\hat{\mu}}},$$

where $c=c(\mu,\psi(s))$. By (40), $\psi(s)$, hence c, can be taken independent of s.

By the fact that B takes values in $\mathscr{M}_{\alpha\mu}(E_1,E_0)$ one has

$$||D_t^\alpha u_2^{(s)}||_{\tilde{E}_0(J')}+||B(s)u_2^{(s)}||_{\tilde{E}_0(J')}\leqslant\tilde{c}||h||_{\tilde{E}_0(J')},$$

and from the fact that $B\in C(J;L(E_1,E_0))$ one concludes that $\tilde{c}$ can be taken independent of s. Hence claim (42) holds.

Choose $n\geqslant 1$ such that with $q=n^{-1}T$ one has

$$c_1\max_{j=1,\dots,n;(j-1)q\leqslant t\leqslant jq}||B(t)-B((j-1)q)||_{L(E_1,E_0)}\leqslant\frac{1}{2}, \tag{43}$$

where c_1 as in (42). Fix $j\in\{1,2,\dots,n\}$, and assume we have a unique solution $\bar{u}_{j-1}$ of (39) on $[0,(j-1)q]$ (for $j=1$, take $\bar{u}_0=u_0$). Then define (recall (11))

$$\tilde{Z}_j=\big\{u\in\tilde{E}_1([0,jq]),u(0)=u_0\mid u(t)=\bar{u}_{j-1}(t), \quad 0\leqslant t\leqslant(j-1)q\big\}.$$

Given an arbitrary $v\in\tilde{Z}_j$, we let u_j be the unique solution of

$$u+g_\alpha*B((j-1)q)u=u_0+g_\alpha*h+g_\alpha*[B((j-1)q)-B(t)]v$$

on $[0,jq]$. Clearly, $[B((j-1)q)-B(t)]v\in BUC_{1-\mu}([0,jq];E_0)$. By uniqueness, $u_j\in\tilde{Z}_j$. Denote the map $v\in\tilde{Z}_j\to u_j\in\tilde{Z}_j$ by F_j. By (42),(43), and observing that $v_1=v_2$ on $[0,(j-1)q]$,

$$||F_j(v_1)-F_j(v_2)||_{\tilde{E}_1([0,jq])}\leqslant\frac{1}{2}||v_1-v_2||_{\tilde{E}_1([0,jq])}.$$

Observe that $\tilde{Z}_j$ is closed in $\tilde{E}_1([0,jq])$, hence it is a complete metric space with respect to the induced metric. Consequently we may apply the Contraction mapping Theorem and conclude that there exists a unique fixed point of F_j in $\tilde{Z}_j$. Denote this fixed point by $\bar{u}_j$. Clearly $\bar{u}_j$ solves (39) on $[0,jq]$.

Proceeding by induction we have the existence of a solution $u \in \tilde{E}_1(J)$ of (39). The induction procedure also gives $c>0$ such that (41) holds. $\square$

7. Local nonlinear theory

We consider the quasilinear equation

$$D_t^\alpha(u-u_0) + A(u)u = f(u) + h(t), \quad t>0, \tag{44}$$

under the following assumptions. Let

$$\mu \in (0,1) \quad \alpha \in (0,2), \quad \alpha+\mu>1, \tag{45}$$

and define $\hat{\mu}$ as earlier by $\hat{\mu} = \alpha^{-1}(\alpha+\mu-1)$. For X, Y Banach spaces, and g a mapping of X into Y, write $g \in C^{1-}(X,Y)$ if every point $x \in X$ has a neighbourhood U such that g restricted to U is globally Lipschitz continuous.

