# MOVING SURFACES AND ABSTRACT PARABOLIC EVOLUTION EQUATIONS

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*Dedicated to Herbert Amann on the occasion of his 60th birthday*

## 1. Introduction

It is the purpose of this paper to give a survey of some recent developments in the theory of classical solutions to elliptic and parabolic problems involving moving surfaces. Problems of this type do not satisfy a superposition principle for solutions and, hence, carry an inherent nonlinear structure. In fact, it turns out that most of the equations describing the evolution of surfaces are of quasilinear or even of fully nonlinear type. Additionally, these equations are often of a nonlocal nature.

From a mathematical point of view it therefore seems tempting - as a first step in the analysis - to look for weak solutions to these nonlinear equations. In many applications, however, the corresponding mathematical model is obtained under the assumption of the existence of a sharp and smooth surface or moving boundary. Of course, one can try to follow a two-step procedure of first constructing weak solutions and then, in a second step, analyzing the actual regularity of a weak solution. Both steps are often closely tied to a comparison principle. Hence, for problems without the luxury of a comparison or maximum principle the above program is not at all obvious to realize. We mention the quasi-stationary Stefan problem with surface tension, the Mullins-Sekerka model, or the surface diffusion flow which do not satisfy a comparison principle. For these problems neither existence of (even weak) solutions nor uniqueness of (even classical) solutions was established until quite recently, see [34, 35, 36, 40, 20, 21, 22]. On the other hand, there are moving boundary problems for which one can guarantee existence of weak solutions (e.g. the Hele-Shaw flow without surface tension, the Stefan problem with Gibbs-Thomson corrections) but for which the actual regularity of weak solutions is still far from being understood.

There is a different approach to problems with moving surfaces in which one seeks a unique classical solution from the very beginning. In this approach one

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transforms a given moving surface problem by means of appropriate diffeomorphisms into a problem on a fixed reference domain and then evolves the diffeomorphisms. In this paper we present several problems with moving surfaces for which the transformed equations lead to nonlinear evolution equations. It turns out that these equations are of parabolic type (in the sense that the corresponding linearized equations can be treated by means of analytic semigroup theory) and hence are accessible to well-established techniques for nonlinear parabolic evolution equations, see [5, 6, 9, 24, 56]. This approach does not rely on a comparison or maximum principle and, in fact, will provide existence and uniqueness results of classical solutions to problems for which neither existence of weak solutions nor uniqueness of classical solutions was previously known, see the results in Sections 2 and 3.

A further advantage of this method is that we are able to specify 'large' function spaces of initial data on which the evolution equations under consideration generate local semiflows. This means in particular that there is no loss of regularity as long as one does not encounter singularities. Furthermore, we show that these semiflows possess a regularizing property in the sense that for positive time the solutions are smooth or even real analytic in space and time.
Any information on the dynamical behavior of these semiflows immediately translates into information about the corresponding moving surface. This approach allows, for instance, to determine stability properties of equilibria, leading to new global existence results [36, 38, 40], even for well-studied problems like the averaged mean curvature flow [39].

As already mentioned above, the analysis of the linearized problems is fundamental for this approach. In particular, we need sharp parameter-dependent a priori estimates for the corresponding linear operators in order to ensure that these operators generate analytic semigroups on appropriate function spaces. However, we cannot rely on well-established a priori estimates for elliptic differential operators, since the linear operators coming from nonlocal moving surface problems are usually not differential operators, but rather pseudo-differential operators. For example, the linearization of the one-sided Hele-Shaw operator without surface tension involves a first order hyperbolic differential operator and a singular Green operator, see [37]. A similar situation, with an additional first order elliptic pseudo-differential operator, is also described in Lemma 2.2 below. In order to obtain sharp resolvent estimates for operators of this type we strongly rely on results and techniques of H. Amann's theory of Fourier multipliers on Besov spaces [6, 7].

## 2. Flows through porous media

We consider the flow of a quasi-incompressible Newtonian fluid through a porous medium. More precisely, let $\mathbb{R}^3$ represent a homogeneous, isotropic, and deformable porous medium. Let furthermore $\Sigma := \mathbb{R}^2 \times \{0\}$ be a fixed impermeable layer in $\mathbb{R}^3$ and assume that some part of the region above the layer

$\Sigma$ is occupied by the fluid. Finally, we assume that there is a sharp interface $\Gamma_t$ separating at every instant $t$ the wet part $\Omega_t$ from the dry part $\mathbb{R}^3 \backslash cl\,(\Omega_t)$.

In order to describe a hydrological model for this flow, let $\rho$, $v$, $p$, and $\varepsilon$ denote the density, the velocity, the pressure, and the strain tensor of the fluid phase, respectively. Under suitable simplifying hypotheses, in particular assuming that spatial changes of $\rho$ and $\varepsilon$ are much smaller than the corresponding temporal changes, the basic mass balance equation of the fluid phase is given by

$$\nabla \cdot v + n\beta\partial_t p + \partial_t \varepsilon = 0, \tag{2.1}$$

where $\beta$ is the so-called fluid's coefficient of compressibility and $n$ is the solid's porosity, cf. [12], p.300. We now assume that

$$\rho = const. \qquad \text{and} \qquad n = const.$$

This assumption particularly permits to introduce the so-called velocity potential or piezometric head

$$u(x,y) := \frac{p(x,y)}{g\rho} + y, \qquad (x,y) \in \mathbb{R}^2 \times \mathbb{R}, \tag{2.2}$$

where $g$ is the gravity acceleration, see [12], p.175, [23], p.32, or [29], p.24. Assuming that the motion is governed according to Darcy's law $v = -k\nabla u$, where $k > 0$ stands for the hydraulic conductivity, the mass balance equation (2.1) becomes

$$-k\Delta u + \rho g n \beta \partial_t u + \partial_t \varepsilon = 0. \tag{2.3}$$

Finally, we introduce the compressibility $\alpha$ of the porous medium (cf. [12], p. 308) and we assume that the dilatation-pressure relation $\varepsilon = \alpha p$ holds. This relation is fulfilled if, e.g., one assumes that displacements of the solid phase occur in vertical direction only, an assumption usually taken for granted in hydrology of groundwater, see [12], p. 310. Using the relation $\varepsilon = \alpha p$ in (2.3) we find

$$S_0 \partial_t u - k\Delta u = 0 \quad \text{in} \quad \Omega_t, \tag{2.4}$$

where $S_0 := \rho g(\alpha + n\beta) \geq 0$ is called the specific storativity. Observe that $S_0 = 0$ for the flow of an incompressible Newtonian fluid through a rigid porous medium, since in this case $\alpha = \beta = 0$.

