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Analyticity of the interface in a free boundary problem

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1 Introduction

Of concern is a class of free boundary problems which arise, for instance, in connection with the flow of an incompressible fluid in porous media. More precisely, we consider the following situation: Let Γ_0 denote a fixed, impermeable layer in a homogeneous and isotropic porous medium. We assume that some part of the region above Γ_0 is occupied with an incompressible Newtonian fluid. In addition, we suppose that there is a sharp interface, Γ_f, separating the wet region Ω_f enclosed by Γ_0 and Γ_f, respectively, from the dry part, i.e., we consider a saturated fluid-air flow. The fluid moves under the influence of gravity and we assume that the motion is governed according to Darcy's law. The standard model encompassing this situation consists of an elliptic equation for a velocity potential, to be solved in a domain with a free boundary, and of an evolution equation for the free boundary. In order to give a concise mathematical description let us introduce the following class of admissible interfaces:

$$\mathscr{U}_0 := \left\{ f \in BC^2(\mathbb{R}^n, \mathbb{R})\,;\ \inf_{x \in \mathbb{R}^n} f(x) > 0 \right\},$$

where $n \geqq 1$ is fixed. Given $f \in \mathscr{U}_0$, let

$$\Omega_f := \{(x,y) \in \mathbb{R}^n \times (0,\infty)\,;\ 0 < y < f(x)\}.$$

Consequently, the boundary of Ω_f consists of

$$\begin{aligned} \Gamma_0 &:= \mathbb{R}^n \times \{0\}, \\ \Gamma_f &:= \operatorname{graph}(f) := \{(x,y) \in \mathbb{R}^n \times (0,\infty)\,;\ y = f(x)\}. \end{aligned}$$

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Let Γ_{f_0} with $f_0 \in \mathscr{U}_0$ be a given initial interface. Then the motion of the fluid is governed by the following system of coupled equations for the velocity potential $u = u(t, \cdot)$ and the free interface $\Gamma_f = \Gamma_{f(t)}$, where $t \in (0, \infty)$:

$$\begin{aligned} \Delta_{n+1} u(t, \cdot) &= 0 \qquad \text{in } \Omega_{f(t)} , \\ u(t, \cdot) | \Gamma_{f(t)} &= f(t) , \\ \partial_{n+1} u(t, \cdot) &= 0 \qquad \text{on } \Gamma_0 , \end{aligned} \tag{1.1}$$

and

$$\partial_t f + \sqrt{1 + |\nabla_n f|^2} \, (\partial_\nu u(t, \cdot))|_{\Gamma_{f(t)}} = 0, \quad f(0) = f_0 . \tag{1.2}$$

Here we use the following notation: Δ_{n+1} stands for the Laplacian in $\mathbb{R}^{n+1}$ and $\partial_{n+1} u$ denotes the partial derivative with respect to the $(n+1)$-coordinate of the space variable. Moreover, $\nabla_n f$ is the gradient of f in $\mathbb{R}^n$. In slight abuse of notation, $(\partial_\nu u)|_{\Gamma_{f(t)}}$ stands for the derivative of u in the direction of the outer unit normal field $\nu = \nu(t)$ on $\Gamma_{f(t)}$. Observe that at each point $(x, f(t,x))$ of $\Gamma_{f(t)}$, the normalized outer unit vector ν is given by

$$\nu(x, f(t,x)) = \frac{(-\nabla_n f(t,x), 1)}{\sqrt{1 + |\nabla_n f(t,x)|^2}} , \quad x \in \mathbb{R}^n, \ t \in [0, \infty) .$$

We complement (1.1)–(1.2) with the additional condition

$$\lim_{|z| \to \infty} u(t,z) = c, \quad t \geqq 0, \quad z \in \Omega_{f(t)} \tag{1.3}$$

for a positive constant c.

An inherent difficulty in treating problem (1.1) comes from the fact that the interface $\Gamma_{f(t)}$, constituting the free boundary of the domain, is a priori unknown. It is to be determined as part of the problem. Note that (1.1) represents an elliptic boundary value problem for the velocity potential u, where t appears as a free parameter, while (1.2) contains an evolution equation for f. Observe that both sets of equations are coupled, such that neither can be solved independently.

In [12, 13], we have obtained existence and uniqueness of a maximal classical Hölder solution of (1.1)–(1.3), provided the initial data $f_0 \in \mathscr{U}_0$ satisfy an additional mild regularity assumption and a suitable parabolicity condition. Moreover, we have proved that solutions conserve the regularity of the initial data and generate a smooth semiflow on an appropriate state space. Since we have to deal with a fully nonlinear evolution equation, these results are far from being immediate.

The purpose of this paper is to show that solutions regularize and are smooth; even analytic in t and x.

In order to formulate our results, we need some preparation. Assume $\alpha \in (0, 1)$ and let $h^{2+\alpha} := h^{2+\alpha}(\mathbb{R}^n)$ be the little Hölder spaces, that is, the closure of $\mathscr{S}(\mathbb{R}^n)$ in $BUC^{2+\alpha}(\mathbb{R}^n)$. We restrict our class $\mathscr{U}_0$ of admissible interfaces to be

$$\mathcal{U} := \{f \in BUC^{2+\alpha}(\mathbb{R}^n)\,;\; f - c \in h^{2+\alpha}(\mathbb{R}^n),\; f \in \mathcal{U}_0\}\,.$$

Observe that $\mathcal{U} \subset \mathcal{U}_0$. Hence the domain Ω_f is well-defined for each $f \in \mathcal{U}$ and has, in particular, a $C^{2+\alpha}$ boundary. It can be shown (see Sect. 2) that the elliptic boundary value problem in the unbounded domain Ω_f

$$\Delta_{n+1} u = 0 \text{ in } \Omega_f,\; \partial_{n+1} u = 0 \text{ on } \Gamma_0,\; u = f \text{ on } \Gamma_f,\; \lim_{|z|\to\infty} u(z) = c$$

has a unique classical solution, named u_f. We now define

$$V := \{f \in \mathcal{U}\,;\; \partial_{n+1} u_f(x, f(x)) < (1 + |\nabla_n f(x)|^2)^{-1},\; x \in \mathbb{R}^n\}\,. \tag{1.4}$$

It is easy to see that $f \equiv c$ belongs to V. Moreover, V is open in $\mathcal{U}$, see Lemma 3.3. We are now ready for our main results.

Theorem 1.1 *Let $f_0 \in V$ be given. Then there exists a unique maximal smooth solution (u, f) of* (1.1)–(1.3) *with*

$$u(t, \cdot\,) \in C^\omega(\overline{\Omega}_{f(t)}), \quad t \in \dot{J}\,,$$

$$f = f(\,\cdot, f_0) \in C(J, V) \cap C^\omega(\dot{J} \times \mathbb{R}^n)\,,$$

where $J = [0, t^+(f_0))$ is the maximal interval of existence and C^ω stands for real analytic. The map $(t, f_0) \mapsto f(t, f_0)$ defines an analytic semiflow on V.

Corollary 1.2 *Given any $f_0 \in V$, the interface $\Gamma_{f(\cdot)}$ is real analytic in $(t,x) \in \dot{J} \times \mathbb{R}^n$.*

Observe that solutions of (1.1)–(1.3) are smooth, even if the initial data have much less regularity. This is a considerable improvement of our previous results in [12, 13]. Note that the smoothing has to come from the evolution equation (1.2) and can not be provided by elliptic regularity theory. Since we have to cope with a fully nonlinear evolution equation involving a nonlocal nonlinearity, (see Sect. 3), this result seems quite surprising. In fact, we get our results from an appropriate invariance property of the nonlinear operator, see our arguments in Lemma 3.2, Theorem 4.4 and Remarks 4.5. Our approach uses results from the theory of maximal regularity due to [9], see also [2, 3, 18]. In addition, we will employ a trick of Angenent, see [3,4]. We also rely on results obtained in [12, 13].

It should be observed that the strong maximum principle yields

$$\partial_{n+1} u_f(x, f(x)) < 1, \quad x \in \mathbb{R}^n\,. \tag{1.5}$$

This can be seen by applying the maximum principle to the function $p(x, y) = u_f(x, y) - y$. (1.5) should now be compared with the condition imposed in (1.4). We do not know if (1.4) is indispensable.

In [17], existence of solutions in the case $n = 1$ is obtained by use of the Nash-Moser implicit function theorem. It should be mentioned that the approach in [17] leads to a serious loss of regularity for solutions, see Remark 3.7b).

There is a different approach to free boundary problems on bounded domains, based on variational inequalities, see [8, 10, 14], and the references mentioned there. In this variational setting one can only get weak solutions, since elliptic theory can not help to improve the regularity of weak solutions. Numerical methods for solving the case $n = 1$ have been presented in [16, 19]. Finally, we would also like to mention [1, 6, 15] for related problems.