Let E_0, E_1 be Banach spaces such that $E_1 \subset E_0$ with dense imbedding and suppose

$$(A,f) \in C^{1-}(E_{\hat{\mu}}, \mathscr{M}_{\alpha\mu}(E_1,E_0) \times E_0), \tag{46}$$

$$u_0 \in E_{\hat{\mu}}, \quad h \in BUC_{1-\mu}([0,T];E_0), \quad \text{for any } T>0. \tag{47}$$

Observe that by (46), for $\tilde{u} \in E_{\hat{\mu}}$ there exists $\omega(\tilde{u}) \geqslant 0$ such that

$$A_\omega(\tilde{u}) \stackrel{\text{def}}{=} A(\tilde{u}) + \omega(\tilde{u})I \in \mathscr{H}_\alpha(E_1,E_0,0) \cap \mathscr{M}_{\alpha\mu}(E_1,E_0).$$

We define a solution u of (44) on an interval $J \subset \mathbb{R}^+$ containing 0 as a function u satisfying $u \in C(J,E_0) \cap C((0,T];E_1)$, $u(0)=u_0$, and such that the fractional derivative of $u-u_0$ of order α satisfies $D_t^\alpha(u-u_0) \in C((0,T];E_0)$ and such that (44) holds on $0<t\leqslant T$.

Our result is:

Theorem 13. *Let* (45)–(47) *hold, where* $E_{\hat{\mu}} = (E_0,E_1)_{\hat{\mu}}^{0,\infty}$ *is a continuous interpolation space. Then there exists a unique maximal solution u defined on the maximal interval of existence* $[0,\tau(u_0))$, *where* $\tau(u_0) \in (0,\infty]$, *and such that for every* $T<\tau(u_0)$ *one has*

(i) $u \in BUC_{1-\mu}([0,T];E_1) \cap BUC([0,T];E_{\hat{\mu}}) \cap BUC_{1-\mu}^{\alpha}([0,T];E_0)$,
(ii) $u + g_\alpha * A(u)u = u_0 + g_\alpha * (f(u)+h)$, $0 \leqslant t \leqslant T$,
(iii) *If* $\tau(u_0)<\infty$, *then* $u \notin UC([0,\tau(u_0));E_{\hat{\mu}})$,

(iv) *If* $\tau(u_0) < \infty$ *and* $E_1 \subset\subset E_0$, *then*

$$\limsup_{t\uparrow\tau(u_0)} ||u(t)||_{E_\delta} = \infty, \quad \textit{for any } \delta \in (\hat{\mu}, 1).$$

We recall that u defined on an interval J is called a maximal solution if there does not exist a solution v on an interval J' strictly containing J such that v restricted to J equals u. If u is a maximal solution, then J is called the maximal interval of existence.

In this section, we prove existence and uniqueness of u satisfying (i), (ii) for some $T > 0$. The continuation is dealt with in Section 8.

Proof of Theorem 13 (i), (ii). Choose ω such that $A_\omega(u_0) \in \mathscr{H}_\alpha(E_1, E_0, 0)$. Then $A_\omega(u_0) \in \mathscr{M}_\alpha(E_1, E_0)$ and there exists a constant c_{u_0}, independent of F, such that if $F \in \tilde{E}_0(J)$ and $u = u(F)$ solves

$$D_t^\alpha u + A_\omega(u_0)u = F(t), \quad 0 < t \leqslant T,$$

with $u(0) = 0$, then

$$||u||_{\tilde{E}_1([0,T])} \leqslant c_{u_0}(\Gamma(1-\alpha))^{-1}||F||_{\tilde{E}_0(J)}. \tag{48}$$

Define

$$B(u) = A(u_0) - A(u), \quad u \in E_{\hat{\mu}}.$$

Then $B \in C^{1-}(E_{\hat{\mu}}, L(E_1, E_0))$, and so, by (46) there exists $\rho_0 > 0$, $L \geqslant 1$ such that

$$||(B,f)(z_1) - (B,f)(z_2)||_{L(E_1,E_0)\times E_0} \leqslant L||z_1 - z_2||_{E_{\hat{\mu}}}, \tag{49}$$

for $z_1, z_2 \in \bar{B}_{E_{\hat{\mu}}}(u_0, \rho_0)$, and such that

$$||B(z)||_{L(E_1,E_0)} \leqslant \frac{1}{12c_{u_0}}; \quad z \in \bar{B}_{E_{\hat{\mu}}}(u_0, \rho_0). \tag{50}$$

