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Maximal regularity for evolution equations in weighted L_p-spaces

By

Jan Prüss and Gieri Simonett

Abstract. Let X be a Banach space and let A be a closed linear operator on X. It is shown that the abstract Cauchy problem

$$\dot{u}(t) + Au(t) = f(t),\ t > 0, \quad u(0) = 0,$$

enjoys maximal regularity in weighted L_p-spaces with weights $\omega(t) = t^{p(1-\mu)}$, where $1/p < \mu$, if and only if it has the property of maximal L_p-regularity. Moreover, it is also shown that the derivation operator $D = d/dt$ admits an $\mathcal{H}^\infty$-calculus in weighted L_p-spaces.

Introduction. Let X be a Banach space and let A be a closed linear operator on X with domain $\mathcal{D}(A)$. We consider the abstract Cauchy problem

$$\dot{u}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = 0, \tag{1.1}$$

where $f \in L_{1,\text{loc}}(\mathbb{R}^+; X)$. In the following we say that the Cauchy problem (1.1) has the property of *maximal L_p-regularity* if for each function $f \in L_p(\mathbb{R}^+; X)$ there exists a unique solution $u \in W_p^1(\mathbb{R}^+; X) \cap L_p(\mathbb{R}^+; \mathcal{D}(A))$. We define $\mathcal{MR}_p(X)$ to be the class of all operators A that admit maximal L_p-regularity for (1.1) in X.

Let us recall some well-known facts about this class. If $A \in \mathcal{MR}_p(X)$ for some $p \in (1, \infty)$, then $A \in \mathcal{MR}_q(X)$ for all $q \in (1, \infty)$. This was first observed by Sobolevskii [9], and was then rediscovered several times, e.g. by Cannarsa and Vespri [3].

If $A \in \mathcal{MR}_p(X)$ for some $p \in (1, \infty)$, then A generates an exponentially stable analytic C_0-semigroup in X. A proof of this fact is contained in Hieber and Prüss [6], see also Prüss [7, Section 10].

In this note we consider the question of maximal regularity for the weighted L_p-spaces

$$L_{p,\mu}(\mathbb{R}^+; X) := \{f : \mathbb{R}^+ \to X : t^{1-\mu} f \in L_p(\mathbb{R}^+; X)\}.$$

We say that A has *maximal $L_{p,\mu}$-regularity* if for each $f \in L_{p,\mu}(\mathbb{R}^+; X)$ there is a unique function $u \in L_{p,\mu}(\mathbb{R}^+; X)$ such that $\dot{u}$, $Au \in L_{p,\mu}(\mathbb{R}^+; X)$, and such that u solves (1.1).

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We will show that $A \in \mathcal{MR}_p(X)$ implies that A also has maximal $L_{p,\mu}$-regularity, that is $A \in \mathcal{MR}_{p,\mu}(X)$, provided $\mu > 1/p$. Moreover, we show that the reverse conclusion is also true.

The restriction on μ comes from several facts. The first one is the embedding

$$L_{p,\mu}(\mathbb{R}^+; X) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^+; X),$$

which is valid for $\mu > 1/p$. The second one is due to Hardy's inequality which reads

$$\int_0^\infty \left| t^{-\mu} \int_0^t f(s)ds \right|^p dt \leqq \frac{1}{(\mu - 1/p)^p} \int_0^\infty |t^{1-\mu} f(t)|^p dt. \tag{1.2}$$

It is valid if $\mu > 1/p$. The third reason comes from the fact that functions in $W^1_{p,\mu}(\mathbb{R}^+; X)$ have a well-defined trace in case that $\mu > 1/p$.

Having established maximal $L_{p,\mu}$-regularity, it not difficult to characterize initial data which lead to solutions in the class $\mathbb{E}_1(\mathbb{R}^+) := W^1_{p,\mu}(\mathbb{R}^+; X) \cap L_{p,\mu}(\mathbb{R}^+; \mathcal{D}(A))$. It is in fact well-known that

$$e^{-tA}x \in \mathbb{E}_1(\mathbb{R}^+) \quad \text{if and only if} \quad x \in D_A(\mu - 1/p, p),$$

where $D_A(\mu-1/p, p)$ denotes the real interpolation space between X and $\mathcal{D}(A)$ of exponent $\mu-1/p$. The case $p = \infty$ has been studied before in the context of functions u which satisfy $t^{1-\mu}u(t) \in BUC((0, T]; X)$ with $\lim_{t\to 0^+} t^{1-\mu}u(t) = 0$. It has been shown that this class allows for maximal regularity if the space X is replaced by $(X, \mathcal{D}(A))_{\theta,0}$, the continuous interpolation space of order $\theta \in (0, 1)$ between X and $\mathcal{D}(A)$. For this theory we refer to the papers of Angenent [2] and Clément and Simonett [4], as well as to the monograph of Amann [1].

We refer to Remark 3.3 for a short discussion of our motivation for studying maximal $L_{p,\mu}$-regularity.

2. Maximal $L_{p,\mu}$-Regularity. Let X be a Banach space and assume that $p \in (1, \infty)$ and $1/p < \mu \leqq 1$. We set

$$L_{p,\mu}(\mathbb{R}^+; X) := \{f : \mathbb{R}^+ \to X : \ t^{1-\mu} f \in L_p(\mathbb{R}^+; X)\}$$

and equip it with the norm $\|f\|_{L_{p,\mu}(\mathbb{R}^+;X)} := (\int_0^\infty |t^{1-\mu} f(t)|^p dt)^{1/p}$. We also define

$$W^1_{p,\mu}(\mathbb{R}^+; X) := \{u \in L_{p,\mu}(\mathbb{R}^+; X) \cap W^1_{1,\mathrm{loc}}((0, \infty); X) : \ \dot{u} \in L_{p,\mu}(\mathbb{R}^+; X)\}.$$

$W^1_{p,\mu}(\mathbb{R}^+; X)$ will always be given the norm

$$\|u\|_{W^1_{p,\mu}} = (\|u\|^p_{L_{p,\mu}(\mathbb{R}^+;X)} + \|\dot{u}\|^p_{L_{p,\mu}(\mathbb{R}^+;X)})^{1/p},$$

which turns it into a Banach space.

In the following we use the notation $E \hookrightarrow F$ if E and F are topological vector spaces such that E is continuously embedded in F.