2 Transformation of the problem

In this section we transform the original problem into a problem on a fixed domain. We give a representation of the transformed differential operators in the new coordinates. As a consequence, it turns out that the transformed operators will depend nonlinearly upon the unknown function f. Here, we follow [12, 13]. As a new result, we show analytic dependence of the mappings upon f. In the sequel, we take the liberty to replace f with $g := f - c$, where c is the constant appearing in (1.3).

In the following, $\alpha \in (0, 1)$ is fixed. Let

$$\mathcal{O} := \left\{ g \in h^{2+\alpha}(\mathbb{R}^n);\ \inf_{x \in \mathbb{R}^n} (c + g(x)) > 0 \right\} .$$

Note that $\mathcal{O}$ is open in $h^{2+\alpha}$. Given $g \in \mathcal{O}$, define

$$\phi(x, y) := \phi_g(x, y) := (x, (1 - y)(c + g(x))) \quad \text{for } (x, y) \in \Omega ,$$

with $\Omega := \mathbb{R}^n \times (0, 1)$. It is easily verified that ϕ_g is a $C^{2+\alpha}$-diffeomorphism from Ω onto Ω_f, where $f = c + g$. Let

$$\phi^* u := \phi_g^* u := u \circ \phi_g \qquad \text{for } u \in C(\overline{\Omega}_f) ,$$

$$\phi_* v := \phi_*^g v := v \circ \phi_g^{-1} \qquad \text{for } v \in C(\overline{\Omega}) ,$$

denote the pull back and push forward operators, respectively, induced by ϕ. Given $g \in \mathcal{O}$ and $v \in C^2(\overline{\Omega})$, we define the following transformed operators:

$$\mathcal{A}(g)v := -\phi_g^* \Delta_{n+1}(\phi_*^g v)$$

$$\mathcal{B}_i(g)v := \phi_g^*(\gamma_i \nabla_{n+1}(\phi_*^g v) | n_i), \quad i = 0, 1 ,$$

where γ_0 and γ_1 stand for the trace operators and $n_0 = (-\nabla_n g, 1)$ and $n_1 = (0, \ldots, 0, -1)$ denote the outer normal according to Γ_{c+g} and Γ_0, respectively.

We set $\Gamma_i := \mathbb{R}^n \times \{i\}$, $i = 0, 1$. Let $g_0 \in \mathcal{O}$ be given and consider the following transformed problem

$$\begin{aligned} \mathcal{A}(g)v &= 0 \quad \text{in } \Omega , \\ v &= g \quad \text{on } \Gamma_0 , \\ \mathcal{B}_1(g)v &= 0 \quad \text{on } \Gamma_1 , \\ \lim_{|z| \to \infty} v(t, z) &= 0 \quad t \geqq 0 , \end{aligned} \tag{2.1}$$

and

$$\partial_t g + \mathscr{B}_0(g)v = 0, \quad g(0) = g_0 := f_0 - c\,. \tag{2.2}$$

Observe that the functions g and v both depend on t. In order to keep the notation simple, we have suppressed its dependence. Note that (2.1) and (2.2) are the transformed versions of (1.1)+(1.3) and (1.2), respectively. It is clear that solutions are also transformed under the diffeomorphisms introduced above.

We will give a representation of the transformed operators $\mathscr{A}(g)$ and $\mathscr{B}_i(g)$ in local coordinates. Let $\pi(x,y) := 1 - y$ for $(x,y) \in \overline{\Omega}$.

Lemma 2.1 *Given $g \in \mathcal{O}$, we have*

$$\mathscr{A}(g) = -\sum_{j,k=1}^{n+1} a_{j,k}(g)\partial_j\partial_k + \sum_{j}^{n+1} a_j(g)\partial_j, \quad \mathscr{B}_i(g) = \sum_{j=1}^{n+1} b_{j,i}(g)\gamma_i\partial_j, \quad i = 0, 1\,,$$

with

$$a_{j,k}(g) = \delta_{jk}, \quad 1 \leqq j,k \leqq n, \quad a_{j,n+1}(g) = a_{n+1,j}(g) = \frac{\pi\partial_j g}{c+g}, \quad 1 \leqq j \leqq n\,,$$

$$a_{n+1,n+1}(g) = \frac{1}{(c+g)^2}(1 + \pi^2|\nabla_n g|^2)\,,$$

$$a_j(g) = 0, \quad 1 \leqq j \leqq n, \quad a_{n+1}(g) = \frac{\pi}{c+g}\left(\frac{2|\nabla_n g|^2}{c+g} - \Delta_n g\right),$$

$$b_{j,0}(g) := -\partial_j g, \quad 1 \leqq j \leqq n, \quad b_{n+1,0}(g) := -\frac{1}{c+g}(1 + |\nabla_n g|^2)\,,$$

$$b_{j,1}(g) := 0, \quad 1 \leqq j \leqq n, \quad b_{n+1,1}(g) := \frac{1}{c+g}\,.$$

Proof. This follows by similar arguments as in the proof of Lemma 2.2 in [12], where the case $n = 1$ is considered. □

We will now study the mapping properties of the differential operators $\mathscr{A}(\cdot)$ and $\mathscr{B}_i(\cdot)$ with respect to g. To do so, we first have to introduce some function spaces. For $m \in \mathbb{N}$ let $\mathscr{S}(\mathbb{R}^m)$ denote the Schwartz space, that is, the Fréchet space of all rapidly decreasing smooth functions on $\mathbb{R}^m$. Moreover, assume that $k \in \mathbb{N}$ and that U is an open subset of $\mathbb{R}^m$. Then $BUC^{k+\alpha}(U)$ denote the classical Hölder spaces of functions having bounded derivatives up to order k, and such that the k-th derivatives satisfy a uniform α-Hölder condition. We define the *little Hölder spaces* of order $k+\alpha$, to be

$$h^{k+\alpha}(U) := \text{closure of } r_U(\mathscr{S}(\mathbb{R}^m)) \text{ in } BUC^{k+\alpha}(U)\,.$$

Here, r_U is the restriction map with respect to U, that is, $r_U v := v|_U$ for $v \in C(\mathbb{R}^m)$. As a special case we obtain

$$h^{k+\alpha} := h^{k+\alpha}(\mathbb{R}^n) = \text{closure of } \mathscr{S}(\mathbb{R}^n) \text{ in } BUC^{k+\alpha}(\mathbb{R}^n)\,. \tag{2.3}$$

Given $a \in \mathbb{R}^n$, let τ_a denote the (left) translation by the vector a, i.e., $(\tau_a g)(x) := g(x+a)$ for $g \in C(\mathbb{R}^n)$. Then the little Hölder spaces have the following property

Lemma 2.2 *$\{\tau_a\,;\, a \in \mathbb{R}^n\}$ is a strongly continuous group of contractions on $h^{k+\alpha}(\mathbb{R}^n)$, $k \in \mathbb{N}$. Moreover,*

$$\tau_a g - \tau_{a_0} g = \int_0^1 \tau_{a_0+s(a-a_0)}(a-a_0\,|\,\nabla_n g)\,ds \quad in \quad h^{k+\alpha}(\mathbb{R}^n)\,, \tag{2.4}$$

$g \in h^{k+1+\alpha}(\mathbb{R}^n)$ and $a, a_0 \in \mathbb{R}^n$.

Proof. Let $k \in \mathbb{N}$ be fixed.

a) It is easily verified that $\{\tau_a\,;\, a \in \mathbb{R}^n\}$ is a group of contractions on $BUC^{k+\alpha}(\mathbb{R}^n)$. (Observe that translations commute with differentiation). Note that $\tau_a(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$ for $a \in \mathbb{R}^n$. It follows from (2.3) that $\{\tau_a\,;\, a \in \mathbb{R}^n\}$ is a group of contractions on $h^{k+\alpha}$.

b) Assume $g \in \mathcal{S}(\mathbb{R}^n)$ and let a, $a_0 \in \mathbb{R}^n$ be given. It is not difficult to see that

$$\|\tau_a g - \tau_{a_0} g\|_{k+\alpha} \leqq c(n)|a-a_0|\,\|g\|_{k+1+\alpha}\ .$$

It follows from (2.3) and part a) that $\{\tau_a\,;\, a \in \mathbb{R}^n\}$ is strongly continuous on $h^{k+\alpha}$.

c) Let $g \in h^{k+1+\alpha}$ and $a, a_0 \in \mathbb{R}^n$ be given. Note that $[s \mapsto \tau_{s(a-a_0)}(a-a_0\,|\,\nabla_n g)] \in C([0,1], h^{k+\alpha})$, owing to part b). Hence the integral on the right side of (2.4) exists in $h^{k+\alpha}$. (2.4) is now a consequence of the mean value theorem. □

We need some further function spaces. Assume that U is an open subset in $\mathbb{R}^m$, $m \in \dot{\mathbb{N}}$. Then we set

$$buc^{k+\alpha}(U) := \text{closure of } BUC^{\infty}(U) \text{ in } BUC^{k+\alpha}(U)\,.$$

Finally, we use the notation

$$buc^{k+\alpha} := buc^{k+\alpha}(\mathbb{R}^n)\,.$$

Assume U is either $\mathbb{R}^n$ or Ω. It is not difficult to verify that pointwise multiplication, i.e., the map $[(u,v) \mapsto uv]$, is bilinear and continuous on

$$h^{k+\alpha}(U) \times h^{k+\alpha}(U) \to h^{k+\alpha}(U)\,,$$

$$buc^{k+\alpha}(U) \times buc^{k+\alpha}(U) \to buc^{k+\alpha}(U)\,,$$

$$buc^{k+\alpha}(U) \times h^{k+\alpha}(U) \to h^{k+\alpha}(U)\,. \tag{2.5}$$

This shows in particular that the spaces $h^{k+\alpha}(U)$ and $buc^{k+\alpha}(U)$ are (continuous) multiplication algebras. Note that $\|uv\|_{k+\alpha} \leqq m(k)\|u\|_{k+\alpha}\|v\|_{k+\alpha}$ for each case in (2.5), where $m(k) > 1$ if $k > 0$.