Define b by

$$||f(z) + \omega(u_0)z||_{E_0} \leqslant b, \quad z \in \bar{B}_{E_{\hat{\mu}}}(u_0, \rho_0), \tag{51}$$

and

$$\varepsilon_0 = \min\left(\rho_0, \frac{1}{12c_{u_0}L}\right). \tag{52}$$

Let $\tilde{u}$ solve

$$D_t^\alpha(\tilde{u} - u_0) + A_\omega(u_0)\tilde{u} = 0, \quad \text{on } [0, T]. \tag{53}$$

Take $\tau > 0$ small enough so that ($\tilde{u}$ as in (53))

$$||\tilde{u} - u_0||_{E_{\hat{\mu}}} \leqslant \frac{\varepsilon_0}{2}, \quad t \in [0, \tau], \tag{54}$$

$$||\tilde{u}||_{\tilde{E}_1(J_\tau)} \leqslant \frac{\varepsilon_0}{2}, \tag{55}$$

$$\Gamma(1-\alpha)\tau^{1-\mu} \leqslant \min\left(\frac{\varepsilon_0}{12c_{u_0}b}, \frac{1}{12c_{u_0}(L+\omega(u_0))}\right), \tag{56}$$

$$||h||_{\tilde{E}_0(J_\tau)} \leqslant \frac{\varepsilon_0}{12c_{u_0}}, \tag{57}$$

where $J_\tau = [0, \tau]$. Define

$$W_{u_0}(J_\tau) = \left\{ v \in \tilde{E}_1(J_\tau) \mid v(0) = u_0, ||v - u_0||_{C(J_\tau, E_{\hat{\mu}})} \leqslant \varepsilon_0 \right\} \cap \bar{B}_{\tilde{E}_1(J_\tau)}(0, \varepsilon_0) \tag{58}$$

and give this set the topology of $\tilde{E}_1(J_\tau)$. Then $W_{u_0}(J_\tau)$ is a closed subset of $\tilde{E}_1(J_\tau)$, and therefore a complete metric space. Moreover, $W_{u_0}(J_\tau)$ is nonempty, because $\tilde{u} \in W_{u_0}(J_\tau)$.

Consider now the map

$$G_{u_0} : W_{u_0}(J_\tau) \to \tilde{E}_1(J_\tau)$$

defined by $u = G_{u_0}(v)$; $v \in W_{u_0}(J_\tau)$, where u solves

$$D_t^\alpha(u - u_0) + A_\omega(u_0)u = B(v)v + f(v) + \omega(u_0)v + h(t). \tag{59}$$

Our first claim is that this map is well defined. To see this, note that as $B \in C^{1-}(E_{\hat{\mu}}, L(E_1, E_0))$ and v is continuous in $E_{\hat{\mu}}$, and by the assumption on f, h it follows that the right-hand side of (59) is in $C((0, \tau]; E_0)$. Also, by (50), (51),(53), (56)–(58),

$$\begin{aligned}
&\sup_{0<t\leqslant\tau} t^{1-\mu}||B(v(t))v(t) + f(v(t)) + \omega(u_0)v(t) + h(t)||_{E_0} \\
&\quad \leqslant \sup_{0<t\leqslant\tau} (t^{1-\mu}||B(v(t))||_{L(E_1,E_0)}||v(t)||_{E_1}) + \tau^{1-\mu}b + ||h||_{\tilde{E}_0(J_\tau)} \\
&\quad \leqslant \frac{1}{12c_{u_0}}||v||_{\tilde{E}_1(J_\tau)} + \frac{\varepsilon_0}{12c_{u_0}} + \frac{\varepsilon_0}{12c_{u_0}} \leqslant \frac{\varepsilon_0}{4c_{u_0}}.
\end{aligned} \tag{60}$$

So the right-hand side of (59) is in $\tilde{E}_0(J_\tau)$, and hence, by (21),(48), (53), the map is well defined.