Lemma 2.1. *Suppose* $p \in (1, \infty)$ *and* $1/p < \mu$. *Then*

(a) $L_{p,\mu}(\mathbb{R}^+; X) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^+; X)$.
(b) $W^1_{p,\mu}(\mathbb{R}^+; X) \hookrightarrow W^1_{1,\mathrm{loc}}(\mathbb{R}^+; X)$.
(c) *Every function* $u \in W^1_{p,\mu}(\mathbb{R}^+; X)$ *has a well-defined trace, that is,* $u(0)$ *is well-defined in* X.

Proof. (a) The first assertion follows from

$$\int_0^T |f(t)|dt \leqq \left(\int_0^T t^{-p'(1-\mu)}dt\right)^{1/p'} \left(\int_0^T |t^{1-\mu} f(t)|^p dt\right)^{1/p} \leqq c\|f\|_{L_{p,\mu}(\mathbb{R}^+;X)}$$

which is valid provided that $\mu > 1/p$.

(b) This follows from the definition of $W^1_{p,\mu}(\mathbb{R}^+; X)$ and from (a).

(c) We conclude from (b) that every function $u \in W^1_{p,\mu}(\mathbb{R}^+; X)$ is locally absolutely continuous, and this yields the assertion in (c). □

In the following we set

$${}_0W^1_{p,\mu}(\mathbb{R}^+; X) := \{u \in W^1_{p,\mu}(\mathbb{R}^+; X) : u(0) = 0\}.$$

It follows that the derivation operator

$$D_\mu u(t) := \dot{u}(t) := \frac{d}{dt}u(t), \quad t > 0, \qquad \mathcal{D}(D_\mu) := {}_0W^1_{p,\mu}(\mathbb{R}^+; X) \tag{2.1}$$

is well-defined on $L_{p,\mu}(\mathbb{R}^+; X)$. It is natural to introduce the mapping

$$\Phi_\mu : L_{p,\mu}(\mathbb{R}^+; X) \to L_p(\mathbb{R}^+; X), \qquad (\Phi_\mu u)(t) := t^{1-\mu}u(t), \quad t > 0.$$

We show that Φ_μ also maps ${}_0W^1_{p,\mu}(\mathbb{R}^+; X)$ into ${}_0W^1_p(\mathbb{R}^+; X)$, provided $\mu > 1/p$.

Proposition 2.2. *Let* $p \in (1, \infty)$ *and let* $1/p < \mu \leqq 1$. *Then*

(a) $\Phi_\mu : L_{p,\mu}(\mathbb{R}^+; X) \to L_p(\mathbb{R}^+; X)$ *is an isometric isomorphism.*
(b) $\Phi_\mu : {}_0W^1_{p,\mu}(\mathbb{R}^+; X) \to {}_0W^1_p(\mathbb{R}^+; X)$ *is a (topological) isomorphism.*

Proof. (a) The assertion in (a) is clear.

(b) (i) We will first show that Φ_μ^{-1} maps ${}_0W^1_p(\mathbb{R}^+; X)$ into ${}_0W^1_{p,\mu}(\mathbb{R}^+; X)$. In order to see this, let $v \in {}_0W^1_p(\mathbb{R}^+; X)$ be given. An easy computation shows that the function $t^{\mu-1}v$ is in $W^1_{p,\mathrm{loc}}((0,\infty); X)$ and that

$$t^{1-\mu}\frac{d}{dt}[t^{\mu-1}v](t) = \dot{v}(t) - (1-\mu)\frac{v(t)}{t}, \quad t > 0. \tag{2.2}$$

By means of Hardy's inequality we can verify that the function v/t belongs to $L_p(\mathbb{R}^+; X)$. Indeed, we infer from $v(t) = \int_0^t \dot v(s)ds$ that

$$\left(\int_0^\infty |t^{-1}v(t)|^p dt\right)^{1/p} = \left(\int_0^\infty \left|t^{-1}\int_0^t \dot v(s)ds\right|^p dt\right)^{1/p}$$

$$\leqq p'\left(\int_0^\infty |\dot v(s)|^p ds\right)^{1/p}. \tag{2.3}$$

We conclude from (2.2)–(2.3) that $\Phi_\mu^{-1}v$ belongs to $W^1_{p,\mu}(\mathbb{R}^+; X)$, and also that the mapping Φ_μ^{-1} is linear and bounded between the indicated spaces.

(ii) Next we show that $u = \Phi_\mu^{-1}v$ has trace zero. Observing that

$$u(t) = t^{\mu-1}v(t) = t^{\mu-1}\int_0^t \dot v(s)\,ds$$

we obtain by Hölder's inequality that $|u(t)| \leqq t^{\mu-1/p}(\int_0^t |\dot v(s)|^p ds)^{1/p}$. This shows that $u(t) \to 0$ as $t \to 0^+$.

(iii) Similar arguments show that Φ_μ maps ${}_0W^1_{p,\mu}(\mathbb{R}^+; X)$ into ${}_0W^1_p(\mathbb{R}^+; X)$, and that the mapping is bounded and linear. □

Before we can state our main result we need some additional preparation.

Proposition 2.3. *Let $p \in (1,\infty)$ and let $1/p < \mu \leqq 1$. Let Y be a Banach space and suppose that $K \in C((0,\infty); \mathcal{L}(X,Y))$ satisfies $|K(t)|_{\mathcal{L}(X,Y)} \leqq M/t$ for $t > 0$, where M is a positive constant. Let*

$$(Tf)(t) := \int_0^t K(t-s)[(t/s)^{1-\mu} - 1]f(s)\,ds, \quad f \in L_p(\mathbb{R}^+; X). \tag{2.4}$$

Then $T \in \mathcal{L}(L_p(\mathbb{R}^+; X), L_p(\mathbb{R}^+; Y))$ and $\|T\| \leqq cM$, where $c = c(p,\mu)$.