Lemma 2.3 *The mappings*

$$\left[g \mapsto \frac{1}{c+g}\right] : \mathcal{O} \to buc^{2+\alpha} ,$$

$$[g \mapsto (\mathscr{A}(g), \mathscr{B}_i(g))] : \mathcal{O} \to \mathscr{L}(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{1+\alpha}), \quad i = 0, 1 ,$$

are (real) analytic.

Proof. a) Pick $g_0 \in \mathcal{O}$. It is easy to see that $\frac{1}{c+g_0} \in buc^{2+\alpha}$. Let $\delta_0 := \|1/(c+g_0)\|_{2+\alpha}$ denote its norm and set $r_0 := (m^2\delta_0)^{-1}$, where $m = m(2)$ is the norm of the bilinear form(s) in (2.5) with $k = 2$. Let $g \in h^{2+\alpha}$ satisfy $\|g - g_0\|_{2+\alpha} < r_0$. Then it is not difficult to verify that $g \in \mathcal{O}$. It follows from (2.5) and the remarks after (2.5) that the series

$$\sum_{k=0}^{\infty} \left(\frac{1}{c+g_0}\right)^k (g - g_0)^k$$

converges in $buc^{2+\alpha}$ for each $g \in h^{2+\alpha}$ with $\|g - g_0\|_{2+\alpha} < r_0$. Moreover, it is easy to see that

$$\frac{1}{c+g} = \frac{1}{c+g_0} \sum_{k=0}^{\infty} \left(\frac{-1}{c+g_0}\right)^k (g - g_0)^k .$$

This proves the first assertion.

b) It follows from a), (2.5), and Lemma 2.1 that

$$[g \mapsto (a_{j,k}(g),\, a_j(g),\, b_{j,i}(g))] : \mathcal{O} \to buc^\alpha(\Omega) \times buc^\alpha(\Omega) \times buc^{1+\alpha}$$

is an analytic map for each $1 \leqq j, k \leqq n+1$ and $i = 0, 1$. Now the second assertion is a consequence of (2.5) and the fact that

$$[a \to a\partial^\beta] : buc^\alpha(\Omega) \to \mathscr{L}(h^{2+\alpha}(\Omega), h^\alpha(\Omega)), \quad \beta \in \mathbb{N}^{n+1},\ |\beta| \leqq 2 ,$$

$$[b \to b\gamma_i\partial_j] : buc^{1+\alpha} \to \mathscr{L}(h^{2+\alpha}(\Omega), h^{1+\alpha}), \quad j = 1, \ldots, n+1,\ i = 0, 1$$

are linear. □

The next Lemma gives an isomorphism property for the elliptic boundary value problem (2.1) in little Hölder spaces.

Lemma 2.4 *Let $g \in \mathcal{O}$ be given. Then*

$$(\mathscr{A}(g), \gamma_0, \mathscr{B}_1(g)) \in Isom(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{2+\alpha} \times h^{1+\alpha}) .$$

Proof. We refer to Theorem 3.5 and Appendix C in [12], where the case $n = 1$ is treated. The proof uses the classical results of Agmon, Douglis, and Nirenberg, the maximum principle, and the continuity method. The same ideas carry over to $n > 1$. □

Given $g \in \mathcal{O}$, we define

$$\mathscr{T}(g) := (\mathscr{A}(g), \gamma_0, \mathscr{B}_1(g))^{-1} \,|\, \{0\} \times h^{2+\alpha} \times \{0\} . \tag{2.6}$$

Assume that $g \in \mathcal{O}$, $h \in h^{2+\alpha}$, and put $u := \mathcal{T}(g)h$. Then u is the unique solution in $h^{2+\alpha}(\Omega)$ of the following elliptic boundary value problem

$$\mathcal{A}(g)u = 0 \text{ in } \Omega, \ \gamma_0 u = h \text{ on } \Gamma_0, \ \mathcal{B}_1(g)u = 0 \text{ on } \Gamma_1 .$$

Lemma 2.5 *The mapping* $[g \mapsto \mathcal{T}(g)] : \mathcal{O} \to \mathcal{L}(h^{2+\alpha}, h^{2+\alpha}(\Omega))$ *is analytic.*

Proof. To shorten the notation, let $F_0 := h^{\alpha}(\Omega)$, $F_2 := h^{2+\alpha}(\Omega)$, $E_1 := h^{1+\alpha}$, and $E_2 := h^{2+\alpha}$.

a) Recall that $\mathcal{O}$ is an open subset of E_2. Moreover, letting

$$A(g) := (\mathcal{A}(g), \gamma_0, \mathcal{B}_1(g)), \quad g \in \mathcal{O} ,$$

it follows from Lemmas 2.3 and 2.4 that

$$A \in C^{\omega}(\mathcal{O}, Isom(F_2, F_0 \times E_2 \times E_1)) .$$

b) Given $A \in Isom(F_2, F_0 \times E_2 \times E_1)$, define $j(A) := A^{-1}$. Then $Isom(F_2, F_0 \times E_2 \times E_1)$ is open in $\mathcal{L}(F_2, F_0 \times E_2 \times E_1)$, and it is known that

$$j \in C^{\omega}(Isom(F_2, F_0 \times E_2 \times E_1), \mathcal{L}(F_0 \times E_2 \times E_1, F_2)) .$$

c) Let $R \in \mathcal{L}(F_0 \times E_2 \times E_1, F_2)$ be given, and define $p(R) \in \mathcal{L}(E_2, F_2)$ by

$$p(R)x_2 := R(0, x_2, 0) \quad \text{for } x_2 \in E_2 .$$

Then $p \in \mathcal{L}(\mathcal{L}(F_0 \times E_2 \times E_1, F_2), \mathcal{L}(E_2, F_2))$ and consequently

$$p \in C^{\omega}(\mathcal{L}(F_0 \times E_2 \times E_1, F_2), \mathcal{L}(E_2, F_2)) .$$

Now the assertion follows from the identity $\mathcal{T} = p \circ j \circ A$ and the fact that the composition of analytic maps is analytic too. □

3 The nonlinear evolution equation

In this section we fuse the coupled system of equations (2.1)–(2.2) into a single (fully) nonlinear evolution equation. To do so, we first introduce a nonlinear, nonlocal pseudo-differential operator which will be instrumental to our approach.

Given $g \in \mathcal{O}$, we define

$$\Phi(g) := \mathcal{B}_0(g)\mathcal{T}(g)g . \tag{3.1}$$

For g fixed, $\mathcal{B}_0(g)\mathcal{T}(g)$ is a nonlocal pseudo-differential operator, the so-called *generalized Dirichlet-Neumann operator*, see [11]. The mapping Φ depends nonlinearly upon g. Observe that Lemma 2.3 and Lemma 2.5 yield

Proposition 3.1 *The mapping* $[g \mapsto \Phi(g)] : \mathcal{O} \to h^{1+\alpha}$ *is analytic.*

The following Lemma will be important for obtaining the smoothing property of solutions.