Next, we assert that $u \in W_{u_0}(J_\tau)$. We show first

$$\sup_{t\in[0,\tau]} ||G_{u_0}(v)(t) - u_0||_{E_{\hat{\mu}}} \leqslant \varepsilon_0. \tag{61}$$

Split $G_{u_0}(v)$:

$$G_{u_0}(v) = \tilde{u} + \tilde{G}_{u_0}(v), \tag{62}$$

where $\tilde{G}_{u_0}(v)$ solves (zero initial value)

$$D_t^\alpha(\tilde{G}_{u_0}(v)) + A_\omega(u_0)\tilde{G}_{u_0}(v) = B(v)v + f(v) + \omega(u_0)v + h(t).$$

By (31), (48), (60),

$$\begin{aligned}\sup_{t\in[0,\tau]} ||\tilde{G}_{u_0}(v)(t)||_{E_{\hat{\mu}}} &\leqslant 2\Gamma(1-\alpha)||\tilde{G}_{u_0}(v)||_{\tilde{E}_1(J_\tau)} \\ &\leqslant 2c_{u_0}||B(v)v + f(v) + \omega(u_0)v + h||_{\tilde{E}_0(J_\tau)} \leqslant 2c_{u_0}\frac{\varepsilon_0}{4c_{u_0}} = \frac{\varepsilon_0}{2}. \end{aligned} \tag{63}$$

Combining (54) and (63) we have (61).

Next, we assert that

$$||G_{u_0}(v)||_{\tilde{E}_1(J_\tau)} \leqslant \varepsilon_0.$$

To show this, split as in (62) and recall (55),(63). So $G_{u_0}(v) \in W_{u_0}(J_\tau)$.

Finally, we claim that G_{u_0} is a contraction. We have, by linearity and (31), (48), (49), (50),

$$\begin{aligned} &||G_{u_0}(v_1) - G_{u_0}(v_2)||_{\tilde{E}_1(J_\tau)} \\ &\quad \leqslant c_{u_0}||B(v_1)v_1 - B(v_2)v_2||_{\tilde{E}_0(J_\tau)} + c_{u_0}||f(v_1) - f(v_2)||_{\tilde{E}_0(J_\tau)} \\ &\qquad + c_{u_0}\omega(u_0)||v_1 - v_2||_{\tilde{E}_0(J_\tau)} \\ &\quad \leqslant c_{u_0}||[B(v_1) - B(v_2)]v_1||_{\tilde{E}_0(J_\tau)} + c_{u_0}||B(v_2)[v_1 - v_2]||_{\tilde{E}_0(J_\tau)} \\ &\qquad + c_{u_0}\tau^{1-\mu}[L + \omega(u_0)] \sup_t ||v_1(t) - v_2(t)||_{E_{\hat{\mu}}} \\ &\quad \leqslant c_{u_0}L||v_1 - v_2||_{\tilde{E}_1(J_\tau)} 2\Gamma(1-\alpha)||v_1||_{\tilde{E}_1(J_\tau)} + \frac{1}{12}||v_1 - v_2||_{\tilde{E}_1(J_\tau)} \\ &\qquad + 2\Gamma(1-\alpha)c_{u_0}\tau^{1-\mu}[L + \omega(u_0)]||v_1 - v_2||_{\tilde{E}_1(J_\tau)} \leqslant \frac{1}{2}||v_1 - v_2||_{\tilde{E}_1(J_\tau)}, \end{aligned}$$

where the last step follows by (52) and(56). Thus the map $v \to G(u_0)v$ is a contraction and has a unique fixed point.

We conclude that there exists u satisfying (i), (ii), for some $T > 0$.

We proceed to the proof of uniqueness. Assume there exist two functions u_1, u_2, both satisfying (i), (ii) on $[0, T]$ for some $T > 0$ and $u_1(t)$ not identically equal to $u_2(t)$ on $[0, T]$.