Equation (2.4) governs the motion of the fluid in the domain $\Omega_t$. Let us now discuss the boundary conditions on $\partial\Omega_t = \Sigma \cup \Gamma_t$. First, recall that $\Sigma$ is assumed to be impermeable for the fluid. Hence we have the no-flux condition

$$\partial_3 u = 0 \quad \text{on} \quad \Sigma.$$

Next, we assume that the interface separating the fluid from the air is given as a graph over $\mathbb{R}^2$ of a time-dependent function $f$, i.e.,

$$\Gamma_t := \Gamma_{f(t)} := \mathrm{graph}(f(t,\cdot)).$$

Moreover, the free interface $\Gamma_{f(t)}$ is characterized by

$$(x,y) \in \Gamma_{f(t)} \quad \Longleftrightarrow \quad y = \sup\{z \in \mathbb{R}\,;\, p(t,x,z) > 0\}.$$

If we neglect surface tension effects the pressure $p$ is continuous, see [29], p. 18. Hence, normalizing the atmospheric pressure to be equal to 0, we find

$$p(t,x,y)=0 \quad \text{for} \quad (x,y)\in\Gamma_{f(t)}. \tag{2.5}$$

Because of $y=f(t,x)$, we obtain from (2.2) and (2.5) the boundary condition

$$u(t,x,f(t,x))=f(t,x), \quad (x,t)\in\mathbb{R}^2\times[0,\infty). \tag{2.6}$$

Finally, setting $F(t,z):=y-f(t,x)$ for $z=(x,y)\in\mathbb{R}^2\times\mathbb{R}$ and $t\geq 0$, the interfaces $\{\Gamma_{f(t)}\,;\,t\geq 0\}$ are characterized by the conservation property that $F$ is identically equal to 0 on $\{\Gamma_{f(t)}\,;\,t\geq 0\}$. Consequently, we get

$$\frac{d}{dt}F(t,z)=\partial_t F(t,z)+\nabla f(t,z)z'=0 \quad \text{on} \quad \Gamma_{f(t)}. \tag{2.7}$$

Again by Darcy's law the velocity $z'$ is given by $-k\nabla u$. Since $\partial_t F=-\partial_t f$ and $\nabla F=(-\nabla_x f,1)$, equation (2.7) becomes

$$\partial_t f+k\sqrt{1+|\nabla f|^2}\partial_\nu u=0 \quad \text{on} \quad \Gamma_{f(t)},$$

where $\partial_\nu$ stands for differentiation with respect to the outer unit normal. Summarizing, we obtain the following set of equations:

$$\begin{cases} S_0\partial_t u-k\Delta u=0 & \text{in} \quad \Omega_t \\ \partial_3 u=0 & \text{on} \quad \Sigma \\ u=f & \text{on} \quad \Gamma_t \\ \partial_t f+k\sqrt{1+|\nabla f|^2}\partial_\nu u=0 & \text{on} \quad \partial\Gamma_t \end{cases} \tag{2.8}$$

where the constants $k$, $S_0$ satisfy $k>0$ and $S_0\geq 0$. Of course, system (2.8) has to be completed by initial conditions and, as it turns out, by a normalization at $\infty$. We will specify appropriate initial conditions later depending on whether $S_0>0$ or $S_0=0$.

Let us first take (2.8) as a model for the following general situation: Let $n\in\mathbb{N}$ with $n\geq 1$ be fixed and set

$$A_0:=\{f\in BC^1(\mathbb{R}^n,\mathbb{R})\,;\,\inf_{x\in\mathbb{R}^n}f(x)>0\}.$$

Given $f\in A_0$, define

$$\Omega_f:=\{(x,y)\in\mathbb{R}^{n+1}\,;\,0<y<f(x)\}.$$

The boundary of this unbounded $C^1$-domain is given by $\Sigma\cup\Gamma_f$, where $\Sigma:=\mathbb{R}^n\times\{0\}$ and $\Gamma_f:=\text{graph}(f)$.

2.1. **Incompressible fluids in rigid porous media.** If we consider the flow of an incompressible fluid in a rigid porous medium, the specific storativity $S_0$ vanishes, cf. (2.4). In this case (2.8) is a model case for the following moving boundary problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_f \\ \partial_\nu u = 0 & \text{on} \quad \Sigma \\ u = f & \text{on} \quad \Gamma_f \\ \lim_{|(x,y)|\to\infty} u(\cdot,(x,y)) = c & \text{on} \quad [0,T] \\ \partial_t f + k\sqrt{1+|\nabla f|^2}\partial_\nu u = 0 & \text{on} \quad \Gamma_f \\ f(0,\cdot) = f_0 & \text{on} \quad \mathbb{R}^n, \end{cases} \tag{2.9}$$

where $f_0 \in A_0$ is a given initial data and $c$ is a given positive constant. To formulate our results for system (2.9), let $h^{k+\alpha}(\mathbb{R}^n)$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, be the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in the usual Hölder norm of $BUC^{k+\alpha}(\mathbb{R}^n)$. It is also convenient to write $h_c^{k+\alpha} := h^{k+\alpha} + c$. Let now

$$U := \{f \in A_0 \,;\, f \in h_c^{2+\alpha}(\mathbb{R}^n)\}$$

and consider for a given $f \in U$ the following elliptic boundary value problem on the unbounded domain $\Omega_f$

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_f \\ \partial_\nu u = 0 & \text{on} \quad \Sigma \\ u = f & \text{on} \quad \Gamma_f \\ \lim\limits_{|(x,y)|\to\infty} u(x,y) = c. \end{cases} \tag{2.10}$$

It can be shown that there exists a unique classical solution $u_f$ of (2.10), see [33], Section 2. We are now prepared to introduce the following domain of parabolicity for problem (2.9):

$$V := \{f \in U \,;\, \partial_{n+1} u_f(x, f(x)) < (1+|\nabla f(x)|^2)^{-1}, \quad x \in \mathbb{R}^n\}. \tag{2.11}$$

Obviously, $f \equiv c$ belongs to $V$. Additionally, it can be shown that $V$ is an open neighborhood of $c$ in $BUC^{2+\alpha}(\mathbb{R}^n)$ and that its diameter in $BUC^{2+\alpha}(\mathbb{R}^n)$ is unbounded. Our main result for the moving boundary problem (2.9) reads as follows:

**Theorem 2.1.** *Given $f_0 \in V$, there exists a unique maximal classical solution to (2.9), i.e., there exists a maximal $t^+ = t^+(f_0) > 0$ and a unique pair $(u, f)$ such that*

$$u(t,\cdot) \in h_c^{2+\alpha}(\Omega_{f(t)}),\ t \in [0,t^+),$$
$$f \in C([0,t^+), V) \cap C^1([0,t^+), h_c^{1+\alpha}(\mathbb{R}^n)).$$

*In addition, we have the following smoothing property for the solution:*

$$u(t,\cdot)\in C^\omega(\overline{\Omega}_{f(t)}),\quad t\in(0,t^+),\quad f\in C^\omega((0,t^+)\times\mathbb{R}^n),$$

*where $C^\omega$ stands for the set of all real analytic functions.*

Let us explain some steps of the proof of Theorem 2.1. The full details can be found in [32, 33]. First we reduce system (2.9) to a single evolution equation for the moving interface. For this let $V_c := V - c$, and observe that $V_c$ is an open neighborhood of 0 in $h^{2+\alpha}(\mathbb{R}^n)$. Given $g\in V_c$, define

$$\varphi_g(x,y) := (x, (1-y)(c+g(x)))\quad \text{for}\quad (x,y)\in\Omega,$$

where $\Omega := \mathbb{R}^n\times(0,1)$ is now a fixed reference domain. It is not difficult to see that $\varphi_g$ is a diffeomorphism of class $C^{2+\alpha}$, mapping $\Omega$ onto $\Omega_{c+g}$. Let $\varphi_g^* := u\circ\varphi_g$ and $\varphi_*^g v := v\circ\varphi_g^{-1}$ denote the corresponding pull-back and push-forward operator, respectively, and let

$$A(g)v := -\varphi_g^*\Delta(\varphi_*^g v),\qquad B_i(g)v := k\varphi_g^*(\gamma_i\nabla(\varphi_*^g v)|n_i),\quad i=0,\,1,\tag{2.12}$$