Lemma 3.2 *The mapping Φ commutes with translations, i.e.,*

$$\tau_a \Phi(g) = \Phi(\tau_a g), \quad g \in \mathcal{O}, \ a \in \mathbb{R}^n .$$

Proof. It follows from Lemma 2.2 and the definition of $\mathcal{O}$ that $\mathcal{O}$ and the spaces $h^{k+\alpha}$ are invariant under translations. Let $\tau_{(a,0)}$ denote the left translation by $(a,0) \in \mathbb{R}^{n+1}$, i.e., $(\tau_{(a,0)}v)(x,y) := v(x+a,y)$ for $v \in C(\mathbb{R}^{n+1})$. Then the spaces $h^{k+\alpha}(\Omega)$ are also invariant under $\tau_{(a,0)}$. Let now $g \in \mathcal{O}$ be fixed.

a) A simple computation reveals that

$$a_{j,k}(\tau_a g) = \tau_{(a,0)}\, a_{j,k}(g), \quad a_j(\tau_a g) = \tau_{(a,0)}\, a_j(g)\,, \tag{3.2}$$

where $a_{j,k}(\,\cdot\,)$ and $a_j(\,\cdot\,)$, $1 \leqq j,k \leqq n+1$, are the coefficients of the differential operator $\mathscr{A}(\,\cdot\,)$, see Lemma 2.1. Similarly, we get

$$b_{j,i}(\tau_a g) = \tau_a\, b_{j,i}(g), \quad j = 1,\ldots,n+1, \quad i = 0,1\,, \tag{3.3}$$

for the coefficients of the boundary differential operators $\mathscr{B}_i$, $i = 0,1$. Next, note that

$$\gamma_i\,(\partial_j \tau_{(a,0)} v) = \tau_a\,(\gamma_i\, \partial_j v), \quad j = 1,\ldots,n+1, \quad i = 0,1 \tag{3.4}$$

for any function $v \in C^1(\overline{\Omega})$. Here, γ_i is the trace operator with respect to Γ_i, i.e., $(\gamma_i\, v)(x) := v(x,i)$ for $v \in C(\overline{\Omega})$ and $i = 0,1$.

b) Let $v := \mathscr{T}(g)g$. By the definition of $\mathscr{T}(g)$ in (2.6), v is a solution of

$$\mathscr{A}(g)v = 0 \text{ in } \Omega, \ \ \gamma_0 v = g \text{ on } \Gamma_0, \ \ \mathscr{B}_1(g)v = 0 \text{ on } \Gamma_1\,. \tag{3.5}$$

We claim that $\tau_{(a,0)}v = \mathscr{T}(\tau_a g)\tau_a g$, which amounts to showing that $u = \tau_{(a,0)}v$ solves the elliptic boundary value problem

$$\mathscr{A}(\tau_a g)u = 0 \text{ in } \Omega, \quad \gamma_0 u = \tau_a g \text{ on } \Gamma_0, \quad \mathscr{B}_1(\tau_a g)u = 0 \text{ on } \Gamma_1\,.$$

Using (3.2) and the fact that differentiation commutes with translations, we obtain

$$a_{j,k}(\tau_a g)\partial_j \partial_k \tau_{(a,0)}v = \tau_{(a,0)}\,(a_{j,k}(g)\partial_j\partial_k v), \quad a_j(\tau_a g)\partial_j \tau_{(a,0)}v = \tau_{(a,0)}\,(a_j(g)\partial_j v)\,.$$

This and (3.5) show that $\mathscr{A}(\tau_a g)\tau_{(a,0)}v = \tau_{(a,0)}\,\mathscr{A}(g)v = 0$. We infer from $\gamma_0\, \tau_{(a,0)}v = \tau_a\, \gamma_0 v$, and from (3.3)–(3.5) that

$$\gamma_0\, \tau_{(a,0)}v = \tau_a g \text{ on } \Gamma_0, \quad \mathscr{B}_1(\tau_a g)\tau_{(a,0)}v = 0 \text{ on } \Gamma_1\,.$$

We have proved that

$$\mathscr{T}(\tau_a g)\tau_a g = \tau_{(a,0)}\, \mathscr{T}(g)g\,. \tag{3.6}$$

c) It remains to combine (3.3)–(3.4) and (3.6) to complete the proof. □

Next we show that the Fréchet derivative $\partial\Phi(g)$ of Φ is the negative generator of a strongly continuous analytic semigroup on $h^{1+\alpha}$, provided an additional condition is imposed on $\mathcal{O}$. For $g \in \mathcal{O}$, set $v_g := \mathscr{T}(g)g$. We introduce the following set

$$W := \left\{ g \in \mathcal{O}\,;\ \left(\frac{1}{c+g(x)}\partial_{n+1}v_g(x,0) + \frac{1}{1+|\nabla_n g(x)|^2}\right) > 0,\, x \in \mathbb{R}^n \right\}.$$

It is not difficult to see that $g \in W$ implies $f := c + g \in V$, and vice versa, where V was introduced in (1.4). We note some properties of W.

Lemma 3.3

a) *W is an open subset of $\mathcal{O}$ and $0 \in W$.*

b) *W is invariant under translations, i.e., $\tau_a(W) \subset W$.*

Proof. a) It is easy to see that $0 \in W$. Let

$$B(g) := \inf_{x \in \mathbb{R}^n} \left(\frac{1}{c + g(x)} \partial_{n+1} v_g(x, 0) + \frac{1}{1 + |\nabla_n g(x)|^2} \right), \quad g \in \mathcal{O} .$$

It is not difficult to verify that $g \in W$ implies $B(g) > 0$. (Use the definition of $\mathcal{O}$ and Lemma 2.4). Therefore, $W = B^{-1}(0, \infty)$. It is a consequence of (2.5) and Lemma 2.5 that $[g \mapsto B(g)] \in C(\mathcal{O}, \mathbb{R})$. Hence $W = B^{-1}(0, \infty)$ is open in $\mathcal{O}$.

b) Let $g \in W$ be given. It follows that

$$\frac{1}{c + \tau_a g(x)} \partial_{n+1} v_{\tau_a g}(x, 0) + \frac{1}{1 + |\nabla_n \tau_a g(x)|^2} > 0, \quad x \in \mathbb{R}^n ,$$

since $v_{\tau_a g} = \tau_{(a,0)} v_g$ by (3.6). This shows that $\tau_a g \in W$. □

We are ready for the core results of this section.

Theorem 3.4 *Let $g \in W$ be given and let $I := [0, T]$ for some $T > 0$. Then*

a) *$-\partial\Phi(g)$ is the generator of a strongly continuous analytic semigroup on $h^{1+\alpha}$.*

b) *$(C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha}), C(I, h^{1+\alpha}))$ is a pair of maximal regularity for $\{\partial\Phi(g); g \in W\}$, that is,*

$$(\partial_t + \partial\Phi(g), \gamma) \in Isom\,(C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha}), C(I, h^{1+\alpha}) \times h^{2+\alpha}), \quad g \in W ,$$

where $\gamma h := h(0)$ for $h \in C(I, h^{2+\alpha})$. If $K \subset W$ is compact, there exists a positive constant $c := c(I, K)$ such that

$$\|(\partial_t + \partial\Phi(g), \gamma)^{-1}\| \leqq c, \quad g \in K .$$

Proof. a) The result was proved in [12], Theorem 6.2 and Corollary 6.3 (see also [13]), under the assumption that $g \in \mathcal{O}$ satisfies the stronger condition

$$\left(\frac{1}{c + g(x)} \partial_{n+1} v_g(x, 0) + \kappa_g(x) \right) > 0, \quad x \in \mathbb{R}^n ,$$

where

$$\kappa_g := \frac{(c + g)^2}{(1 + (c + g)^2 + |\nabla_n g|^2)(1 + |\nabla_n g|^2)} .$$

We shall show that the problem does have a scaling invariance, which can be used to improve the previous result contained in [12, 13]. To do so, let $\lambda > 0$ be a real and let σ_λ denote the dilation of a function by $1/\lambda$, that is, $\sigma_\lambda f(x) := f(x/\lambda)$ for $f \in C(\mathbb{R}^n)$. It is easy to see that σ_λ defines an isomorphism on the spaces $h^{k+\alpha}(\mathbb{R}^n)$. For convenience, let us set $f_\lambda := \lambda\,\sigma_{1/\lambda} f$. Using similar arguments as in the proof of Lemma 3.2 it can be shown that

$$v_{g_\lambda}(x,y) = \lambda\, v_g(x/\lambda, y), \quad (x,y) \in \Omega\,.$$

Let us now assume that $g \in W$ is given. Then it can be verified that the rescaled function g_λ satisfies

$$\left(\frac{1}{(c+g)_\lambda(x)}\partial_{n+1} v_{g_\lambda}(x,0) + \kappa_{g_\lambda}(x)\right) > 0, \quad x \in \mathbb{R}^n\,, \tag{3.7}$$

provided λ is chosen large enough. It is also not difficult to verify that the nonlinear mapping Φ has the scaling property $\Phi(g_\lambda) = \sigma_{1/\lambda}\Phi(g)$ for $g \in \mathcal{O}$. It then follows from the chain rule that

$$\partial\Phi(g) = \sigma_\lambda \partial\Phi(g_\lambda)\,\lambda\,\sigma_{1/\lambda}\,. \tag{3.8}$$

Let λ be fixed such that (3.7) is satisfied. We conclude from our previous results in [12, 13] that the operator $-\partial\Phi(g_\lambda)$ generates a strongly continuous analytic semigroup on $h^{1+\alpha}$. But so does $-\lambda\,\partial\Phi(g_\lambda)$. We can now infer from (3.8) that $-\partial\Phi(g)$ generates a strongly continuous analytic semigroup on $h^{1+\alpha}$.

b) It can be shown that $-\partial\Phi(g)$ generates a strongly continuous analytic semigroup on $h^{1+\beta}$ too, where $\beta \in (0,\alpha)$ and $g \in W$. The statements in b) then follow from the interpolation result

$$(h^{1+\beta}, h^{2+\beta})^0_{\alpha-\beta,\infty} = h^{1+\alpha}\,,$$

where $(\,\cdot\,,\,\cdot\,)^0_{\theta,\infty}$, $\theta \in (0,1)$, denotes the continuous interpolation method, from [9], see also [2, 3, 18], and from the results in [12] which state that all bounds are uniform on compact subsets of W. □

Suppose that $g_0 \in W$. We consider the nonlinear evolution equation

$$\partial_t g + \Phi(g) = 0, \quad g(0) = g_0\,. \tag{3.9}$$

Note that the elliptic equation (2.1) and the evolution equation (2.2) are now united in a single equation, involving only the unknown function g, which determines the free boundary. The next Lemma shows that solutions of (3.9) lead to solutions of (2.1)–(2.2), and vice versa.