Define

$$\tau_1 = \sup\big\{t \in [0, T] \mid \text{ (44) has a unique solution in } \tilde{E}_1([0, t])\big\}.$$

Then $0 \leqslant \tau_1 < T$. Also, for any $\tau \in (\tau_1, T]$ there exists a solution u of (44) on $J_\tau \stackrel{\text{def}}{=} [0, \tau]$, such that $u(t) = u_1(t)$ on $[0, \tau_1]$ but u does not equal u_1 everywhere on $\tau_1 < t \leqslant \tau$. Let, for $\tau \in (\tau_1, T]$, $J_\tau = [0, \tau]$,

$$W_{u_1}(J_\tau) = \Big\{v \in \tilde{E}_1(J_\tau) \mid v(t) = u_1(t),\ 0 \leqslant t \leqslant \tau_1, \\ ||v - u_1||_{C(J_\tau; E_{\hat{\mu}})} \leqslant \varepsilon_0\Big\} \cap \bar{B}_{\tilde{E}_1(J_\tau)}(u_1(t), \varepsilon_0).$$

Give this set the topology of $\tilde{E}_1(J_\tau)$. Then $W_{u_1}(J_\tau)$ is a complete metric space which is nonempty because $u_1 \in W_{u_1}(J_\tau)$.

Consider the map $G_{u_1} : W_{u_1}(J_\tau) \to \tilde{E}_1(J_\tau)$ defined bu $u = G_{u_1}(v)$ for $v \in W_{u_1}(J_\tau)$, where u solves

$$D_t^\alpha(u - u_0) + A_\omega(u_1(\tau_1))u(t) = B(v(t))v(t) + f(v(t)) + \omega(u_1(\tau_1))v(t) + h(t),$$

with $B(v(t)) \stackrel{\text{def}}{=} A(u_1(\tau_1)) - A(v(t))$ and where we have chosen $\omega(u(\tau_1))$ such that $A_\omega(u_1(\tau_1)) \in \mathscr{H}_\alpha(E_1, E_0, 0)$. By (46), $A_\omega(u_1(\tau_1)) \in \mathscr{M}_{\alpha\mu}(E_1, E_0)$. Proceed as in the existence part to show that the map G_{u_1} is welldefined, and that for τ sufficiently close to τ_1 one has that G_{u_1} maps $W_{u_1}(J_\tau)$ into itself. Finally show that the map is a contraction if $\tau - \tau_1$ is sufficiently small and so the map has a unique fixed point. On the other hand, any solution of (44) is a fixed point of the map, provided τ (depends on the particular solution) is taken sufficiently close to τ_1. A contradiction results and uniqueness follows.

Thus we have shown that (i), (ii), and uniqueness hold for some $T > 0$.

8. Continuation of solutions

We proceed to the final part of the proof of Theorem 13.

Suppose we have a unique solution u of (44) on $J_\tau = [0, \tau]$, for some $\tau > 0$, such that

$$u \in C(J_\tau; E_{\hat{\mu}}) \cap \tilde{E}_1(J_\tau).$$

Take $T > \tau$ and let

$$Z \stackrel{\text{def}}{=} \Big\{ w \in C([0,T];E_{\hat{\mu}}) \mid w(t) = u(t),\ \ t \in [0,\tau],$$
$$(t-\tau)^{1-\mu} D_t^{\alpha}(w-u_0) \in BUC((\tau,T];E_0), ||[t-\tau]^{1-\mu} D_t^{\alpha}(w-u_0)||_{E_0} \to 0,\ \ t \downarrow \tau,$$
$$[t-\tau]^{1-\mu} w \in BUC((\tau,T];E_1);\ \ ||[t-\tau]^{1-\mu} w||_{E_1} \to 0,\ \ t \downarrow \tau \Big\}. \tag{64}$$

Choose ε_0 sufficiently small. Define

$$Z_u \stackrel{\text{def}}{=} \{ w \in Z \mid ||w - u(\tau)||_{C([\tau,T];E_{\hat{\mu}})} \leqslant \varepsilon_0,\ \ ||w||_{\tilde{E}_1([\tau,T])} \leqslant \varepsilon_0 \}. \tag{65}$$

Choose $\omega(u(\tau))$ so that $A_{\omega}(u(\tau)) \in \mathscr{H}_{\alpha}(E_1, E_0, 0)$. For $v \in Z_u$, consider $(0 \leqslant t \leqslant T)$,

$$\begin{aligned} &D_t^{\alpha}(u-u_0) + A_{\omega}(u(\tau))u(t) \\ &\quad = A(u(\tau))v(t) - A(v(t))v(t) + f(v(t)) + \omega(u(\tau))v(t) + h(t). \end{aligned}$$

Let u_v be the corresponding solution. If $u_v = v$, then we have a solution of (44) on $[0,T]$, identically equal to u on $[0,\tau]$. This solution may however have a singularity for $t \downarrow \tau$.