Proof. Let $f \in L_p(\mathbb{R}^+; X)$ be given. To shorten notation we set

$$\varphi(r) := (1+r)^{1-\mu} - 1.$$

It is not difficult to establish the elementary estimate

$$\varphi(r) \leqq \min\{r^{1-\mu}, (1-\mu)r\}, \quad r > 0. \tag{2.5}$$

Observe that

$$|(Tf)(t)|_Y \leqq M \int\limits_0^t \frac{1}{(t-s)} \varphi\left(\frac{t-s}{s}\right) |f(s)|_X \, ds.$$

Therefore, T is pointwise bounded by the scalar integral operator S given by

$$(Su)(t) := M \int\limits_0^t \frac{1}{(t-s)} \varphi\left(\frac{t-s}{s}\right) u(s)\, ds, \quad u \in L_p(\mathbb{R}^+).$$

Hölder's inequality then yields

$$\begin{aligned} |(Su)(t)| &\leqq M\|u\|_p \left(\int\limits_0^t \left[\varphi\left(\frac{t-s}{s}\right) \frac{1}{(t-s)} \right]^{p'} ds \right)^{1/p'} \\ &= M\|u\|_p \left(\int\limits_0^1 \left[\varphi\left(\frac{1-\sigma}{\sigma}\right) \frac{1}{(1-\sigma)} \right]^{p'} d\sigma \right)^{1/p'} \cdot t^{-1/p} \end{aligned}$$

for any $u \in L_p(\mathbb{R}^+)$. Here we have to observe that the integral $\int\limits_0^1 [\varphi(\frac{1-\sigma}{\sigma})\frac{1}{(1-\sigma)})]^{p'} d\sigma$ is finite. In fact, this follows from (2.5) due to

$$\begin{aligned} &\int\limits_0^{1/2} \left[\frac{(1-\sigma)^{1-\mu}}{\sigma^{1-\mu}} \cdot \frac{1}{1-\sigma} \right]^{p'} d\sigma + (1-\mu) \int\limits_{1/2}^1 \left[\frac{1-\sigma}{\sigma} \cdot \frac{1}{1-\sigma} \right]^{p'} d\sigma \\ &\quad \leqq c(p,\mu). \end{aligned}$$

We conclude that $S : L_p(\mathbb{R}^+) \to L_{p,weak}(\mathbb{R}^+)$ is bounded for each $p > 1/\mu$. By the Marcinkiewicz interpolation theorem, S is bounded in $L_p(\mathbb{R}^+)$ for each $p > 1/\mu$, with bound dominated by $c(p,\mu)M$, where $c(p,\mu)$ depends only on p and μ. Consequently, T is bounded with the same bound. □

We are now ready for the main result of this section.

Theorem 2.4. *Let* $p \in (1,\infty)$ *and let* $1/p < \mu \leqq 1$. *Then*

$$A \in \mathcal{MR}_p(X) \text{ if and only if } A \in \mathcal{MR}_{p,\mu}(X).$$

Proof. In the following we shall use the notation $X_0 := X$ and $X_1 := \mathcal{D}(A)$, where X_1 is equipped with the norm $\|A\cdot\|_{X_0}$. It follows that X_1 is a Banach space which is densely embedded in X_0.

(i) Suppose that $A \in \mathcal{MR}_p(X)$. Then we know that $-A$ generates a bounded analytic semigroup $\{e^{-tA}; t \geqq 0\}$ on X_0. Let $f \in L_{p,\mu}(\mathbb{R}^+; X_0)$ be given. Let us consider the function u defined by the variation of constants formula

$$u(t) := \int_0^t e^{-(t-s)A} f(s)ds, \quad t > 0. \tag{2.6}$$

It follows from Lemma 2.1(a) that this integral exists in X_0. We will now rewrite equation (2.6) in the following way

$$\begin{aligned} u(t) &= t^{\mu-1} \int_0^t e^{-(t-s)A} s^{1-\mu} f(s)ds \\ &\quad + t^{\mu-1} \int_0^t e^{-(t-s)A}[(t/s)^{1-\mu} - 1]s^{1-\mu} f(s)ds \\ &= \Phi_\mu^{-1}[(D_1 + A)^{-1}\Phi_\mu f + T_A \Phi_\mu f] = \Phi_\mu^{-1}[v_1 + v_2]. \end{aligned}$$

Here we use the same notation for A and its canonical extension on $L_p(\mathbb{R}^+; X_0)$, given by $(Au)(t) := Au(t)$ for $t > 0$. By definition, T_A is the integral operator

$$(T_A g)(t) := \int_0^t e^{-(t-s)A}[(t/s)^{1-\mu} - 1]g(s)ds, \quad g \in L_p(\mathbb{R}^+; X_0).$$

Observe that the kernel $K_A(t) := e^{tA}$ satisfies the assumptions of Proposition 2.3 with $Y = X_1$. We conclude that

$$T_A \in \mathcal{L}(L_p(\mathbb{R}^+; X_0), L_p(\mathbb{R}^+; X_1)) \quad \text{and} \quad AT_A \in \mathcal{L}(L_p(\mathbb{R}^+; X_0)). \tag{2.7}$$

It is a consequence of (2.7) that v_2 has a derivative almost everywhere, given by

$$\dot{v}_2 = -AT_A \Phi_\mu f + (1-\mu)t^{-\mu} \int_0^t e^{-(t-s)A} f(s)ds. \tag{2.8}$$

It follows from Hardy's inequality that

$$\begin{aligned} &\int_0^\infty \left| t^{-\mu} \int_0^t e^{-(t-s)A} f(s)ds \right|^p dt \\ &\leqq M \int_0^\infty \left(t^{-\mu} \int_0^t |f(s)|ds \right)^p dt \leqq cM \|f\|_{L_{p,\mu}} \end{aligned} \tag{2.9}$$

and we infer from (2.7)–(2.9) that

$$v_2 \in {}_0W^1_p(\mathbb{R}^+; X_0) \cap L_p(\mathbb{R}^+; X_1). \tag{2.10}$$

It follows from our assumption that v_1 enjoys the same regularity properties as v_2 and consequently, v satisfies (2.10) as well. Lemma 2.1 then shows that

$$u \in {}_0W^1_{p,\mu}(\mathbb{R}^+; X_0) \cap L_{p,\mu}(\mathbb{R}^+; X_1). \tag{2.11}$$

It is now easy to verify that u is in fact a solution of the Cauchy problem (1.1). We have thus shown that $A \in \mathcal{MR}_{p,\mu}(X)$.