Lemma 3.5 *Let $g_0 \in W$ be given.*

a) *Suppose that $g \in C(J,W) \cap C^1(J,h^{1+\alpha})$ is a solution of* (3.9) *on an interval $J = [0,T)$. Let $v(t,\,\cdot\,) := \mathcal{T}(g(t))g(t)$. Then the pair (v,g) is a classical solution of* (2.1)–(2.2) *with*

$$\begin{aligned} &g \in C(J,W) \cap C^1(J,h^{1+\alpha}) \\ &v(t,\,\cdot\,) \in h^{2+\alpha}(\Omega), \quad t \in J\,. \end{aligned} \tag{3.10}$$

b) *Suppose that (v,g) is a classical solution of* (2.1)–(2.2) *on J, satisfying* (3.10). *Then $g \in C(J,W) \cap C^1(J,h^{1+\alpha})$ is a solution of* (3.9) *on J.*

Proof. The proof follows from our definition of $\mathcal{T}(g)$ in (2.6). □

We show the existence and uniqueness of solutions for (3.9).

Theorem 3.6 *Given any g_0 in W, there exists a unique maximal solution*

$$g(\cdot, g_0) \in C(J_{g_0}, W) \cap C^1(J_{g_0}, h^{1+\alpha})$$

for the nonlinear evolution equation (3.9), *where* $J_{g_0} := [0, t^+(g_0))$ *denotes the maximal interval of existence. The map* $(t, g_0) \mapsto g(t, g_0)$ *defines an analytic semiflow on* W.

Proof. The proof is based on Theorem 3.4, which enables us to apply the theory of maximal regularity. For the analyticity of the semiflow see [3]. We refer to [12, 13] for some additional information on the behavior of solutions as t approaches $t^+(g_0)$. □

Remarks 3.7 (a) It should be noted that the property of maximal regularity, as stated in Theorem 3.4b), is quite restrictive. In fact, a result of Baillon [5] shows that maximal regularity can only be expected in Banach spaces containing an isomorphic copy of c_0, the space of all sequences that converge to 0. On the other side, the results of Da Prato and Grisvard [9] ensure the existence of Banach spaces where maximal regularity does occur. (Of concern are, of course, unbounded operators).

(b) Observe that solutions of (3.9) preserve the regularity of the initial values. Theorem 3.6 should be compared with the main result in [17], where a Nash-Moser type approximation technique is used. With this approach, the authors are only able to guarantee existence of a (local) solution with much less space regularity than the initial values are assumed to have. In fact, these authors consider the case $n = 1$ and suppose that the initial values are in the Sobolev space $H^{18}(\mathbb{R})$. Then they guarantee the existence of a local solution $g \in C([0, \tau], H^{14}(\mathbb{R})) \cap C^1([0, \tau], H^1(\mathbb{R}))$.

4 The smoothing property

In this section we will prove the much stronger result that solutions regularize for $t > 0$. In order to obtain the results on analytic dependence, we will rely on a trick invented by Angenent (see [3] and [4]). This trick consists of introducing some additional parameters in the evolution equation, and then to use the implicit function theorem to exploit the analytic dependence on the parameters. Maximal regularity will again be instrumental in pushing through this idea.

Let $g_0 \in W$ be given and let

$$g := g(\cdot, g_0) \in C(J, W) \cap C^1(J, h^{1+\alpha}) \tag{4.1}$$

be the solution of (3.9), where $J := [0, t^+(g_0))$. Assume $T \in \dot{J}$ is fixed and set $I := [0, T]$. Let $(\lambda, \mu) \in (1 - \varepsilon, 1 + \varepsilon) \times \mathbb{R}^n$ with ε sufficiently small be given and define $g_{\lambda,\mu}$ by

$$g_{\lambda,\mu}(t) := \tau_{t\mu} g(\lambda t), \quad t \in I \,. \tag{4.2}$$

Note that $g_{\lambda,\mu}(t,x) = g(\lambda t, x + t\mu)$ for $t \in I$ and $x \in \mathbb{R}^n$.

Lemma 4.1 *Given $(\lambda, \mu) \in (1-\varepsilon, 1+\varepsilon) \times \mathbb{R}^n$, the function $g_{\lambda,\mu}$ satisfies*

$$g_{\lambda,\mu} \in C(I, W) \cap C^1(I, h^{1+\alpha})$$

and $g_{\lambda,\mu}$ solves the evolution equation

$$\partial_t h + \Phi_{\lambda,\mu}(h) = 0, \quad t \in I, \quad h(0) = g_0 \,, \tag{4.3$_{\lambda,\mu}$}$$

with

$$\Phi_{\lambda,\mu}(g) := \lambda\, \Phi(g) - (\mu \,|\, \nabla g) \quad \text{for } g \in W \,,$$

where $(\cdot \,|\, \cdot)$ denotes the inner product in $\mathbb{R}^n$.

Proof. Choose ε sufficiently small, such that $\lambda t \in J$ for all $\lambda \in (1-\varepsilon, 1+\varepsilon)$ and $t \in I$.

a) We then infer from (4.1) and Lemma 3.3b) that $g_{\lambda,\mu}(I) \subset W$. We show that $g_{\lambda,\mu} \in C(I, h^{2+\alpha})$. If $t \in I$ is fixed and $h \in \mathbb{R}$ is sufficiently small,

$$\begin{aligned} g_{\lambda,\mu}(t+h) - g_{\lambda,\mu}(t) &= \tau_{(t+h)\mu} g(\lambda(t+h)) - \tau_{t\mu} g(\lambda t) \\ &= \tau_{(t+h)\mu}(g(\lambda(t+h)) - g(\lambda t)) + (\tau_{(t+h)\mu} - \tau_{t\mu}) g(\lambda t) \,, \end{aligned}$$

and the assertion follows from Lemma 2.2 and (4.1).

b) Next we show that $g_{\lambda,\mu}(\cdot)$ is differentiable in $h^{1+\alpha}$, with derivative

$$\partial_t g_{\lambda,\mu}(t) = \lambda\, \tau_{t\mu} \partial_t g(\lambda t) + (\mu \,|\, \nabla g_{\lambda,\mu}(t)) \,. \tag{4.4}$$

Let $t \in I$ be given. If follows from the mean value theorem and from Lemma 2.2 that

$$\begin{aligned} h^{-1}(g_{\lambda,\mu}(t+h) - g_{\lambda,\mu}(t) - h\,(\lambda\, \tau_{t\mu} \partial_t g(\lambda t) + (\mu \,|\, \nabla g_{\lambda,\mu}(t)))) \\ = I_h^1(t) + I_h^2(t) + I_h^3(t) \,, \end{aligned}$$

where

$$I_h^1(t) = \lambda \int_0^1 \tau_{(t+h)\mu}(\partial_t g(\lambda t + \lambda s h) - \partial_t g(\lambda t))\, ds \,,$$

$$I_h^2(t) = \lambda\, (\tau_{(t+h)\mu} - \tau_{t\mu}) \partial_t g(\lambda t) \,,$$

$$I_h^3(t) = \int_0^1 (\tau_{(t+sh)\mu} - \tau_{t\mu})(\mu \,|\, \nabla g(\lambda t))\, ds \,.$$

We infer from Lemma 2.2 and (4.1) that $I_h^j(t)$ converges towards 0 in $h^{1+\alpha}$ as $h \to 0$, $j = 1,2,3$. This proves our claim. To verify that the derivative of $g_{\lambda,\mu}(\cdot)$ is continuous, it suffices to observe that

$$\lambda\, \tau_{t\mu} \partial_t g(\lambda t) + (\mu \,|\, \nabla g_{\lambda,\mu}(t)) = \tau_{t\mu}(\lambda \partial_t g(\lambda t) + (\mu \,|\, \nabla g(\lambda t))) \,.$$