We may repeat the existence proof above to obtain a unique fixed point (of the map $v \to u_v$) $\hat{u}(t)$, $0 \leqslant t \leqslant T$, in Z_u if T is sufficiently close to τ. Clearly, $\hat{u} = u$ on $[0,\tau]$.

Moreover, $\hat{u} \in C([0,T];E_{\hat{\mu}})$ and so, by (46), $A(\hat{u}(t))$, $t \in [0,T]$, is a compact subset of $\mathscr{H}_{\alpha}(E_1, E_0)$. Now use the arguments of [1, Corollary 1.3.2 and proof of Theorem 2.6.1; 9, p. 10] to deduce that there exists a fixed $\hat{\omega} \geqslant 0$ such that

$$A_{\hat{\omega}}(\hat{u}(t)) \stackrel{\text{def}}{=} A(\hat{u}(t)) + \hat{\omega} I \in \mathscr{H}_{\alpha\mu}(E_1, E_0, 0)$$

for every $t \in [0,T]$.

Also,

$$A_{\hat{\omega}}(t) \stackrel{\text{def}}{=} A_{\hat{\omega}}(\hat{u}(t)) \in C([0,T];L(E_1,E_0))$$

and so $A_{\hat{\omega}}(t)$ satisfies (40) (recall that $\alpha + \mu > 1$ is assumed.) In addition,

$$\hat{f}(t) \stackrel{\text{def}}{=} f(\hat{u}(t)) \in BUC([0,T];E_0) \subset \tilde{E}_0([0,T]),$$
$$\hat{\omega}\hat{u}(t) \in C([0,T];E_{\hat{\mu}}) \subset \tilde{E}_0([0,T]).$$

Then note that $\hat{u}$ solves

$$D_t^{\alpha}(u-u_0) + \hat{A}_{\hat{\omega}}(t)u(t) = \hat{f}(t) + \hat{\omega}\hat{u}(t) + h(t), \quad t \in [0,T], \tag{66}$$

and that the earlier result on nonautonomous linear equations can be applied. But by this result there is a unique function $\hat{u}_1(t)$ in $BUC_{1-\mu}([0,T];E_1)$ solving (66) on $[0,T]$.

Moreover, there certainly exists $T_1 > \tau$ such that $\hat{u}_1$ considered on $[0, T_1]$ is contained in Z_u (in the definition of Z_u, take $T = T_1$). Thus we must have $\hat{u}_1 = \hat{u}$ on $[\tau, T_1]$ and so $\hat{u}$ does not have a singularity as $t \downarrow \tau$. The solution u may therefore be continued to $[0, T_1]$, for some $T_1 > \tau$, so that (i), (ii) are satisfied on $[0, T_1]$.

(iii) Suppose $0 < \tau(u_0) < \infty$, and assume $u \in UC([0, \tau(u_0)); E_{\hat{\mu}})$. Then $\lim_{t \uparrow \tau(u_0)}$ exists in $E_{\hat{\mu}}$. Define

$$\tilde{u}(t) = u(t), \ \ t \in [0, \tau(u_0)); \quad \tilde{u}(t) = \lim_{t \uparrow \tau(u_0)} u(t), \ \ t = \tau(u_0).$$

Then $\tilde{u} \in C([0, \tau(u_0)]; E_{\hat{\mu}})$. Define, for $\hat{\omega}$ sufficiently large,

$$B(t) = A_{\hat{\omega}}(\tilde{u}(t)), \quad \tilde{f}(t) = f(\tilde{u}(t)) + \hat{\omega}\tilde{u}(t), \quad 0 \leqslant t \leqslant \tau(u_0).$$

By (46) and the compactness arguments above we have that $B(t)$ satisfies the assumptions required in our nonautonomous result. Consider then

$$D_t^{\alpha}(v - u_0) + B(t)v = \tilde{f}(t) + h(t), \quad 0 \leqslant t \leqslant \tau(u_0).$$