(b) Suppose now that $A \in \mathcal{MR}_{p,\mu}(X_0)$. As in the case $\mu = 1$ one can show that A generates a bounded analytic semigroup $\{e^{-tA}; t \geqq 0\}$ on X_0. Let $f \in L_p(\mathbb{R}^+; X_0)$ be given. Here we use the representation

$$\begin{aligned} u(t) &= t^{1-\mu} \int_0^t e^{-(t-s)A} s^{\mu-1} f(s)ds - \int_0^t e^{-(t-s)A}[(t/s)^{1-\mu} - 1] f(s)ds \\ &= \Phi_\mu (D_\mu + A)^{-1} \Phi_\mu^{-1} f - T_A f, \end{aligned}$$

with T_A as above. The assertion follows now by similar arguments as in (a). □

3. Trace spaces and maximal regularity. We will now consider the Cauchy problem

$$\dot{u}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0, \tag{3.1}$$

where A is a closed linear operator on X. As in the last section we use the notation $X_0 := X$ and $X_1 := \mathcal{D}(A)$, where X_1 is equipped with the norm $\|A \cdot\|_{X_0}$. We also introduce the function spaces

$$\begin{aligned} \mathbb{E}_0 &:= \mathbb{E}_0(\mathbb{R}^+) := L_{p,\mu}(\mathbb{R}^+; X_0), \\ \mathbb{E}_1 &:= \mathbb{E}_1(\mathbb{R}^+) := W^1_{p,\mu}(\mathbb{R}^+; X_0) \cap L_{p,\mu}(\mathbb{R}^+; X_1). \end{aligned} \tag{3.2}$$

It is not difficult to verify that the norm

$$\|u\|_{\mathbb{E}_1} := (\|u\|^p_{L_{p,\mu}(\mathbb{R}^+;X_1)} + \|\dot{u}\|^p_{L_{p,\mu}(\mathbb{R}^+;X_0)})^{1/p} \tag{3.3}$$

turns $\mathbb{E}_1(\mathbb{R}^+)$ into a Banach space. It follows from the representation

$$u(0) = \int_0^1 u(s)ds - \int_0^1 (1-s)\dot{u}(s)ds$$

that the trace map

$$\gamma : \mathbb{E}_1(\mathbb{R}^+) \to X_0, \quad \gamma u = u(0)$$

is continuous. Consequently, the *trace space* $(\gamma\mathbb{E}_1, \|\cdot\|_{\gamma\mathbb{E}_1})$ of $\mathbb{E}_1$, given by,

$$\gamma\mathbb{E}_1 := \gamma(\mathbb{E}_1(\mathbb{R}^+)), \quad \|x\|_{\gamma\mathbb{E}_1} := \inf\{\|u\|_{\mathbb{E}_1} : u \in \mathbb{E}_1, \gamma u = x\}, \tag{3.4}$$

is a well-defined Banach space with $\gamma\mathbb{E}_1 \subset X_0$. In our next result we characterize $\gamma\mathbb{E}_1$ in terms of the real interpolation space $(X_0, X_1)_{\mu-1/p,p}$. We remind here that $(X_0, X_1)_{\mu-1/p,p}$ is sometimes also denoted by $D_A(\mu - 1/p, p)$.

Proposition 3.1. *Suppose* $p \in (1, \infty)$ *and* $1/p < \mu \leqq 1$. *Then*

(a) $\gamma\mathbb{E}_1 = (X_0, X_1)_{\mu-1/p,p}$ *with equivalent norms.*
(b) $\mathbb{E}_1(\mathbb{R}^+) \hookrightarrow BUC(\mathbb{R}^+; (X_0, X_1)_{\mu-1/p,p})$.
(c) $\mathbb{E}_1(\mathbb{R}^+) \hookrightarrow C((0, \infty), (X_0, X_1)_{1-1/p,p})$, *where the latter space is given the Fréchet topology of uniform convergence on compact subsets of* $(0, \infty)$.

P r o o f. (a) (i) We recall that $A \in \mathcal{MR}_p$ implies that $-A$ generates a strongly continuous exponentially stable analytic C_0-semigroup in X_0. It is then well-known that

$$x \in (X_0, X_1)_{\mu-1/p,p} \text{ if and only if } e^{-tA}x \in \mathbb{E}_1(\mathbb{R}^+), \tag{3.5}$$

see for instance [11, Theorem 1.14.5]. Moreover,

$$\|Ae^{-tA}x\|_{L_{p,\mu}(\mathbb{R}^+;X_0)} \text{ defines a norm on } (X_0, X_1)_{\mu-1/p,p}\,. \tag{3.6}$$

(ii) Let $x \in (X_0, X_1)_{\mu-1/p,p}$ be given and let $u := e^{-tA}x$. It follows from (3.5) that $u \in \mathbb{E}_1$. Since $u(0) = x$ we conclude that $x \in \gamma\mathbb{E}_1$. Equation (3.6) finally yields $(X_0, X_1)_{\mu-1/p,p} \hookrightarrow \gamma\mathbb{E}_1$.

(iii) Conversely, let us assume that $x \in \gamma\mathbb{E}_1$. By definition this means that there exists a function $u \in \mathbb{E}_1$ such that $x = u(0)$. We know that

$$x = u(t) - \int_0^t \dot{u}(s)ds, \quad t > 0$$

in X_0, and we then conclude that

$$\begin{aligned} t^{1-\mu}|Ae^{-tA}x|_{X_0} &\leqq t^{1-\mu}|Ae^{-tA}u(t)|_{X_0} + t^{1-\mu}\left|Ae^{-tA}\int_0^t \dot{u}(s)ds\right|_{X_0} \\ &\leqq c\left(t^{1-\mu}|u(t)|_{X_1} + t^{-\mu}\int_0^t |\dot{u}(s)|_{X_0}\,ds\right). \end{aligned}$$

It now follows from Hardy's inequality, stated in (1.2), that

$$\|Ae^{-tA}x\|_{L_{p,\mu}(\mathbb{R}^+;X_0)} \leqq c\|u\|_{\mathbb{E}_1}.$$

Since this is true for any function $u \in \mathbb{E}_1$ with $u(0) = x$ we infer from (3.6) that $\gamma\mathbb{E}_1 \hookrightarrow (X_0, X_1)_{\mu-1/p,p}$.