Then (4.1) and an analogous argument as in a) give the assertion.

c) Using (4.4) and the fact that g is a solution of (3.9), we immediately get

$$\partial_t g_{\lambda,\mu}(t) = -\lambda\,\tau_{t\mu}\Phi(g(\lambda t)) + (\mu\,|\,\nabla g_{\lambda,\mu}(t)) .$$

Now, we involve Lemma 3.2 to obtain

$$\partial_t g_{\lambda,\mu}(t) = -\lambda\Phi(\tau_{t\mu}g(\lambda t)) + (\mu\,|\,\nabla g_{\lambda,\mu}(t)) .$$

Since $\tau_{t\mu}g(\lambda t) = g_{\lambda,\mu}(t)$, see (4.2), we have proved the assertions of Lemma 4.1. □

Now we turn our attention to the parameter dependent evolution equation $(4.3)_{\lambda,\mu}$. Invoking the implicit function theorem and maximal regularity (see Theorem 3.4), we will show that solutions of $(4.3)_{\lambda,\mu}$ depend analytically on the parameters λ and μ. This result is then used to prove that the functions $g_{\lambda,\mu}$, which are solutions of $(4.3)_{\lambda,\mu}$ by the previous arguments, admit much better regularity properties with respect to t and x than obtained in Theorem 3.6.

Lemma 4.2 *Let Λ be open in $(1-\varepsilon, 1+\varepsilon)\times\mathbb{R}^n$ with $(1,0)\in\Lambda$. Then the mapping*

$$C(I,W)\cap C^1(I,\,h^{1+\alpha})\times\Lambda \to C(I,\,h^{1+\alpha})\times h^{2+\alpha}$$

$$(g,(\lambda,\mu)) \mapsto F(g,(\lambda,\mu)) := (\partial_t g + \lambda\Phi(g) \ - (\mu\,|\,\nabla g), g(0)-g_0)$$

is analytic. Moreover,

$$\partial_1 F(g,(1,0)) \in Isom\,(C(I,\,h^{2+\alpha})\cap C^1(I,\,h^{1+\alpha}), C(I,\,h^{1+\alpha})\times h^{2+\alpha})$$

for $g\in C(I,W)\cap C^1(I,\,h^{1+\alpha})$, where $\partial_1 F$ is the derivative with respect to g.

Proof. Observe first that $C(I,W)$ is an open subset of $C(I,\,h^{2+\alpha})$. Indeed, this follows from the fact that W is open in $h^{2+\alpha}$ and from the compactness of I. Therefore, $dom(F)$, the domain of definition of F, is open in $C(I,\,h^{2+\alpha})\cap C^1(I,\,h^{1+\alpha})\times\Lambda$.

a) Note that $\Phi\in C^\omega(W,h^{1+\alpha})$ induces a mapping $\Phi\in C^\omega(C(I,W), C(I,\,h^{2+\alpha}))$, where we use the same notation without fearing too much confusion. Its derivative is given by

$$(\partial\Phi(g)h)(t) = \partial\Phi(g(t))h(t) . \tag{4.5}$$

Also note that the mapping

$$[g\mapsto(\partial_t g,\nabla g, g(0))] : C(I,W)\cap C^1(I,\,h^{1+\alpha}) \to C(I,\,h^{1+\alpha})^{n+1}\times h^{2+\alpha} \tag{4.6}$$

is analytic, being the restriction of a continuous linear operator to an open subset. It is now easy to see that F is analytic.

b) We infer from (4.5)–(4.6) that

$$\partial_1 F(g,(\lambda,\mu))h = (\partial_t h + \lambda\,\partial\Phi(g)h - (\mu\,|\,\nabla h), h(0)) \tag{4.7}$$

for $g \in C(I, W) \cap C^1(I, h^{1+\alpha})$ and $h \in C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha})$. We show that the linear inhomogeneous evolution equation

$$\partial_t h + \partial\Phi(g(t))h = f(t), \quad h(0) = x \tag{4.8}$$

has for each $(f, x) \in C(I, h^{1+\alpha}) \times h^{2+\alpha}$ a unique solution $h \in C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha})$. Indeed, this follows from Theorem 3.4 and an additional consideration. (See [2] Remark III 3.4.2b), [3] p. 100, or [9] p. 351). (4.7)–(4.8) and the open mapping theorem complete the proof. □

Proposition 4.3 *There exists an open neighborhood $\Lambda \subset (1-\varepsilon, 1+\varepsilon) \times \mathbb{R}^n$ of $(1,0)$ such that the mapping*

$$[(\lambda, \mu) \mapsto g_{\lambda,\mu}] \in C^\omega(\Lambda, C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha})) . \tag{4.9}$$

The derivatives $\partial_\lambda g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)}$ and $\partial_{\mu_j} g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)}$ are the solutions of

$$\partial_t h + \partial\Phi(g(t))h - \partial_t g(t) = 0, \quad h(0) = 0 ,$$

and

$$\partial_t h + \partial\Phi(g(t))h - \partial_{x_j} g(t) = 0, \quad h(0) = 0 ,$$

respectively, where $g = g(\,\cdot\,, g_0)$ is the solution of (3.9).

Proof. Observe that $F(g, (\lambda, \mu)) = (0, 0)$ holds true if, and only if, g is a solution of $(4.3)_{\lambda,\mu}$. Now all statements follow from Lemma 4.1, Lemma 4.2, and from the Implicit Function Theorem in Banach spaces. □

We are in the position to prove the smoothing property of solutions.

Theorem 4.4 *Let $g := g(\,\cdot\,, g_0)$ be the solution of (3.9), defined on $J := [0, t^+(g_0))$. Then*

$$g \in C^\infty(\dot{J} \times \mathbb{R}^n) .$$

Moreover,

$$t^{k+|\beta|} \partial_t^k \partial_x^\beta g \in C(J, h^{2+\alpha}) \cap C^1(J, h^{1+\alpha}), \quad (k, \beta) \in \mathbb{N} \times \mathbb{N}^n$$

and

$$t^{k+|\beta|} \partial_t^k \partial_x^\beta g(t) = \partial_\lambda^k \partial_\mu^\beta g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)}(t), \quad t \in J. \tag{4.10}$$

In addition,

$$t^{k+\ell} g \in C^k(J, h^{\ell+2+\alpha}) \cap C^{k+1}(J, h^{\ell+1+\alpha}), \quad k, \ell \in \mathbb{N}. \tag{4.11}$$

Proof. a) Let $T \in (0, t^+(g_0))$ and $m \in \mathbb{N}$ be fixed. Then there is an $\varepsilon := \varepsilon_m > 0$ such that

$$(1 + 2\varepsilon)^{m+1} T < t^+(g_0) .$$

Set $I_i := [0, (1+2\varepsilon)^{m-i} T]$, where $i \in \{0, \ldots, m\}$. Assume $(\lambda, \mu) \in ((1-\varepsilon_m, 1+\varepsilon_m) \times \mathbb{R}^n) \cap \Lambda$, where Λ is determined by Proposition 4.3. In the following we will show by an induction argument that

$$t^i \partial_t^k \partial_x^\beta g_{\lambda,\mu} = \partial_\lambda^k \partial_\mu^\beta g_{\lambda,\mu} \in C(I_i, h^{2+\alpha}) \cap C^1(I_i, h^{1+\alpha}), \quad k + |\beta| = i, \ i = 0, \ldots, m \tag{4.12$_i$}$$

and

$$t^i \partial_t^k \partial_x^\beta g_{\lambda,\mu} \in C(I_i, h^{1+\alpha}), \quad k + |\beta| = i + 1, \ i = 0, \dots, m, \qquad (4.13)_i$$

where $k \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. To do so, observe first that Lemma 4.1 ensures

$$g_{\lambda,\mu} \in C(I_0, h^{2+\alpha}) \cap C^1(I_0, h^{1+\alpha}), \qquad (4.12)_0$$

and hence

$$\partial_t g_{\lambda,\mu}, \ \partial_{x_j} g_{\lambda,\mu} \in C(I_0, h^{1+\alpha}). \qquad (4.13)_0$$

We infer from $(4.13)_0$ and the mean value theorem that

$$\omega(t,h) := h^{-1}(g_{\lambda+h,\mu}(t) - g_{\lambda,\mu}(t) - th\partial_t g_{\lambda,\mu}(t))$$
$$= t\int_0^1 \tau_{t\mu}(\partial_t g(\lambda t + sht) - \partial_t g(\lambda t))\,ds$$

for $t \in I_1$, where $h \in \dot{\mathbb{R}}$ with $|h| \leqq \varepsilon_m \wedge (1 - \varepsilon_m)$. Using Lemma 2.2, $(4.13)_0$ and the compactness of I_0, we can conclude that $\|\omega(t,h)\|_{1+\alpha}$ converges towards 0 as $h \to 0$, uniformly in $t \in I_1$, i.e.,

$$\omega(\cdot, h) \to 0 \quad \text{in } C(I_1, h^{1+\alpha}) \text{ as } h \to 0.$$

On the other hand, (4.9) ensures that the derivative of $[s \mapsto g_{s,\mu}]$ exists in the stronger topology of $C(I_1, h^{2+\alpha}) \cap C^1(I_1, h^{1+\alpha})$. Since this latter space is embedded in $C(I_1, h^{1+\alpha})$ and the derivative is uniquely determined, we have proved that $\partial_\lambda g_{\lambda,\mu} = t\partial_t g_{\lambda,\mu}$ in $C(I_1, h^{1+\alpha})$, and therefore that $t\partial_t g_{\lambda,\mu} \in C(I_1, h^{2+\alpha}) \cap C^1(I_1, h^{1+\alpha})$.