for $v\in h^{2+\alpha}(\Omega)$ be the corresponding transformed operators. Here, $\gamma_i$ is the trace operator with respect to $\Sigma$ and $\Gamma_{c+g}$, respectively, and $n_1 := (-\nabla g, 1)$ and $n_0 := (0,-1)$ stand for the outer normal direction on $\Gamma_{c+g}$ and on $\Sigma$, respectively. The solution operator for the transformed elliptic boundary value problem on $\Omega$

$$\begin{cases} A(g)v = w & \text{in } \Omega\\ v = h & \text{on } \Gamma_0\\ B_1(g)v = 0 & \text{on } \Gamma_1\\ \lim_{|(x,y)|\to\infty} v(x,y) = 0, & \end{cases}\tag{2.14}$$

is denoted by $R(g)$, where $\Gamma_i := \mathbb{R}^n\times\{i\}$, $i=0,1$. More precisely, given $(w,h)\in h^\alpha(\Omega)\times h^{2+\alpha}(\Gamma_0)$, it can be shown by classical methods (cf. Appendix C in [32]) that there exists a unique solution $R(g)(w,h)$ of (2.14) in $h^{2+\alpha}(\Omega)$. It is convenient to split the operator $R(g)$ into $R(g) = S(g)\oplus T(g)$, where $S(g) := R(g)|h^\alpha(\Omega)\times\{0\}$ and $T(g) := R(g)|\{0\}\times h^{2+\alpha}(\Gamma_0)$. Of course, using the notation from (2.10), we have $\varphi_g^* u_f - c = T(g)g$ with $g = f-c$. Hence it is not difficult to see that the moving boundary problem (2.9) is equivalent to the abstract evolution equation

$$\frac{d}{dt}g + \Phi(g) = 0,\qquad g(0) = g_0\qquad \text{in}\quad h^{1+\alpha}(\mathbb{R}^n),\tag{2.15}$$

where $\Phi(g) := B_0(g)T(g)g$, $g_0 := f_0 - c$, and where we identify in the following $\Gamma_0$ with $\mathbb{R}^n$. More precisely, if $(u,f)$ is a classical solution to (2.9) with initial data $f_0\in V$, then $g := f-c$ is a solution to (2.15) with initial data $g_0 := f_0 - c$, and vice versa: if $g$ is a solution to (2.15) with initial data $g_0\in V_c$ belonging to $C([0,t^+),V_c)\cap C^1([0,t^+),h^{1+\alpha}(\mathbb{R}^n))$, then the pair $(\varphi_*^g T(g)g + c, g + c)$ is a classical solution to (2.9) with initial data $f_0 := g_0 + c$. Consequently, we will focus our attention on the abstract evolution equation (2.15).

Observe that the diffeomorphism $\varphi_g$ depends analytically on $g$. Hence, expressing the coefficients of the transformed operators $A(g)$ and $B_0(g)$ in terms of $\varphi_g$ and using the inverse function theorem, one can verify that $\Phi$ depends analytically on $g \in V_c$. More precisely, Proposition 3.1 [33] and Lemma 4.3 [32] yield:

**Lemma 2.2.** *We have $\Phi \in C^\omega(V_c, h^{1+\alpha}(\mathbb{R}^n))$ with*

$$\partial\Phi(g)h = B_0(g)T(g)h + \partial B_0(g)[h, T(g)g] - B_0(g)S(g)\partial A(g)[h, T(g)g],$$

*for $g \in V_c$ and $h \in h^{2+\alpha}(\mathbb{R}^n)$. Here*

$$\partial B_0(g)[h, v] := \frac{d}{ds} B_0(g + sh)v\big|_{s=0}, \quad \partial A(g)[h, v] := \frac{d}{ds} A(g + sh)v\big|_{s=0},$$

*for $v \in h^{2+\alpha}(\Omega)$.*

In order to further investigate the linearization of $\Phi$, it is convenient to introduce the following notation. Given two Banach spaces $E_1$ and $E_0$ such that $E_1$ is continuously and densely injected into $E_0$, we denote by $\mathcal{H}(E_1, E_0)$ the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as an unbounded operator in $E_0$ with domain $E_1$, generates a strongly continuous analytic semigroup on $E_0$.

In the following we fix $g \in V_c$. The operator $B_0(g)T(g)$ is a pseudo-differential operator of first order and it is called the generalized Dirichlet-Neumann operator, [8, 30, 48]. It can be shown that $B_0(g)T(g) \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^n), h^{1+\alpha}(\mathbb{R}^n))$, see Corollary 6.3 in [32]. The operator $\partial B_0(g)[\cdot, T(g)g]$ is a first order hyperbolic differential operator, whereas $B_0(g)S(g)\partial A(g)[\cdot, T(g)g]$ is a first order pseudo-differential operator. Hence, the full linearization $\partial\Phi(g)$ is the sum of three first order pseudo-differential operators. However, the following result holds true:

**Theorem 2.3.** *Given $g \in V_c$, we have*

$$\partial\Phi(g) \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^n), h^{1+\alpha}(\mathbb{R}^n)).$$

The main ideas of the proof of Theorem 2.3 can be summarized as follows: First we associate to each of the operators

$$B_0(g)T(g), \quad \partial B_0(g)[\cdot, T(g)g], \quad \text{and} \quad - B_0(g)S(g)\partial A(g)[\cdot, T(g)g]$$

Fourier multiplier operators $F_1$, $F_2$, and $F_3$ by freezing the spatial variable $x \in \mathbb{R}^n$. Using then the fact that each $g \in V_c$ satisfies

$$\partial_{n+1}[T(g)g](x, 0) + \frac{c + g(x)}{1 + |\nabla g(x)|^2} > 0, \quad x \in \mathbb{R}^n,$$

cf. (2.14), it can be shown that $F := F_1 + F_2 + F_3$ is a regularly elliptic Fourier multiplier of first order in the sense of H. Amann, see [6, 7]. For details we refer to Section 3 in [33]. Subtle perturbation techniques then allow to carry over the generation property of $F$ to $\partial\Phi(g)$, cf. Section 6 in [32].

Based on Theorem 2.3 we can now rely on maximal regularity results in the sense of Da Prato and Grisvard [24], see also [9, 56, 65], to find a unique solution

$$g \in C([0,t^+),V_c) \cap C^1([0,t^+),h^{1+\alpha}(\mathbb{R}^n))$$

of the nonlinear equation (2.15). The regularizing property of the solution, i.e., the fact that $g$ is actually real analytic in $(0,t^+) \times \mathbb{R}^n$, follows from the equivariance of problem (2.9) with respect to spatial translations, and again from maximal regularity. We will explain this method, which relies on an idea of S.B. Angenent [9, 10], in detail for the flow through a deformable porous medium discussed below.

**Remarks 2.4.** **a)** The first analytic results for problem (2.9) are due to Kawarada and Koshigoe [51], see also [52]. These authors construct in the case $n = 1$ for suitable initial data in the Sobolev space $H^{18}(\mathbb{R})$ a solution in $C([0,T],H^{14}(\mathbb{R})) \cap C^1([0,T],H^1(\mathbb{R}))$ by a Nash-Moser iteration. This approach leads to a serious loss of regularity for solutions. In addition, there are no uniqueness results in [51, 52].