By the earlier result on linear nonautonomous equations, there exists a unique $v \in \tilde{E}_1([0, \tau(u_0)])$ which solves this equation on $[0, \tau(u_0)]$. By uniqueness, $v(t) = u(t)$, $0 \leqslant t < \tau(u_0)$. But $v \in UC([0, \tau(u_0)]; E_{\hat{\mu}})$ and so $v(\tau(u_0)) = \tilde{u}(\tau(u_0))$, hence $v(t) = \tilde{u}(t)$, $0 \leqslant t \leqslant \tau(u_0)$. Thus

$$D_t^{\alpha}(v - u_0) + A(v(t))v(t) = f(v(t)) + h(t), \quad 0 \leqslant t \leqslant \tau(u_0).$$

By earlier results we may now continue the solution past $\tau(u_0)$ and so a contradiction follows.

(iv) Suppose $\tau(u_0) < \infty$ and assume $\limsup_{t \uparrow \tau(u_0)} ||u(t)||_{E_\delta} < \infty$ for some $\delta > \hat{\mu}$. Consider the set $u([0, \tau(u_0)))$. This set is bounded in E_δ, hence its closure is compact in $E_{\hat{\mu}}$.

Take any $\bar{t} \in (0, \tau(u_0))$. Consider

$$\begin{aligned} &D_t^{\alpha}(u - u_0) + A_{\omega}(u(\bar{t})) \\ &\quad = [A(u(\bar{t})) - A(v(t))]v(t) + f(v(t)) + \omega(u(\bar{t}))v(t) + h(t), \end{aligned}$$

and the solution u (which we have on $[0, \tau(u_0))$) on $[0, \bar{t}]$. Now let $\bar{t}$ play the role of τ in (64), and define the set from which v is picked as in (65). Then, as in the considerations following (64), (65), we obtain a continuation of $u(t)$ to $[\bar{t}, \bar{t} + \delta]$, where $\delta = \delta(u(\bar{t})) > 0$. (By uniqueness, on $[\bar{t}, \tau(u_0))$ this is of course the solution we already have.) On the other hand, δ depends continuously on $u(\bar{t})$. But the closure of $\bigcup_{0 \leqslant \bar{t} < \tau(u_0)} u(\bar{t})$ is compact in $E_{\hat{\mu}}$, and so $\delta(u(\bar{t}))$ is bounded away from zero for $0 \leqslant \bar{t} < \tau(u_0)$. Hence the solution may be continued past $\tau(u_0)$ (take $\bar{t}$ sufficiently close to $\tau(u_0)$) and a contradiction follows.

9. An example

In this last section we indicate briefly how our results may be applied to the quasilinear equation

$$u = u_0 + g_\alpha * (\sigma(u_x)_x + h), \quad t \geqslant 0, \quad x \in (0,1), \tag{67}$$

with $u = u(t,x)$, and

$$u(t,0) = u(t,1) = 0, \quad t \geqslant 0; \quad u(0,x) = u_0(x).$$

As was indicated in the Introduction, this problem occurs in viscoelasticity theory, see [10].

We require

$$\sigma \in C^3(R), \quad \text{with } \sigma(0) = 0, \tag{68}$$

and impose the growth condition

$$0 < \sigma_0 \leqslant \sigma'(y) \leqslant \sigma_1, \quad y \in R, \tag{69}$$

for some positive constants σ_0 σ_1.

Take

$$F_0 = \{u \in C[0,1] \mid u(0) = u(1) = 0\},$$

and

$$F_1 = \{u \in C^2[0,1] \mid u^{(i)}(0) = u^{(i)}(1) = 0; \; i = 0,2\}.$$

We fix $\hat{\mu} = \frac{1}{2}$, then $\mu = 1 - \frac{\alpha}{2}$, and $\alpha + \mu > 1$ holds. With $\theta \in (0, \frac{1}{2})$, let

$$E_0 = (F_0, F_1)_\theta^{0,\infty} = \{u \mid u \in h^{2\theta}[0,1]; u(0) = u(1) = 0\}, \tag{70}$$

and

$$E_1 = \{u \in F_1 \mid u'' \in E_0\}. \tag{71}$$

Then

$$E_{\hat{\mu}} = E_{\frac{1}{2}} = \{u \mid u \in h^{1+2\theta}[0,1]; u(0) = u(1) = 0\}.$$