(b) Let $\{\lambda_s : s \geqq 0\}$ be the semigroup of left translations on $L_{1,\mathrm{loc}}(\mathbb{R}^+; X_0)$, i.e.,

$$\lambda_s u(t) := u(s+t), \quad u \in L_{1,\mathrm{loc}}(\mathbb{R}^+; X_0), \quad s, t \in \mathbb{R}^+.$$

It is easy to see that $\{\lambda_s : s \geqq 0\}$ acts as a semigroup of contractions on $\mathbb{E}_1(\mathbb{R}^+)$. Moreover, since the space $C_c((0,\infty), X_j)$ is dense in $L_{p,\mu}(\mathbb{R}^+ X_j)$ for $j = 0, 1$, we conclude that $\{\lambda_s : s \geqq 0\}$ is strongly continuous on $\mathbb{E}_1(\mathbb{R}^+)$. It is a consequence of [1, Proposition III.1.4.2] that $\mathbb{E}_1(\mathbb{R}^+) \hookrightarrow BUC(\mathbb{R}^+; \gamma\mathbb{E}_1)$, and the assertion in (b) follows from (a).

(c) Let $\tau > 0$ be a fixed number. Then we have

$$\begin{aligned}\mathbb{E}_1(\mathbb{R}^+) &\hookrightarrow W_p^1([\tau,\infty); X_0) \cap L_p([\tau,\infty); X_1)\\ &\hookrightarrow BUC([\tau,\infty), (X_0, X_1)_{1-1/p,p}).\end{aligned}$$

Since this is true for any number $\tau > 0$ we obtain the assertion in (c). □

Suppose that $A \in \mathcal{MR}_p(X)$. Our next result shows that the Cauchy-problem (3.1) admits maximal regularity in $L_{p,\mu}(\mathbb{R}^+; X_0)$, provided u_0 belongs to $(X_0, X_1)_{\mu-1/p,p}$. Moreover, it shows that the solution $u = u(f, u_0)$ depends continuously on (f, u_0).

Theorem 3.2. *Let $p \in (1,\infty)$ and $1/p < \mu \leqq 1$. Suppose that $A \in \mathcal{MR}_p(X)$. Then*

$$\left(\frac{d}{dt} + A, \gamma\right) \in \mathrm{Isom}(\mathbb{E}_1(\mathbb{R}^+), \mathbb{E}_0(\mathbb{R}^+) \times (X_0, X_1)_{\mu-1/p,p}).$$

Proof. (i) We first observe that $(\frac{d}{dt} + A) \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$. Proposition 3.1(a) then yields

$$\left(\frac{d}{dt} + A, \gamma\right) \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0 \times (X_0, X_1)_{\mu-1/p,p}). \tag{3.7}$$

(ii) Theorem 2.4 shows that the operator $(D_\mu + A)$ with domain

$$\mathcal{D}(D_\mu + A) = \mathcal{D}(D_\mu) \cap \mathcal{D}(A) = \{u \in \mathbb{E}_1(\mathbb{R}^+) : u(0) = 0\}$$

is invertible. Let $(f, x) \in \mathbb{E}_0 \times (X_0, X_1)_{\mu-1/p,p}$ be given and let

$$u := (D_\mu + A)^{-1} f + e^{-tA} u_0. \tag{3.8}$$

It follows from (3.5) that $u \in \mathbb{E}_1$. Clearly, u solves the Cauchy problem (3.1). Therefore, $(\frac{d}{dt} + A, \gamma)$ maps $\mathbb{E}_1$ onto $\mathbb{E}_0$. It is easy to see that this mapping is an injection. The assertion follows now from (3.7) and the open mapping theorem. □

Remarks 3.3. (a) Theorem 3.2 shows that the Cauchy problem (3.1) admits a (unique) solution $u \in \mathbb{E}_1(\mathbb{R}^+)$, provided $(f, u_0) \in \mathbb{E}_0(\mathbb{R}^+) \times (X_0, X_1)_{\mu-1/p,p}$. It follows from Proposition 3.1 that u enjoys the additional regularity

$$u \in BUC(\mathbb{R}^+; (X_0, X_1)_{\mu-1/p,p}) \cap C((0,\infty); (X_0, X_1)_{1-1/p,p}). \tag{3.9}$$

This shows that $u(t)$ has, for every $t > 0$, more regularity than the initial value u_0. It is important to observe that this regularizing effect cannot be obtained in the usual setting of L_p-maximal regularity.

(b) The maximal regularity results of this paper can be used to establish existence and uniqueness of solutions for quasilinear parabolic evolution equations

$$\dot{u} + A(u)u = f(u), \quad u(0) = u_0, \tag{3.10}$$

under the assumption that there exists a pair of Banach spaces (X_0, X_1), with X_1 densely embedded in X_0, such that the nonlinear mappings (A, f)

$$(A, f) : (X_0, X_1)_{\mu-1/p,p} \to \mathcal{L}(X_1, X_0) \times X_0 \tag{3.11}$$

are locally Lipschitz-continuous, and such that

$$A(v) \in \mathcal{MR}_p(X_0), \quad v \in (X_0, X_1)_{\mu-1/p,p} . \tag{3.12}$$

To be more precise, one can construct a unique solution u in the function space $\mathbb{E}_1([0, T])$ for initial values $u_0 \in (X_0, X_1)_{\mu-1/p}$, provided $T = T(u_0)$ is sufficiently small. The solution has the additional regularity properties stated in equation (3.9), where $\mathbb{R}^+$ is of course to be replaced by $[0, T]$. In addition, one can show that the quasilinear equation (3.10) generates a semiflow in the natural phase-space $(X_0, X_1)_{\mu-1/p,p}$, and one can develop a geometric theory which parallels the theory for ordinary differential equations. We refrain from giving more details here, as the proofs are similar to the case of L_p-maximal regularity, see for instance Prüss [8].

(c) Our results on maximal $L_{p,\mu}$-regularity give more flexibility in dealing with quasilinear and semilinear problems, as compared to the 'classical' case where $\mu = 1$. To be more precise, we have the liberty to work in phase-spaces with little regularity, the only requirement being that the nonlinear mappings A and f satisfy assumption (3.11). This is an important feature for questions related to global existence of solutions, for it allows one to look for a-priori estimates in 'weaker' norms. Moreover, we have the extra benefit that solutions regularize. This property, in turn, is very helpful for questions related to qualitative properties of solutions, such as the study of ω-limes sets.

4. The derivation operator. In this section we assume that X is a Banach space of class $\mathcal{HT}$. By definition this means that the vector-valued Hilbert transform is bounded in $L_q(\mathbb{R}; X)$ for some $q \in (1, \infty)$. It is then well-known that the operator $D = d/dt$ with domain ${}_0W_p^1(\mathbb{R}^+; X)$ admits an $\mathcal{H}^\infty$-calculus on X, see for instance Prüss [7]. Here we show that this is also true for the weighted spaces $L_{p,\mu}$ provided that $1/p < \mu$.