**b)** We treat (2.15) as a fully nonlinear equation, although the operator $\Phi(g) = B_0(g)T(g)g$ has a quasi-linear structure in the sense that, given $g \in V_c$, the mapping $[h \mapsto B_0(g)T(g)h]$ is a first order linear operator. This quasi-linear structure is not useful in our situation, since the nonlinear dependence in $\Phi$ is of first order as well.

**c)** Given $g \in V_c$, let $f = g + c$. Then we have

$$\partial_{n+1}u_f(x,f(x)) < (1+|\nabla f(x)|^2)^{-1}, \qquad x \in \mathbb{R}^n, \tag{2.16}$$

where $u_f$ is the solution of the elliptic problem (2.10). Relation (2.16) should be seen as a parabolicity condition for (2.9) in the sense that the corresponding linearized problem is well-posed and induces a strongly continuous analytic semigroup on $h^{1+\alpha}(\mathbb{R}^n)$, provided (2.16) holds true, whereas the same linearized problem is backwards parabolic if

$$\partial_{n+1}u_f(x,f(x)) > (1+|\nabla f(x)|^2)^{-1}, \qquad x \in \mathbb{R}^n.$$

On the other hand, letting $p(x,y) := u_f(x,y) - y$ for $(x,y) \in \mathbb{R}^n \times [0,\infty)$, it follows from the strong maximum principle that $\partial_{n+1}u_f(x,f(x)) < 1$, since $\nabla p$ is orthogonal to $\Gamma_f$. We do not know whether or not problem (2.9) is still well-posed if condition (2.16) is replaced by the condition $\partial_{n+1}u_f(x,f(x)) < 1$.

**d)** It can be shown that

$$\partial B_0(g)[\cdot,T(g)g] - B_0(g)S(g)\partial A(g)[\cdot,T(g)g] \to 0$$

in $\mathcal{L}(h^{2+\alpha}(\mathbb{R}^n),h^{1+\alpha}(\mathbb{R}^n))$ as $\|g\|_{h^{2+\alpha}(\mathbb{R}^n)} \to 0$. Since $B_0(g)T(g)$ belongs to $\mathcal{H}(h^{2+\alpha}(\mathbb{R}^n),h^{1+\alpha}(\mathbb{R}^n))$ it follows therefore from Lemma 2.2 and well-known perturbation results for the class $\mathcal{H}(h^{2+\alpha}(\mathbb{R}^n),h^{1+\alpha}(\mathbb{R}^n))$ that there is an $\varepsilon_0 > 0$ such that

$$\partial\Phi(g) \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^n),h^{1+\alpha}(\mathbb{R}^n)),$$

provided $||g||_{h^{2+\alpha}(\mathbb{R}^n)} < \varepsilon_0$. Let us emphasize that there are arbitrarily large $g \in V_c$ and that Theorem 2.3 does not rely on a smallness condition.

2.2. **Quasi-incompressible fluids in deformable porous media.** We consider now the flow of a quasi-incompressible fluid in a deformable porous medium. In this case the specific storativity $S_0$ is positive, cf. (2.4), and (2.8) leads us to the problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} \quad \Omega_f \\ \partial_\nu u = 0 & \text{on} \quad \Sigma \\ u = f & \text{on} \quad \Gamma_f \\ \lim_{|(x,y)|\to\infty} u(\cdot,(x,y)) = c & \text{on} \quad [0,T] \\ \partial_t f + k\sqrt{1+|\nabla f|^2}\partial_\nu u = 0 & \text{on} \quad \Gamma_f \\ f(0,\cdot) = f_0 & \text{on} \quad \mathbb{R}^n \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega_{f_0}, \end{cases} \tag{2.17}$$

where now $f_0 \in A_0$ and $u_0 \in BUC^1(\Omega_{f_0})$ are given initial data. As before, $c$ and $k$ are positive constants. In contrast to problem (2.9) we treat system (2.17) in appropriate Sobolev spaces rather than in Hölder spaces. To formulate our results, we fix $p > n+2$ and denote by $W_p^\sigma$ the Sobolev-Slobodeckii spaces of order $\sigma \geq 0$, i.e., $W_p^\sigma$ is the usual Sobolev space if $\sigma \in \mathbb{N}$, whereas $W_p^\sigma$ stands for the Besov space $B_{pp}^\sigma$ if $\sigma > 0$ is not an integer. Furthermore, in the following we fix $s \in (0, 1/p)$, $\alpha \in [1/p, 1-((n+1)/p)]$, and $\gamma \in (0,c)$. Given $g \in W^{2+s-1/p}(\mathbb{R}^n)$, it follows from well-known trace theorems that there exists an extension $\tilde{g} \in W^{2+s}(\mathbb{R}^{n+1})$ of $g$ such that $\mathrm{supp}(\tilde{g}) \subset \mathbb{R}^n \times (-3/4, 3/4)$ and such that $\tilde{g}|\mathbb{R}^n \times \{0\} = g$. For technical reasons we introduce some further notation. Given $f \in A_0$, let $g := f - c$ and set

$$\omega(f) := \sup_{(x,y)\in\Omega} [|\nabla_x \tilde{g}(x,y)|^2 + |(1-y)\partial_{n+1}\tilde{g}(x,y) - \tilde{g}(x,y)|^2]^{1/2}.$$

Later on, the quantity $\omega(f)$ will be useful to construct an appropriate diffeomorphism, mapping again the reference domain $\Omega$ onto $\Omega_{c+g}$. Now let

$$V := \{f \in A_0 \, ; \, f \in W_{p,c}^{2+s-1/p}(\mathbb{R}^n), \, \omega(f) < \min(1,\gamma)\},$$

where, for simplicity, we write $W_{p,c}^\sigma := W_p^\sigma + c$, $\sigma \geq 0$.

**Theorem 2.5.** *Let $\rho > 0$ be given. Then there exists a $\delta > 0$ such that for any $f_0 \in V$ and $u_0 \in W_{p,c}^{2+s}(\Omega_{f_0})$ satisfying*

$$||u_0 - c||_{W^{2+s}(\Omega_{f_0})} < \rho \qquad ||\partial_{n+1}u_0||_{BUC^\alpha(\Omega_{f_0})} < \delta,$$
$$||\partial_{n+1}u_0|\Gamma_{f_0}||_{W_p^{1+s-1/p}(\mathbb{R}^n)} < \delta$$

*and the compatibility conditions*

$$u_0|\Gamma_{f_0} = f_0, \qquad \partial_{n+1}u|\Sigma = 0,$$

*there exists* $t^+ := t^+(u_0, f_0)$ *and a unique maximal classical solution* $(u, f)$ *of problem (2.17) in the class*

$$f \in C([0,t^+),V) \cap C^1([0,t^+), W^{1+s-1/p}_{p,c}(\mathbb{R}^n)) \cap C^\omega((0,t^+)\times\mathbb{R}^n)$$
$$u(\cdot,\cdot) \in C^\infty(\overline{\Omega}_{f,T}), \quad u(t,\cdot) \in BUC^{1+\alpha}(\Omega_{f(t)}),\ t\in[0,T),$$

*where* $\overline{\Omega}_{f,T} := \{(t,(x,y)) \in (0,T)\times\mathbb{R}^{n+1}\,;\ (x,y)\in\overline{\Omega}_{f(t)}\}$.

Before we prove Theorem 2.5, let us add the following comments.

**Remarks 2.6.** **a)** Theorem 2.5 extends earlier results obtained in [31], since we are now able to prove the analyticity of the free interface $\Gamma_f$ and the smoothness of the potential $u$.