By Lemma 2.2 and the same arguments as above, we first find

$$h^{-1}(g_{\lambda,\mu+he_j}(t) - g_{\lambda,\mu}(t) - th\partial_{x_j} g_{\lambda,\mu}(t))$$
$$= t\int_0^1 (\tau_{t(\mu+she_j)} - \tau_{t\mu})\partial_{x_j} g(\lambda t))\,ds, \quad t \in I_1,$$

and we then conclude that $\partial_{\mu_j} g_{\lambda,\mu} = t\partial_{x_j}\partial_{\lambda,\mu}$. In summary, we have shown

$$t\partial_t g_{\lambda,\mu} = \partial_\lambda g_{\lambda,\mu} \in C(I_1, h^{2+\alpha}) \cap C^1(I_1, h^{1+\alpha})$$
$$(4.12)_1$$
$$t\partial_{x_j} g_{\lambda,\mu} = \partial_{\mu_j} g_{\lambda,\mu} \in C(I_1, h^{2+\alpha}) \cap C^1(I_1, h^{1+\alpha}).$$

$(4.13)_1$ is now a consequence of $(4.12)_1$ and $(4.13)_0$. In a next step, we employ the result in $(4.12)_1$ and the mean value theorem to derive

$$\partial_\lambda g_{\lambda+h,\mu}(t) - \partial_\lambda g_{\lambda,\mu}(t) - ht^2\partial_t^2 g_{\lambda,\mu}(t) = t(\partial_t g_{\lambda+h,\mu}(t) - \partial_t g_{\lambda,\mu}(t) - ht\partial_t^2 g_{\lambda,\mu}(t))$$
$$= ht\int_0^1 \tau_{t\mu}(t\partial_t^2 g(\lambda t + sht) - t\partial_t^2 g(\lambda t))\,ds, \quad t \in I_2.$$

Since $t\partial_t^2 g_{\lambda,\mu} \in C(I_1, h^{1+\alpha})$, see $(4.13)_1$, we can use the compactness of I_1 and Lemma 2.2 to see that this function divided by h converges towards 0 in $C(I_2, h^{1+\alpha})$ as $h \to 0$. Proposition 4.3 then implies

$$\partial_\lambda^2 g_{\lambda,\mu} = t^2\partial_t^2 g_{\lambda,\mu} \in C(I_2, h^{2+\alpha}) \cap C^1(I_2, h^{1+\alpha}).$$

The remaining assertions of $(4.12)_2$ are obtained in the same way. We can now repeat the arguments and we arrive, after a finite number of steps, to $(4.12)_m$ and $(4.13)_m$.

b) Note that $I = I_m \subset I_{m-1} \subset \cdots \subset I_0$. Hence the statements of $(4.12)_i$ and $(4.13)_i$ remain true on the fixed interval I for $i = 1, \ldots, m$. By choosing $(\lambda, \mu) = (1, 0)$ we have, in particular,

$$t^i \partial_t^k \partial_x^\beta g = \partial_\lambda^k \partial_\mu^\beta g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)} \in C(I,\, h^{2+\alpha}) \cap C^1(I,\, h^{1+\alpha}), \quad k + |\beta| = i \tag{4.14}$$

for $i = 0, \ldots, m$, and

$$t^i \partial_t^k \partial_x^\beta g_{\lambda,\mu} \in C(I,\, h^{1+\alpha}), \quad k + |\beta| = i + 1, \quad i = 0, \ldots, m\,. \tag{4.15}$$

Since the interval $I = [0, T] \subset J$ and $m \in \mathbb{N}$ can be chosen arbitrarily, (4.10) is an immediate consequence of (4.14). Using the fact that $C(\dot{J}, h^{2+\alpha})$ is embedded in $C(\dot{J} \times \mathbb{R}^n)$, we have also proved that $g \in C^\infty(\dot{J} \times \mathbb{R}^n)$.

c) Observe that $g \in h^{\ell+2+\alpha}$ if, and only if, $\partial_x^\beta g \in h^{2+\alpha}$ for $|\beta| \leqq \ell$. Therefore, $t^{k+\ell} g \in C^k(J, h^{\ell+2+\alpha})$ holds true if and only if

$$\partial_t^j (t^{k+\ell} \partial_x^\beta g) \in C(J, h^{2+\alpha}), \quad j = 0, \ldots, k, \quad |\beta| \leqq \ell\,. \tag{4.16}$$

Let $j \in \{0, \ldots, k\}$ and $|\beta| \leqq \ell$ be fixed. It is easily verified that $\partial_t^j (t^{k+\ell} \partial_x^\beta g(t))$ consists of a finite linear combination of terms

$$t^{k-j+\ell-|\beta|}\, t^{i+|\beta|} \partial_t^i \partial_x^\beta g(t), \quad i = 0, \ldots, j\,.$$

Now, (4.16) follows from (4.10).

d) It follows from c) that $t^{k+\ell} g \in C^k(J, h^{\ell+1+\alpha})$. Hence, it remains to prove that

$$\partial_t^{k+1} (t^{k+\ell} \partial_x^\beta g) \in C(J, h^{1+\alpha}), \quad |\beta| \leqq \ell\,.$$

This can be achieved by similar considerations as in part c), where (4.15) is being used instead of (4.10). □

Remarks 4.5 Let $g := g(\,\cdot\,, g_0)$ be the solution of (3.9), defined on $J = [0, t^+(g_0))$.

(a) It has been proved in Theorem 4.4 that g enjoys better regularity properties than given in (4.1), in particular $t\partial_t g \in C(J, h^{2+\alpha}) \cap C^1(J, h^{1+\alpha})$. In addition, $t\partial_t g$ is the solution of the linear equation

$$\partial_t h + \partial\Phi(g(t))h - \partial_t g(t) = 0, \quad t \in J, \quad h(0) = 0\,. \tag{4.17}$$

This can be shown by using Proposition 4.3 and Theorem 4.4, or by a direct computation. We will briefly indicate a different and more elementary proof of the regularizing property of solutions to the nonlinear equation (3.9). Set $I := [0, T]$ for some fixed $T \in \dot{J}$ and choose $\varepsilon_0 > 0$ such that $[0, T + \varepsilon_0] \subset J$. Set

$$v(t, \varepsilon) := \varepsilon^{-1}(tg(t + \varepsilon) - tg(t)), \quad t \in I, \quad \varepsilon \in (0, \varepsilon_0]\,.$$

Then it is not difficult to see that $v(\cdot, \varepsilon)$ is a solution of the linear equation

$$\partial_t v + (A(t) + B(t,\varepsilon))v + f(t,\varepsilon) = 0, \quad t \in I, \quad v(0) = 0\,, \tag{4.18$_\varepsilon$}$$

where $A(t) = \partial\Phi(g(t))$ and

$$B(t,\varepsilon) = \int_0^1 (\partial\Phi(g(t) + s(g(t+\varepsilon) - g(t))) - \partial\Phi(g(t)))\,ds\,,$$

$$f(t,\varepsilon) = -\varepsilon^{-1}(g(t+\varepsilon) - g(t))\,.$$

Using (4.1), it can be verified that

$$B(\cdot,\varepsilon) \to 0 \text{ in } C(I, \mathscr{L}(h^{2+\alpha}, h^{1+\alpha})), \quad f(\cdot,\varepsilon) \to \Phi(g(\cdot)) \text{ in } C(I, h^{1+\alpha})\,,$$

as $\varepsilon \to 0$. It follows with the help of maximal regularity (see Theorem 3.4, and [3] p. 100) that the inhomogeneous linear evolution equation (4.17) has a unique solution $h \in C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha})$. The same argument ensures that $(4.18)_\varepsilon$ has, for ε sufficiently small, a unique solution $v = v_\varepsilon$, possessing the same regularity as h. Moreover, it can be shown that

$$v_\varepsilon \to h \text{ in } C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha}) \text{ as } \varepsilon \to 0\,.$$