Let us first introduce some notation. Suppose that A is a closed, linear, and densely defined operator on a Banach space X. Then we denote by $N(A)$ and $R(A)$ the kernel and the range of A, respectively. The operator A is called *sectorial* if

- $N(A) = 0$ and $R(A)$ is dense in X,
- $(-\infty, 0) \subset \rho(A)$ and $\|t(t + A)^{-1}\| \leqq M$ for $t > 0$,

where $\rho(A)$ is the resolvent set of A. The set of all sectorial operators in X will be denoted by $\mathcal{S}(X)$. If $A \in \mathcal{S}(X)$ then there is some $\psi \in (0, \pi)$ such that the sector $\Sigma_{\pi-\psi} = \{z \in \mathbb{C}, z \neq 0, |\arg(z)| < \pi - \psi\}$ is contained in $\rho(-A)$, and there is a positive constant M_ψ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leqq M_\psi, \quad \lambda \in \Sigma_{\pi-\psi}. \tag{4.1}$$

The *spectral angle* ϕ_A of A is defined as the infimum of all angles ψ such that (4.1) holds.

We will now consider the derivation operator D_μ defined in (2.1). Thanks to Proposition 2.2 the operator

$$\bar{D}_\mu := \Phi_\mu D_\mu \Phi_\mu^{-1}, \quad \mathcal{D}(\bar{D}_\mu) := {}_0W_p^1(\mathbb{R}^+; X), \tag{4.2}$$

which acts on the function space $L_p(\mathbb{R}^+; X)$, is well-defined. It follows from (2.2) that

$$\bar{D}_\mu = D_1 + B, \quad \text{where} \quad (Bv)(t) := -(1-\mu)v(t)/t. \tag{4.3}$$

Observe that $\bar{D}_\mu$ and D_μ coincide if $\mu = 1$. Moreover, note that D_μ in $L_{p,\mu}(\mathbb{R}^+; X)$ is similar to $D_1 + B$ in $L_p(\mathbb{R}^+; X)$. It follows from equation (2.3) that B is relatively bounded with respect to D_1, with bound smaller than 1, provided $(1-\mu)p' < 1$, i.e. for $1 \geqq \mu > 1/p$. It is now easy to see that the operators D_μ and $\bar{D}_\mu$ share the following properties.

Proposition 4.1. *Suppose* $1 < p < \infty$ *and* $1/p < \mu \leqq 1$. *Then*

(i) $\bar{D}_\mu$ *is closed and densely defined in* $L_p(\mathbb{R}^+; X)$. *Moreover,* $N(\bar{D}_\mu) = 0$ *and* $R(\bar{D}_\mu)$ *is dense in* $L_p(\mathbb{R}^+; X)$.

(ii) D_μ *is closed and densely defined in* $L_{p,\mu}(\mathbb{R}^+; X)$. *Moreover,* $N(D_\mu) = 0$ *and* $R(D_\mu)$ *is dense in* $L_{p,\mu}(\mathbb{R}^+; X)$.

Proof. (i) It is well-known that the linear operator D_1 has all the properties listed in the proposition. Since B is relatively bounded with respect to D_1 with relative bound strictly smaller than 1, we obtain from (4.3) that $\bar{D}_\mu$ enjoys the same properties, see for instance [5, Section 1.3].

(ii) The assertions in (ii) follow from (i) by employing the isomorphism Φ_μ. □

In the sequel we take the liberty to work with D_μ and $\bar{D}_\mu$ interchangeably, that is, we will use the representation that is the most convenient one.

Remark 4.2. It is well-known that the operator $-D_1$ generates a positive C_0-semigroup $\{T(t) : t \in \mathbb{R}^+\}$ of contractions on $L_p(\mathbb{R}^+; X)$ which is given by

$$[T(t)u](s) := \begin{cases} u(s-t) & \text{if } s > t, o \\ 0 & \text{if } s < t. \end{cases} \tag{4.4}$$

This implies the resolvent estimate

$$\|(\lambda + D_1)^{-1}\|_{\mathcal{L}(L_p(\mathbb{R}^+; X))} \leqq \frac{1}{\mathrm{Re}\lambda}, \quad \mathrm{Re}\lambda > 0.$$

However, note that $\{T(t) : t \in \mathbb{R}^+\}$ does not induce a C_0-semigroup on $L_{p,\mu}(\mathbb{R}^+; X)$ for $\mu < 1$, as $T(t)$ does not map $L_{p,\mu}(\mathbb{R}^+; X)$ into $L_{p,\mu}(\mathbb{R}^+; X)$ for $t > 0$. Nevertheless, we can show that D_μ admits the resolvent estimate

$$\|(\lambda + D_\mu)^{-1}\|_{\mathcal{L}(L_{p,\mu}(\mathbb{R}^+;X))} \leqq \frac{c_{p,\mu}}{\mathrm{Re}\lambda}, \quad \mathrm{Re}\lambda > 0,$$

for some positive constant $c_{p,\mu}$ (which necessarily must be strictly greater than 1).

We begin with a useful auxiliary result.

Lemma 4.3. *Let $1/p < \mu \leqq 1$ and suppose that $k \in L_1(\mathbb{R}^+; \mathcal{L}(X, Y))$ satisfies $|k(t)| \leqq \kappa(t)$, where $\kappa \in L_1(\mathbb{R}^+)$ is nonnegative and nonincreasing, and where Y is a Banach space. Then we have*

(i) $\|\int_0^t k(t-s)(t/s)^{1-\mu}v(s)ds\|_p \leqq c_{p,\mu}\|\kappa\|_1\|v\|_p$ *for* $v \in L_p(\mathbb{R}^+; X)$, *where* $c_{p,\mu} = 2^{1-\mu}[1 + (1 - p'(1-\mu))^{-p/p'}]^{1/p}$.