**b)** There is a different approach in weighted Hölder spaces to problem (2.17) outlined in [14, 13]. Recently, a (different) $L_p$-theory for problems of type (2.17) was proposed in [66, 67].

**c)** Our regularity result for the moving boundary $\Gamma_f$ also extends results in [53], where, under the assumption of the existence of classical $C^1$-solutions, the $C^\infty$-smoothness of the moving boundary is proved. No existence results are presented in [53]. In contrast, Theorem 2.5 guarantees the existence of classical solutions and we further establish the actual regularity of the moving boundary.

**d)** Observe that the constant function $(u,f)\equiv(c,c)$ is a solution to (2.17). This trivial solution should be regarded as an equilibrium of system (2.17). Furthermore it is not difficult to verify that $V$ is an open neighborhood of $c$ in $W^{2+s-1/p}_{p,c}(\mathbb{R}^n)$.

**Proof of Theorem 2.5** (i) In a first step we provide an appropriate extension for $g \in W^{k+s-1/p}_p(\mathbb{R}^n)$ to a function on the whole of $\mathbb{R}^{n+1}$. For this, let $\gamma_0$ be the trace operator with respect to $\Sigma$ and let

$$\gamma_0^c \in \mathcal{L}(W^{k+s-1/p}_p(\mathbb{R}^n), W^{k+s}_p(\mathbb{R}^{n+1})), \quad k\in\mathbb{N}^* \tag{2.18}$$

be the coretraction of $\gamma_0$ constructed in [69], Theorem 2.7.2. It follows from formula (2.7.2.42) in [69] that $\gamma_0^c$ possesses the following translation equivariance

$$\tau_a\gamma_0^c g = \gamma_0^c\tau_a g, \quad a\in\mathbb{R}^n,\ g\in W^{k+s-1/p}_p(\mathbb{R}^n),$$

where

$$(\tau_a g)(x) := g(x+a) \quad \text{and} \quad (\tau_a v)(x,y) := v(x+a,y)$$

for $g \in W^{k+s-1/p}_p(\mathbb{R}^n)$, $v\in W^{k+s}_p(\mathbb{R}^{n+1})$, and $(x,y)\in\mathbb{R}^n\times\mathbb{R}$. Next we pick $\eta\in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\eta)\subset(-3/4,3/4)$ and $\eta(y)=1$ for $y\in[-1/2,1/2]$. Given $g\in W^{k+s-1/p}_p(\mathbb{R}^n)$, set

$$(\rho_0 g)(x,y) := \eta(y)(\gamma_0^c g)(x,y), \quad (x,y)\in\Omega. \tag{2.19}$$

Recall that $\Omega := \mathbb{R}^n\times(0,1)$. Obviously, we have

$$\tau_a\rho_0 g = \rho_0\tau_a g, \quad a\in\mathbb{R}^n,\ g\in W^{k+s-1/p}_p(\mathbb{R}^n). \tag{2.20}$$

For simplicity we often write $\tilde{g} := \rho_0 g$.

(ii) Let $V_c := V - c$, pick $g \in V_c$, and set

$$\varphi_g(x,y) := (x, (1-y)(c+\tilde{g}(x,y))), \quad (x,y) \in \Omega.$$

As in [31] Lemma 2.2 one shows that $V_c$ is an open neighborhood of 0 in $W_p^{2+s-1/p}(\mathbb{R}^n)$, and that, given $g \in V_c$, we have

$$\varphi_g \in \mathit{Diff}^{1+\alpha}(\Omega, \Omega_{c+g}), \quad \gamma_0 \varphi_g \in \mathit{Diff}^{1+\alpha}(\Sigma, \Gamma_{c+g}).$$

As in (2.12) we now define the transformed operators

$$A(g)v := -\varphi_g^*(\Delta \varphi_*^g v), \quad B_i(g)v := k\varphi_g^*(\gamma_i \nabla(\varphi_*^g)|n_i), \ i = 0, 1$$

for $v \in W_p^2(\Omega)$, where $n_0 := (-\nabla g, 1)$ and $n_1 := (0,-1)$ denote the outer normals on $\Gamma_{c+g}$ and $\Gamma_0$, respectively. To express the coefficients of these operators in terms of $g \in V_c$, let $G(g) := c + \tilde{g} - \pi \partial_{n+1}\tilde{g}$, where $\pi(x,y) := 1-y$ for $(x,y) \in \Omega$, and denote by $G_{jk}(g) := (\partial_j \varphi_g | \partial_k \varphi_g)$, $1 \le j,\, k \le n+1$, the components of the metric tensor induced by $\varphi_g$. It is not difficult to verify that $\det[G_{jk}(g)] = G(g)$ and that the inverse $[G^{jk}(g)]$ of $[G_{jk}(g)]$ is given by

$$[G^{jk}(g)] = \begin{bmatrix} 1 & 0 & \cdots & 0 & \frac{\pi\partial_1\tilde{g}}{G(g)} \\ 0 & 1 & \cdots & 0 & \frac{\pi\partial_2\tilde{g}}{G(g)} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & \frac{\pi\partial_n\tilde{g}}{G(g)} \\ \frac{\pi\partial_1\tilde{g}}{G(g)} & \frac{\pi\partial_2\tilde{g}}{G(g)} & \cdots & \frac{\pi\partial_n\tilde{g}}{G(g)} & \frac{1+|\pi\nabla_x\tilde{g}|^2}{G^2(g)} \end{bmatrix} \tag{2.21}$$

Then we have

$$A(g)v = -\frac{1}{G(g)} \sum_{j,k=1}^{n+1} \partial_j (G(g) G^{jk}(g) \partial_k v), \quad v \in W_p^2(\Omega). \tag{2.22}$$

By the construction of the extension operator $\rho_0$, in particular by the choice of $\eta$, one easily verifies that the boundary operator $B_0$ is represented as

$$B_0(g)v = k \sum_{j=1}^{n} -\partial_j g \gamma_0(\partial_j v) - \frac{k}{\gamma_0 G(g)}(1+|\nabla g|^2)\gamma_0(\partial_{n+1} v), \tag{2.23}$$

whereas $B_1(g)v = (k/c)\gamma_1(\partial_{n+1}v)$. Finally, we need

$$F(v,g) := -\frac{\pi \partial_{n+1} v}{G(g)} \rho_0(B_0(g)v), \quad (v,g) \in W_p^2(\Omega) \times V_c.$$

Assume now that $(u, f)$ is a solution to problem (2.17). Setting $g := f - c$ and $v := \varphi_g^*(u - c)$, one shows that $(v, g)$ is formally a solution to

$$\begin{cases} \partial_t v + A(g)v + F(v,g) = 0 & \text{in} \quad (0,T] \times \Omega \\ v = g & \text{on} \quad [0,T] \times \Gamma_0 \\ \partial_{n+1} v = 0 & \text{on} \quad [0,T] \times \Gamma_1 \\ \lim_{|(x,y)| \to \infty} v(\cdot,(x,y)) = 0 & \text{on} \quad [0,T] \\ \partial_t g + B_0(g)v = 0 & \text{on} \quad (0,T] \times \Gamma_0 \\ v(0,\cdot) = v_0 & \text{in} \quad \Omega \\ g(0,\cdot) = g_0 & \text{on} \quad \Gamma_0, \end{cases} \tag{2.24}$$