Since $v(\cdot,\varepsilon)$ is a solution of $(4.18)_\varepsilon$, we have $v(\cdot,\varepsilon) = v_\varepsilon$, and hence

$$v(\cdot,\varepsilon) \to h \text{ in } C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha}) \text{ as } \varepsilon \to 0\,.$$

On the other side, $v(\cdot,\varepsilon) \to t\partial_t g$ in the topology of $C(I, h^{1+\alpha})$ as $\varepsilon \to 0$. Hence we have proved that

$$t\partial_t g = h \in C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha})\,.$$

(b) We consider now the linear evolution equation

$$\partial_t h + \partial\Phi(g(t))h - \partial_j g(t) = 0, \quad t \in J, \quad h(0) = 0\,. \tag{4.19}$$

Let

$$v(t,\varepsilon) := \varepsilon^{-1}(t\tau_{\varepsilon e_j} g(t) - tg(t)), \quad t \in I, \quad \varepsilon \in (0, \varepsilon_0]\,.$$

It follows from Lemma 2.2, Lemma 3.2, and from (4.1) that $v(\cdot,\varepsilon)$ is a solution of

$$\partial_t v + (A(t) + B(t,\varepsilon))v + f(t,\varepsilon) = 0, \quad t \in I, \quad v(0) = 0\,, \tag{4.20$_\varepsilon$}$$

where $A(t) = \partial\Phi((t))$ and

$$B(t,\varepsilon) = \int_0^1 (\partial\Phi(g(t) + s(\tau_{\varepsilon e_j} g(t) - g(t))) - \partial\Phi(g(t)))\,ds\,,$$

$$f(t,\varepsilon) = -\varepsilon^{-1}(\tau_{\varepsilon e_j} g(t) - g(t))\,.$$

Using Lemma 2.2, (4.1), and a compactness argument, we can verify that

$$B(\,\cdot\,,\varepsilon) \to 0 \text{ in } C(I, \mathscr{L}(h^{2+\alpha}, h^{1+\alpha})), \quad f(\,\cdot\,,\varepsilon) \to -\partial_j g \text{ in } C(I, h^{1+\alpha}),$$

as $\varepsilon \to 0$. It can be shown that the solution v_ε of $(4.20)_\varepsilon$ converges to the solution h of (4.19) in the topology of $C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha})$ as $\varepsilon \to 0$. Now, we can conclude as above that

$$t\partial_j g = h \in C(I, h^{2+\alpha}) \cap C^1(I, h^{1+\alpha}).$$

(c) One can then proceed by induction along the lines of a) and b). In order to prove statements for higher order partial derivatives of g with respect to x, Lemma 3.2 is to be replaced by

$$\tau_a(\partial^k \Phi(g)[h_1, \ldots, h_k]) = \partial^k \Phi(\tau_a g)[\tau_a h_1, \ldots, \tau_a h_k], \quad g \in W, \quad h_1, \ldots, h_k \in h^{2+\alpha}.$$

Here, $a \in \mathbb{R}^n$ and $\partial^k \Phi(g)$ is the Fréchet derivative of Φ of order k at g. It can be proved that the mapping Φ defined in (3.1) satisfies this property.

(d) Angenent's trick provides a very elegant way to prove Theorem 4.4. Note that one can obtain the regularity results of Theorem 4.4 by using the ideas sketched in a)–c).

(e) The proof of the smoothing effect with respect to x relies on the fact that the nonlinear (non-local) operator Φ commutes with translations. Note that this property is always satisfied for local operators, that is, for substitution operators induced by local functions.

(f) Let g be the solution of (3.9). Then we obtain by a formal computation that

$$\partial_t(t\partial_t g(t)) = -t\partial_t \Phi(g(t)) + \partial_t g(t) = -\partial\Phi(g(t)) t\partial_t g(t) + \partial_t g(t).$$

Moreover, by purely formal arguments, we also see that

$$\begin{aligned}\partial_t(t\partial_j g(t)) &= t\partial_j\partial_t g(t) + \partial_j g(t) = -t\partial_j \Phi(g(t)) + \partial_j g(t) \\ &= -\partial\Phi(g(t)) t\partial_j g(t) + \partial_j g(t).\end{aligned}$$

The arguments in (a) and (b) (or Lemma 3.2 and Theorem 4.4) show that all of the steps are justified.

5 Analytic dependence, proof of Theorem 1.1

We will now use the full power of Proposition 4.3 to prove the analytic dependence of g upon t and x.

Theorem 5.1 *Let* $g := g(\,\cdot\,, g_0)$ *be the solution of* (3.9), *defined on* $J := [0, t^+(g_0))$. *Then*

$$g \in C^\omega(\dot{J} \times \mathbb{R}^n).$$

Proof. Note that we have already proved in Theorem 4.4 that g is smooth on $\dot{J} \times \mathbb{R}^n$. Hence, it remains to show that g is represented by its Taylor series in a neighborhood of any point of $\dot{J} \times \mathbb{R}^n$. Thus, let $(t_0, x_0) \in \dot{J} \times \mathbb{R}^n$ be given. We will show that there is a $r_0 = r_0(t_0) > 0$ with

$$g(t,x) = \sum_{k+|\beta|=0}^{\infty} \frac{1}{k!\beta!} \partial_t^k \partial_x^\beta g(t_0,x_0)(t-t_0)^k (x-x_0)^\beta \tag{5.1}$$

for $(t,x) \in \mathbb{B}((t_0,x_0),r_0)$. Let $T \in \dot{J}$ be given with $t_0 < T$ and set $I := [0,T]$. Observe that (4.9) ensures, in particular, that the mapping

$$[(\lambda,\mu) \mapsto g_{\lambda,\mu}] : \Lambda \to C(I,\, h^{2+\alpha}) \cap C(I,\, h^{1+\alpha})$$

can be represented by its Taylor series in a neighborhood of $(\lambda,\mu) = (1,0)$. We can find $r > 0$ with

$$\frac{1}{k!\beta!} \| \partial_\lambda^k \partial_\mu^\beta g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)} \|_{C(I,\, h^{2+\alpha}) \cap C^1(I,\, h^{1+\alpha})} \, r^{k+|\beta|} \leqq M \tag{5.2}$$

for some $M > 0$. It follows from (4.10) that

$$t^{k+|\beta|} \, |\partial_t^k \partial_x^\beta g(t,x)| \leqq \| \partial_\lambda^k \partial_\mu^\beta g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)} \|_{C(I,\, h^{2+\alpha}) \cap C^1(I,\, h^{1+\alpha})} \tag{5.3}$$

for $(t,x) \in I \times \mathbb{R}^n$. Let $\varepsilon_0 \in (0,t_0)$ be fixed and set

$$r_0 := \min\{\varepsilon_0 r,\; t_0 - \varepsilon_0,\; T - t_0\}\,. \tag{5.4}$$

Then it follows from (5.2)–(5.4) that

$$\frac{1}{k!\beta!} |\partial_t^k \partial_x^\beta g(t,x)| \, r_0^{k+|\beta|} \leqq M \tag{5.5}$$

for $t \in (t_0 - r_0, t_0 + r_0)$ and $x \in \mathbb{R}^n$. Hence, the radius of convergence of the Taylor series (5.1) equals at least r_0. Moreover, we deduce from (5.5) that g is being represented in $\mathbb{B}((t_0,x_0),r_0)$ by its power series. And so, the proof of Theorem 5.1 is finished. □

Proof of Theorem 1.1 Let $f_0 \in V$ be given and set $g_0 := f_0 - c$. Then $g_0 \in W$, and Theorem 3.6 and Theorem 5.1 ensure that the nonlinear evolution equation (3.9) has a unique solution $g := g(\cdot, g_0)$, defined on $J := [0, t^+(g_0))$ and satisfying

$$g \in C(J,W) \cap C(\dot{J}, h^\infty) \cap C^\omega(\dot{J} \times \mathbb{R}^n)\,, \tag{5.6}$$

where $h^\infty := \bigcap_{s \in \mathbb{R}_+} h^s$. We define

$$v(t) := \mathcal{T}(g(t))g(t), \quad t \in J\,.$$

Then it follows from (2.6), (5.6), and elliptic regularity theory, that

$$v(t) \in C^\omega(\overline{\Omega}), \quad t \in \dot{J}\,.$$

Lemma 3.5 shows that (v,g) is a classical [smooth] solution of (2.1)–(2.2) on J [on $\dot{J}$]. Next, define

$$(u(t), f(t)) := ((\phi_*^{g(t)} v(t)) + c, g(t) + c), \quad t \in J\,,$$

where ϕ_*^g is introduced in Sect. 2. It follows that (u, g) is a solution of (1.1)–(1.3) having the regularity properties stated in Theorem 1.1. Since each of the steps can be reversed, we have also proved uniqueness. Finally, it follows from Theorem 3.6 and the considerations above that the map $(t, f_0) \to f(t, f_0)$ defines an analytic semiflow on V. This completes the proof of Theorem 1.1. □

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