(ii) *The convolution operator $K := k*$ belongs to $\mathcal{L}(L_{p,\mu}(\mathbb{R}^+; X), L_{p,\mu}(\mathbb{R}^+; Y))$ and $\|K\| \leqq c_{p,\mu}\|\kappa\|_1$.*

Proof. (i) Let $v \in L_p(\mathbb{R}^+; X)$ be given. Then Hölder's inequality implies

$$\left\|\int_0^t k(t-s)(t/s)^{1-\mu}v(s)ds\right\|_p^p \leqq \int_0^\infty \left[\int_0^t \kappa(t-s)(t/s)^{1-\mu}|v(s)|ds\right]^p dt$$

$$\leqq \int_0^\infty \left[\int_0^t \kappa(t-r)r^{-p'(1-\mu)}dr\right]^{p/p'} t^{p(1-\mu)} \int_0^t \kappa(t-s)|v(s)|^p ds dt$$

$$= \int_0^\infty |v(s)|^p \left\{\int_s^\infty t^{p(1-\mu)}\kappa(t-s)\left[\int_0^t \kappa(t-r)r^{-p'(1-\mu)}dr\right]^{p/p'} dt\right\} ds$$

$$\leqq c_{p,\mu}^p\|\kappa\|_1^p\|v\|_p^p,$$

as the following estimates show. On the one hand, we have

$$\int_s^\infty t^{p(1-\mu)}\kappa(t-s)\left[\int_{t/2}^t \kappa(t-r)r^{-p'(1-\mu)}dr\right]^{p/p'} dt$$

$$\leqq 2^{p(1-\mu)}\int_s^\infty \kappa(t-s)\left[\int_{t/2}^t \kappa(t-r)dr\right]^{p/p'} dt$$

$$\leqq 2^{p(1-\mu)}\|\kappa\|_1^{1+p/p'} = 2^{p(1-\mu)}\|\kappa\|_1^p.$$

Since $\kappa(t)$ is nonincreasing and $(1-\mu)p' < 1$ we have, on the other hand,

$$\begin{aligned}
&\int_s^\infty t^{p(1-\mu)}\kappa(t-s)\left[\int_0^{t/2}\kappa(t-r)r^{-p'(1-\mu)}dr\right]^{p/p'}dt\\
&\leqq \int_s^\infty t^{p(1-\mu)}\kappa(t-s)\left[\kappa(t/2)\int_0^{t/2}r^{-p'(1-\mu)}dr\right]^{p/p'}dt\\
&= (1-p'(1-\mu))^{-p/p'}2^{p(1-\mu)}\\
&\quad\cdot\int_s^\infty (t/2)^{p(1-\mu)}\kappa(t-s)[\kappa(t/2)(t/2)^{1-p'(1-\mu)}]^{p/p'}dt\\
&= (1-p'(1-\mu))^{-p/p'}2^{p(1-\mu)}\int_s^\infty \kappa(t-s)[\kappa(t/2)(t/2)]^{p/p'}dt\\
&\leqq (1-p'(1-\mu))^{-p/p'}2^{p(1-\mu)}\|\kappa\|_1^p.
\end{aligned}$$

Note that the last inequality follows from

$$\kappa(t/2)(t/2) = \int_0^{t/2}\kappa(t/2)\,d\tau \leqq \int_0^{t/2}\kappa(\tau)\,d\tau \leqq \|\kappa\|_1,$$

where we have once more used that κ is nonincreasing.

(ii) We conclude from (i) that

$$\begin{aligned}
\|Kv\|_{L_{p,\mu}} &= \left(\int_0^\infty t^{(1-\mu)p}|Kv(t)|^p dt\right)^{1/p}\\
&= \left(\int_0^\infty\left|\int_0^t k(t-s)(t/s)^{1-\mu}s^{1-\mu}v(s)ds\right|^p dt\right)^{1/p}\\
&\leqq c_{p,\mu}\|\kappa\|_1\|s^{1-\mu}v\|_p = c_{p,\mu}\|\kappa\|_1\|v\|_{L_{p,\mu}},
\end{aligned}$$

and the proof of Lemma 4.3 is complete. $\square$

Proposition 4.4. *Let $1/p < \mu \leqq 1$. Then the resolvent set $\rho(D_\mu)$ contains the open negative half-plane $\mathbb{C}_- = -\Sigma_{\pi/2}$, and the estimate*

$$\|(\lambda + D_\mu)^{-1}\|_{\mathcal{L}(L_{p,\mu}(\mathbb{R}^+;X))} \leqq \frac{c_{p,\mu}}{\operatorname{Re}\lambda},\quad \operatorname{Re}\lambda > 0, \tag{4.5}$$

holds. In particular, D_μ is sectorial in $L_{p,\mu}(\mathbb{R}^+;X)$ with spectral angle $\phi_{D_\mu} = \pi/2$.

P r o o f. (i) Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ be fixed and set

$$(4.6) \qquad (K_\lambda f)(t) := \int_0^t e^{-\lambda(t-s)} f(s)\, ds, \quad f \in L_{p,\mu}(\mathbb{R}^+; X).$$

Moreover, let $\kappa(t) := e^{-t\mathrm{Re}\lambda}$. Then K_λ satisfies the assertions of Lemma 4.3, with $\|\kappa\|_1 = 1/\mathrm{Re}\,\lambda$. Consequently, Lemma 4.3 shows that K_λ is a bounded linear operator in $L_{p,\mu}(\mathbb{R}^+; X)$, and that

$$(4.7) \qquad \|K_\lambda\|_{\mathcal{L}(L_{p,\mu}(\mathbb{R}^+;X))} \leqq \frac{c_{p,\mu}}{\mathrm{Re}\,\lambda}\,.$$

(ii) We verify that $(\lambda + D_\mu) : \mathcal{D}(D_\mu) \to L_{p,\mu}(\mathbb{R}^+; X)$ is invertible for any $\mathrm{Re}\lambda > 0$, with

$$(4.8) \qquad [(\lambda + D_\mu)^{-1} f](t) = \int_0^t e^{-\lambda(t-s)} f(s)\, ds, \quad f \in L_{p,\mu}(\mathbb{R}^+; X).$$

Indeed, let $f \in L_{p,\mu}(\mathbb{R}^+; X)$ be given and recall that $L_{p,\mu}(\mathbb{R}^+; X)$ is embedded in $L_{1,\mathrm{loc}}(\mathbb{R}^+; X)$. It is then not difficult to see that the differential equation

$$\left(\lambda + \frac{d}{dt}\right) u = f, \quad u(0) = 0,$$

has a unique solution $u = u_\lambda$ in $W^1_{1,\mathrm{loc}}(\mathbb{R}^+; X)$. It is given by the right-hand side of equation (4.8). It remains to show that $u_\lambda \in \mathcal{D}(D_\mu)$. For this we note that $u_\lambda = K_\lambda f$ and $\dot{u}_\lambda = f - \lambda K_\lambda u_\lambda$. Hence we obtain from (i) that u_λ as well as $\dot{u}_\lambda$ belong to the space $L_{p,\mu}(\mathbb{R}^+; X)$. Since $u_\lambda(0) = 0$ we conclude that $u_\lambda \in \mathcal{D}(D_\mu)$, and this establishes equation (4.8). We have shown that $\rho(D_\mu)$ contains $\mathbb{C}_-$, and the resolvent estimate (4.5) is now a direct consequence of (4.7)–(4.8).