(iii) We shall now focus our attention on the transformed system (2.24). In order to introduce an appropriate notion of solutions to (2.24), let

$$\begin{aligned} E_0 &:= W_p^s(\Omega) \times W_p^{1+s-1/p}(\Gamma_0) \\ E_1 &:= \{(v,g) \in W_p^{2+s}(\Omega) \times W_p^{2+s-1/p}(\Gamma_0)\ ;\ \gamma_0 v = g,\ \gamma_1 \partial_{n+1} v = 0\}. \end{aligned}$$

Each of these spaces is given the natural topology, i.e., the product topology for $E_0$ and the relative topology for $E_1$. It follows from the trace theorem that $E_1$ is a closed subspace of $W_p^{2+s}(\Omega) \times W_p^{2+s-1/p}(\Gamma_0)$ and it can be shown that $E_1$ is dense in $E_0$, see Lemma 3.1 in [31]. Observe further that the "stationary" boundary conditions of (2.24) are incorporated in the space $E_1$.
Next let $D_1 := \{(v,g) \in E_1\ ;\ g \in V_c\}$ and define

$$\Pi(z) := (A(g)v + F(v,g), B_0(g)v) \quad \text{for} \quad z = (v,g) \in D_1.$$

It is not difficult to see that $D_1$ is an open subset of $E_1$. Moreover, using the representations (2.22) and (2.23) one verifies that

$$\Pi \in C^\omega(D_1, E_0). \tag{2.25}$$

Recalling the definition of $E_1$, system (2.24) is equivalent to the abstract evolution equation

$$\frac{d}{dt} z + \Pi(z) = 0, \quad z(0) = z_0 := (v_0, g_0) \quad \text{in} \quad E_0. \tag{2.26}$$

More precisely, given $z_0 = (v_0, g_0) \in D_1$, we call $z = (v, g)$ a classical $W_p^s$-solution to (2.24) if and only if

$$z \in C([0,T], D_1) \cap C^1([0,T], E_0) \tag{2.27}$$

and $z$ satisfies the equations in (2.26) pointwise on $[0,T]$. Using the diffeomorphism $\varphi_g$ one shows that each classical $W_p^s$–solution $z = (v,g)$ of (2.24) gives rise to a classical solution to (2.17) by setting $f := g + c$ and

$$u(t,(x,y)) := (\varphi_*^{g(t)} v(t))(x,y) + c, \quad t \in [0,T],\ (x,y) \in \Omega_{f(t)},$$

cf. Lemma 2.4 in [31].

(iv) We know from (2.25) that, given $z = (v, g) \in D_1$, we have $\partial\Pi(z) \in \mathcal{L}(E_1, E_0)$. Unfortunately, in order to guarantee that $\partial\Pi(z)$ belongs to the class $\mathcal{H}(E_1, E_0)$, we need a smallness assumption for $v$. More precisely, given $\rho > 0$ and $\delta > 0$, let

$$W_{\rho,\delta} := \{(v, g) \in D_1 \; ; \; \|v\|_{2+s,p} < \rho, \; \|\partial_{n+1} v\|_\alpha < \delta, \; \|\gamma_0 \partial_{n+1} v\|_{1+s,p} < \delta\},$$

where $\|\cdot\|_{\sigma,p}$ and $\|\cdot\|_\alpha$ stands for the norm in $W_p^\sigma$ and $BUC^\alpha$, respectively. The following crucial result for $\partial\Pi(z)$ was proved in [31], Corollary 3.6: Given $\rho > 0$, there is a $\delta > 0$ such that

$$\partial\Pi(z) \in \mathcal{H}(E_1, E_0) \quad \text{for all} \quad z \in W_{\rho,\delta}. \tag{2.28}$$

Based on (2.28) we can now apply results of A. Lunardi to find a unique solution of system (2.24). Indeed, given $z_0 = (v_0, g_0) \in W_{\rho,\delta}$, Theorem 2 in [55] and an extension argument (cf. Theorem 3.8 in [31]) show that there exists a unique maximal classical $W_p^s$-solution to (2.24) on $[0, t^+(v_0, g_0))$.

(v) Let us now verify that the interface constructed above depends analytically on the space and time variables. For this we fix $\rho > 0$ and choose $\delta > 0$ such that (2.28) holds. Let $W := W_{\rho,\delta}$. We fix $(v_0, g_0) \in W$ and let $z = (v, g) \in C(I, W) \cap C^1(I, E_0)$ denote the unique solution of (2.24) on $[0, T]$, where $T \in (0, t^+(v_0, g_0))$ is fixed and $I := [0, T]$. Additionally, let $I_0 := (0, T]$. Given $\beta \in (0, 1)$ and a Banach space $E$, let

$$C_\beta^\beta(I, E) := \{u \in BUC(I, E) \cap C^\beta(I_0, E); \lim_{\varepsilon\to 0} \varepsilon^\beta \sup_{\varepsilon \le s < t \le 2\varepsilon} \frac{\|u(s) - u(t)\|_E}{|s - t|^\beta} = 0\},$$

where $C^\beta(I_0, E)$ denotes the space of all locally $\beta$-Hölder continuous functions from $I_0$ to $E$. It is not difficult to verify that $C_\beta^\beta(I, E)$, equipped with the norm

$$\|u\|_{C_\beta^\beta(I,E)} := \|u\|_\infty + \sup_{2\varepsilon \in I_0} \varepsilon^\beta \sup_{\varepsilon \le s < t \le 2\varepsilon} \frac{\|u(s) - u(t)\|_E}{|s - t|^\beta}, \quad u \in C_\beta^\beta(I, E),$$

is a Banach space. The space $C_\beta^{1+\beta}(I, E)$ consists of those $u \in C_\beta^\beta(I, E)$ such that $u'$ belongs to $C_\beta^\beta(I, E)$ too, given the obvious norm. Of course, $C_\beta^{1+\beta}(I, E)$ is also a Banach space. With this notation, it follows from (2.28) and Theorem III.2.5.6 in [6] that

$$(C_\beta^\beta(I, E_0), C_\beta^\beta(I, E_1) \cap C_\beta^{1+\beta}(I, E_0))$$

is a pair of maximal regularity for the family $\{\partial\Pi(w) \; ; \; w \in W\}$, that is,

$$(\frac{d}{dt} + \partial\Pi(w), \mathrm{tr}) \in \mathrm{Isom}(C_\beta^\beta(I, E_1) \cap C_\beta^{1+\beta}(I, E_0), C_\beta^\beta(I, E_0) \times E_1), \tag{2.29}$$

for every $w \in W$, where $\mathrm{tr}u := u(0)$ for $u \in C_\beta^\beta(I, E_1)$. For later purposes we also need the following additional regularity of the solution $z$ of (2.26)

$$z \in C_\beta^\beta(I, E_1) \cap C_\beta^{1+\beta}(I, E_0), \tag{2.30}$$

which is guaranteed by Theorem 8.1.1 in [56].