We take, for $u \in E_{\frac{1}{2}}$, $v \in E_1$,

$$A(u)v = -\sigma'(u_x)v_{xx}.$$

Then one has $A(u)v \in E_0$, and, more generally, that the well defined map $v \to A(u)v$ lies in $L(E_1, E_0)$ for every $u \in E_{\frac{1}{2}}$.

We claim that this map satisfies $A(u) \in \mathscr{M}_{\alpha\mu}(E_1, E_0) \cap \mathscr{H}_\alpha(E_1, E_0, 0)$. To this end one takes (for fixed $u \in E_{\frac{1}{2}}$)

$$\tilde{A}v \stackrel{\text{def}}{=} -\sigma'(u_x)v'', \quad v \in F_1,$$

and observes that this map is an isomorphism $F_1 \to F_0$ and that $\tilde{A}$, as an operator in F_0, is closed, positive, with spectral angle 0. Thus Theorem 11 can be applied and our claim follows.

The only remaining condition to be verified is that $u \to A(u) \in C^{1-}(E_{\frac{1}{2}}, L(E_1, E_0))$. But this follows after some estimates which make use of the smoothness assumption (68) imposed on σ.

We thus have, applying Theorem 13:

Theorem 14. *Let* $\alpha \in (0, 2)$. *Take* $\theta \in (0, \frac{1}{2})$ *and* E_0, E_1 *as in*, (70), (71). *Let* (68), (69) *hold. Assume* $h \in BUC_{\frac{\alpha}{2}}([0, T]; h^{2\theta}[0, 1])$, *with* $h(0) = h(1) = 0$. *Assume* $u_0 \in h^{1+2\theta}[0, 1]$ *with* $u_0(0) = u_0(1) = 0$.

Then (67) *has a unique maximal solution u defined on the maximal interval of existence* $[0, \tau(u_0))$ *where* $\tau(u_0) \in (0, \infty]$ *and such that for any* $T < \tau(u_0)$ *one has*

$$u \in BUC_{\frac{\alpha}{2}}([0, T]; h^{2+2\theta}[0, 1]) \cap BUC([0, T]; h^{1+2\theta}[0, 1]) \cap BUC^{\alpha}_{\frac{\alpha}{2}}([0, T]; h^{2\theta}[0, 1]).$$

If $\tau(u_0) < \infty$, *then* $\limsup_{t \uparrow \tau(u_0)} ||u(t)||_{C^{1+2\theta+\delta}} = \infty$ *for every* $\delta > 0$. *In particular, since* $\theta \in (0, \frac{1}{2})$ *is arbitrary, we conclude that if*

$$\limsup_{t \uparrow \tau(u_0)} ||u(t)||_{C^{1+\delta}} < \infty, \tag{72}$$

for some $\delta > 0$, *then* $\tau(u_0) = \infty$.

Global existence and uniqueness of smooth solutions of (67) under assumptions (68), (69), is thus seen to follow from (72). However, the verification of (72) is in general a very difficult task. For $\alpha < \frac{4}{3}$ this task is essentially solved (see [10]).

By different methods, the existence, but not the uniqueness, of a solution u satisfying

$$u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^+; L^2(0, 1)) \cap L^2_{\text{loc}}(\mathbb{R}^+; W^{2,2}_0(0, 1))$$

was proved in [12], for the range $\alpha \in [\frac{4}{3}, \frac{3}{2}]$. For $\frac{3}{2} < \alpha < 2$, only existence of global weak solutions has been proved [11]. We do however conjecture that unique smooth, global solutions do exist for the entire range $\alpha \in (0, 2)$.

Acknowledgments

The first author acknowledges the support of the Magnus Ehrnrooth foundation (Finland). The second author acknowledges the support of the Nederlandse organisatie voor wetenschappelijk onderzoek (NWO).

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