(iii) Let $\psi \in (\pi/2, \pi)$ be fixed. One readily verifies that

$$\mathrm{Re}\,\lambda \geqq |\lambda||\cos\psi|, \quad \lambda \in \Sigma_{\pi-\psi}.$$

It then follows from (4.5) that equation (4.1) is satisfied for any $\psi \in (\pi/2, \pi)$ and we conclude that $\phi_{D_\mu} \leqq \pi/2$. ϕ_{D_μ} can, on the other hand, not be strictly smaller than $\pi/2$, as this would imply that D_μ generates a (strongly continuous analytic) semigroup on $L_{p,\mu}(\mathbb{R}^+; X)$, which is ruled out by Remark 4.2. The assertion follows now from Proposition 4.1. □

The next result shows that D_μ admits an $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\pi/2$, provided X is a Banach space of class $\mathcal{HT}$.

Theorem 4.5. *Let $p \in (1, \infty)$ and $1/p < \mu \leqq 1$. Suppose that X is of class $\mathcal{HT}$. Then D_μ admits an $\mathcal{H}^\infty$-calculus in $L_{p,\mu}(\mathbb{R}^+; X)$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{D_\mu} = \pi/2$.*

Proof. Let $\phi > \pi/2$ be fixed and let $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ be given, where $\mathcal{H}_0^\infty(\Sigma_\phi)$ denotes the set of all bounded holomorphic functions $g \in \mathcal{H}^\infty(\Sigma_\phi)$ such that there are positive numbers c and α with

$$|g(z)| \leqq c\frac{|z|^\alpha}{1+|z|^{2\alpha}}, \quad z \in \Sigma_\phi.$$

Then $h(D_\mu)$ is well-defined as a Dunford integral

$$h(D_\mu) = \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - D_\mu)^{-1}\, d\lambda$$

with Γ being the contour $\Gamma = (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ for a fixed $\psi \in (\pi/2, \phi)$. It follows from (4.8) that $h(D_\mu)$ is also represented by the convolution

$$[h(D_\mu)v](t) = \int_0^t K_h(t-s)v(s)ds, \quad t > 0, \tag{4.9}$$

where the kernel K_h belongs to $C((0,\infty)) \cap L_1(\mathbb{R}^+)$ and is given by the inverse Laplace-transform of h,

$$K_h(t) = \frac{1}{2\pi i}\int_\Gamma h(\lambda)e^{\lambda t}d\lambda, \quad t > 0.$$

To prove the assertion we have to estimate this convolution in $L_{p,\mu}(\mathbb{R}^+; X)$, i.e. we have to prove an inequality of the form

$$\left\| \int_0^t K_h(t-s)(t/s)^{1-\mu}v(s)ds \right\|_p \leqq C_\phi \|h\|_\infty \|v\|_p \tag{4.10}$$

for $v \in L_p(\mathbb{R}^+; X)$ and $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, with a constant C_ϕ independent of h. This will be done by comparing $h(D_\mu)$ with the functional calculus of D_1 in $L_p(\mathbb{R}^+; X)$, which is well-known to be bounded since X is of class $\mathcal{HT}$; see e.g. Prüss [7]. So we know that there is a constant M_ϕ independent of h such that

$$\|h(D_1)v\|_p = \left\| \int_0^t K_h(t-s)v(s)ds \right\|_p \leqq M_\phi \|h\|_\infty \|v\|_p \tag{4.11}$$

for any $v \in L_p(\mathbb{R}^+; X)$ and $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$. One easily verifies that

$$\begin{aligned}\Phi_\mu h(D)\Phi_\mu^{-1} &= \frac{1}{2\pi i}\int_\Gamma h(\lambda)\Phi_\mu(\lambda - D_1)^{-1}\Phi_\mu^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - (D_1 + B))^{-1}\, d\lambda = h(D_1 + B).\end{aligned}$$

Consequently,

$$(4.12) \qquad (T_h v)(t) := [h(D_1 + B) - h(D_1)]v(t) = \int_0^t K_h(t-s)[(t/s)^{1-\mu} - 1]v(s)ds,$$

where $v \in L_p(\mathbb{R}^+; X)$. Observe that

$$|K_h(t)| \leqq \frac{\|h\|_\infty}{\pi} \int_0^\infty e^{tr\cos\psi} dr \leqq \frac{C_\phi \|h\|_\infty}{t}, \quad h \in \mathcal{H}_0^\infty(\Sigma_\phi).$$

Therefore, the kernel K_h satisfies the assumptions of Proposition 2.3 and we conclude that $T_h \in \mathcal{L}(L_p(\mathbb{R}^+; X))$ with

$$\|T_h\|_{\mathcal{L}(L_p(\mathbb{R}^+;X))} \leqq c(p, \mu, \phi)\|h\|_\infty, \qquad h \in \mathcal{H}_0^\infty(\Sigma_\phi),$$

where the constant $c(p, \mu, \phi)$ does not depend on h. We can now conclude that D_μ has an $\mathcal{H}^\infty$-calculus and that the $\mathcal{H}^\infty$-angle is less or equal to $\pi/2$. It is clear that the angle cannot be strictly smaller than $\pi/2$ and this completes the proof. □

Remark 4.6. We restricted our attention to the case $1/p < \mu \leqq 1$. All results of this section are also valid for $\mu > 1$. They are, in fact, obvious in this case.

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Jan Prüss
Fachbereich Mathematik und Informatik
Martin-Luther-Universität Halle-Wittenberg
D-60120 Halle
Germany
anokd@mathematik.uni-halle.de

Gieri Simonett
Department of Mathematics
Vanderbilt University
Nashville, TN 37240
USA
simonett@math.vanderbilt.edu