(vi) Next we collect some useful properties of the solution $z = (v, g)$ and $\Pi$ concerning translations in space and dilation in time. Given $a \in \mathbb{R}^n$ and $w = (r, h) \in E_0$, let $\tau_a w := (\tau_{(a,0)} r, \tau_a h)$, cf. (2.20). It follows from (2.20)–(2.23) that

$$\tau_a(W) \subset W \quad \text{and} \quad \tau_a \Pi(w) = \Pi(\tau_a w),\ a \in \mathbb{R}^n,\ w \in W. \tag{2.31}$$

Let now $(\lambda, \mu) \in (1-\varepsilon, 1+\varepsilon) \times \mathbb{R}^n$, with $\varepsilon > 0$ sufficiently small, be given and set

$$z_{\lambda,\mu}(t) := \tau_{t\mu} z(\lambda t), \quad t \in I.$$

Since $\tau_a(W) \subset W$, we find that $z_{\lambda,\mu}(I) \subset W$. Additionally, we have

$$z_{\lambda,\mu} \in C(I, W) \cap C^1(I, E_0). \tag{2.32}$$

Examplarily, let us show that the first component $v_{\lambda,\mu} := \tau_{t\mu} v(\lambda t)$ of $z_{\lambda,\mu}$ belongs to $C(I, W_p^{2+s}(\Omega))$. Obviously, we have

$$\begin{aligned} v_{\lambda,\mu}(t+h) - v_{\lambda,\mu}(t) &= \tau_{(t+h)\mu}(v(\lambda(t+h)) - v(\lambda t)) \\ &\quad + (\tau_{(t+h)\mu} - \tau_{t\mu}) v(\lambda t). \end{aligned}$$

Hence, using the fact that the set of all translations $\{\tau_a\ ;\ a \in \mathbb{R}^n\}$ forms a strongly continuous group of contractions on $W_p^{2+s}(\Omega)$, and the fact that $v \in C(I, W_p^{2+s}(\Omega))$, we conclude that $v_{\lambda,\mu} \in C(I, W_p^{2+s}(\Omega))$. Let now

$$\Pi_{\lambda,\mu}(w) := \lambda \Pi(w) - D_\mu w, \quad w \in W,$$

where $D_\mu w := ((\mu|\nabla_x v)\,,\,(\mu|\nabla g))$. Using the translation equivariance of $\Pi$, cf. (2.31), one shows that $z_{\lambda,\mu}$ solves the evolution equation

$$\frac{d}{dt} z_{\lambda,\mu} + \Pi_{\lambda,\mu}(z_{\lambda,\mu}) = 0, \quad t \in I, \quad z_{\lambda,\mu}(0) = z_0. \tag{2.33}$$

Moreover, since $(\mu|\nabla g)$ involves only tangential derivatives and since $[v \mapsto (\mu|\nabla_x v)]$ is a first order operator, it is not difficult to see that solutions to (2.33) are unique in the class (2.32), cf. the proof of Theorem 3.8 in [31].

(vii) In the following, let $\Lambda$ be an open neighborhood of (1,0) in $(1-\varepsilon, 1+\varepsilon) \times \mathbb{R}^n$. Moreover, we set $D := C(I, W) \cap C_\beta^\beta(I, E_1) \cap C_\beta^{1+\beta}(I, E_0)$. Given $(\lambda, \mu) \in \Lambda$ and $w \in D$, we define

$$F(w, (\lambda, \mu)) := (\frac{d}{dt} w + \lambda \Pi(w) - D_\mu w, w(0) - z_0).$$

Recall that $I$ is compact and that $W$ is open in $E_1$. Moreover, one has the embedding $C_\beta^\beta(I, E_1) \hookrightarrow C(I, E_1)$, cf. Proposition III.2.1.1 in [6]. Hence we find that $D$ is open in $C_\beta^\beta(I, E_1) \cap C_\beta^{1+\beta}(I, E_0)$. In summary, we see that $\mathrm{dom}(F) = D \times \Lambda$ is an open subset of $(C_\beta^\beta(I, E_1) \cap C_\beta^{1+\beta}(I, E_0)) \times \mathbb{R}^{n+1}$. In addition, it follows from (2.25) that

$$F \in C^\omega(D \times \Lambda, C_\beta^\beta(I, E_0) \times E_1).$$

The derivative $\partial_1 F$ of $F$ with respect to $w \in D$ is given by

$$\partial_1 F(w,(1,0))h = (\frac{d}{dt}h + \partial\Pi(w)h, h(0)). \tag{2.34}$$

Observe that $F(w,(\lambda,\mu)) = 0$ holds true if and only if $w$ is a solution to the evolution equation (2.33). Observe also that we have $(z,(1,0)) \in D \times \Lambda$, cf. (2.30), with $F(z,(1,0)) = 0$. In addition, it follows from (2.34), (2.29) and Theorem 2.6.1 in [6] that

$$\partial_1 F(z,(1,0)) \in \mathrm{Isom}(C^{\beta}_{\beta}(I,E_1) \cap C^{1+\beta}_{\beta}(I,E_0), C^{\beta}_{\beta}(I,E_0) \times E_1)).$$

Consequently, the implicit function theorem guarantees the existence of an open neighborhood $\Lambda_0$ of $\Lambda$ and a unique mapping

$$[(\lambda,\mu) \mapsto w_{\lambda,\mu}] \in C^{\omega}(\Lambda_0, C(I,W) \cap C^{\beta}_{\beta}(I,E_1) \cap C^{1+\beta}_{\beta}(I,E_0))$$

such that $F(w_{\lambda,\mu},(\lambda,\mu)) = 0$. Since $C^{1+\beta}_{\beta}(I,E_0) \subset C^1(I,E_0)$, we find by the unique solvability of (2.33) in the class (2.32) that

$$[(\lambda,\mu) \mapsto z_{\lambda,\mu}] \in C^{\omega}(\Lambda_0, C(I,W) \cap C^{\beta}_{\beta}(I,E_1) \cap C^{1+\beta}_{\beta}(I,E_0)). \tag{2.35}$$

(viii) We now show that the interface depends analytically on the spatial and temporal variables. For this let $g_{\lambda,\mu} := \tau_{t\mu} g(\lambda t)$ be the second component of $z_{\lambda,\mu}$. By (2.35) we particularly have

$$[(\lambda,\mu) \mapsto g_{\lambda,\mu}] \in C^{\omega}(\Lambda_0, C^1(I, W^{1+s-1/p}_p(\Gamma_0)). \tag{2.36}$$

Moreover, given $(k,\gamma) \in \mathbb{N} \times \mathbb{N}^n$, an induction argument shows that

$$t^{k+|\gamma|}\partial_t^k \partial_x^\gamma g(t) = \partial_\lambda^k \partial_\mu^\gamma g_{\lambda,\mu}|_{(\lambda,\mu)=(1,0)}(t), \quad t \in I,$$

cf. the proof of Theorem 4.4 in [33]. In particular, we have $g \in C^{\infty}((0,T) \times \mathbb{R}^n)$. Also the fact that

$$g \in C^{\omega}((0,T) \times \mathbb{R}^n) \tag{2.37}$$

follows from (2.36). Indeed, representing $g_{\lambda,\mu}$ by its Taylor series, and using Sobolev's embedding theorem, one shows that, given $(t_0,x_0) \in (0,T) \times \mathbb{R}^n$, there are positive constants $r$ and $M$ such that

$$|\partial_t^k \partial_x^\gamma g(t,x)| r^{k+|\gamma|} \leq M k! \gamma!$$

for all $(k,\gamma) \in \mathbb{N} \times \mathbb{N}^n$ and all $(t,x) \in \mathbb{B}((t_0,x_0),r)$.
To verify the smoothness of $v$, consider the semilinear parabolic equation

$$\left\{\begin{aligned} \partial_t v + A(g)v + F(v,g) &= 0 && \text{in} \quad (0,T] \times \Omega \\ v &= g && \text{on} \quad [0,T] \times \Gamma_0 \\ \partial_{n+1} v &= 0 && \text{on} \quad [0,T] \times \Gamma_1 \\ v(0,\cdot) &= v_0 && \text{in} \quad \Omega \end{aligned}\right.$$

for the function $v$. Combining (2.37) with (2.18), (2.19), (2.22), and (2.23) we find that all coefficients of $A(g)$ and $F(\cdot,g)$ are smooth. Hence, it follows from well-known regularity results for semilinear parabolic initial boundary value problems, cf. Corollary 9.4 in [4], that $v$ belongs to $C^\infty((0,T)\times\overline{\Omega})$. This completes the proof. ∎

**Remark 2.7.** Although we prove that the function $g$ appearing in (2.24) (and yielding the interface $f = g + c$ in (2.17)) is real analytic, we only get $C^\infty$-regularity of the corresponding potential $v$. Since we use a cut-off function in our construction of the extension operator $\rho_0$, the regularity of $v$ cannot be improved to real analyticity in the framework presented here. We leave it as an open problem to find an extension operator sharing all properties of $\rho_0$ but implying in addition the analyticity of the coefficients $A(g)$ and $F(\cdot, g)$. If such an extension operator exists the analyticity of $v$ follows from [41] Theorem 3.3.1.

## 3. Volume preserving mean curvature flows

Let $\Gamma = \{\Gamma(t); t \geq 0\}$ be a family of closed compact hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 1$. In this section we consider the evolution of $\Gamma$ under the assumption that the normal velocity $V$ of $\Gamma$ is given as a function of the mean curvature $H(t)$ of each individual surface $\Gamma(t)$, i.e., $V(t) = F(H(t))$. Of course, both quantities $V$ and $H$ have to be given an appropriate orientation. More precisely, we adopt the sign convention that $V(t)$ and $H(t)$ are positive for a locally expanding family of hypersurfaces and for a locally convex hypersurface, respectively. Let $A(t) = \int_{\Gamma(t)} d\sigma(t)$ denote the surface area of $\Gamma(t)$ and assume that each surface $\Gamma(t)$ is smooth. Then it follows from the first variation of the area functional, cf. Theorem 4 in [54], that

$$\frac{d}{dt}A(t) = n \int_{\Gamma(t)} H(t)V(t)d\sigma(t) = n \int_{\Gamma(t)} H(t)F(H(t))d\sigma(t), \; t \geq 0. \tag{3.1}$$

In particular, we observe that $A(t)$ is monotone as a function of $t$ if $\int H(t)V(t)$ has a sign. The most prominent and most simple example of this type is certainly the mean curvature flow, where the normal velocity is given by $V(t) = -H(t)$. We shall present below some further examples for which $\int V(t)H(t)$ has a sign.
Assume now furthermore that each surface $\Gamma(t)$ encloses a well-defined domain $\Omega(t)$ in $\mathbb{R}^{n+1}$ and let $vol(t) := \int_{\Omega(t)} dx$ denote the volume of $\Omega(t)$. Then one has

$$\frac{d}{dt}vol(t) = \int_{\Gamma(t)} V(t)d\sigma(t) = \int_{\Gamma(t)} F(H(t))d\sigma(t), \; t \geq 0, \tag{3.2}$$

cf. Theorem 2E in [47]. Therefore the evolution equation $V = F(H)$ is volume preserving, provided $\int_{\Gamma(t)} F(H)d\sigma(t) = 0$. This holds true for instance if $F = div(X)$ for some vector field $X$, since $\Gamma(t)$ is closed.

3.1. **The averaged mean curvature flow.** We consider the evolution equation

$$V(t) = \overline{H}(t) - H(t), \quad \Gamma(0) = \Gamma_0, \tag{3.3}$$

where $\overline{H} := |\Gamma(t)|^{-1} \int_{\Gamma(t)} H \, d\sigma(t)$ is the average of the mean curvature. Obviously, this flow is volume preserving. Moreover, (3.1) shows that

$$\frac{1}{n}\frac{d}{dt}A(t) = \int\limits_{\Gamma(t)} H(\overline{H} - H)d\sigma(t) = -\int\limits_{\Gamma(t)} (H - \overline{H})^2 d\sigma(t) \leq 0,$$

since $\overline{H}\int(H - \overline{H})d\sigma = 0$. Thus the flow (3.3) decreases the area $A(t)$. The averaged mean curvature flow has been identified as the singular limit of a nonlocal Ginzburg-Landau equation [15].

3.2. **The surface diffusion flow.** We consider now the evolution equation

$$V(t) = \Delta_{\Gamma(t)} H(t), \quad \Gamma(0) = \Gamma_0, \tag{3.4}$$

where $\Delta_{\Gamma(t)}$ stands for the Laplace-Beltrami operator on $\Gamma(t)$. Again this flow is volume preserving, i.e.,

$$\frac{d}{dt}vol(t) = \int\limits_{\Gamma(t)} V \, d\sigma(t) = \int\limits_{\Gamma(t)} div_{\Gamma(t)} \, grad_{\Gamma(t)} \, H(t) \, d\sigma = 0,$$

and area decreasing

$$\frac{1}{n}\frac{d}{dt}A(t) = \int\limits_{\Gamma(t)} HV \, d\sigma(t) = \int\limits_{\Gamma(t)} [\Delta_{\Gamma(t)} H] H \, d\sigma(t) = -\int\limits_{\Gamma(t)} |grad_{\Gamma(t)} \, H|^2 d\sigma(t) \leq 0.$$

The surface diffusion flow (3.4) was first introduced by Mullins [60] to model surface dynamics for phase interfaces when the evolution is governed only by mass diffusion in the interface. It has also been examined in a more general mathematical and physical context by Davi and Gurtin [26], and by Cahn and Taylor [19].

3.3. **The intermediate surface diffusion flow.** The surface diffusion flow (3.4) and the averaged mean curvature flow (3.3) are formally connected by the so-called intermediate surface diffusion flow given by

$$V(t) = \Delta_{\Gamma(t)}(\alpha - \beta\Delta_{\Gamma(t)})^{-1}(H(t) - \overline{H}(t)), \ t > 0, \quad \Gamma(0) = \Gamma_0, \tag{3.5}$$

where $\alpha$ and $\beta$ are positive constants. Indeed, fix $t \geq 0$, set $\Sigma := \Gamma(t)$ and let $\Delta$ denote the $L_2(\Sigma)$-realization of $\Delta_\Sigma$. Furthermore, let $H := 1^\perp$ and write $\Delta_\perp := \Delta | H$.
We first consider the case $\beta = 1$ and let $\alpha > 0$ tend to 0. Since $0 \notin \sigma(\Delta_\perp)$ we have

$$\lim_{\alpha \to 0}(\alpha - \Delta_\perp)^{-1} = -\Delta_\perp^{-1} \quad \text{in} \quad \mathcal{L}(H),$$

suggesting that solutions to (3.5) should converge to solutions of the averaged mean curvature flow (3.3) as $\alpha \to 0$ and $\beta = 1$.

To consider the case $\alpha = 1$ and $\beta \to 0$, recall that $-\Delta_\perp$ is m-accretive on $H$. Hence the Lumer-Phillips theorem implies that

$$(1-\beta\Delta_\perp)^{-1} = \frac{1}{\beta}(\frac{1}{\beta} - \Delta_\perp)^{-1} \to 1_H \quad \text{in} \quad \mathcal{L}_s(H)$$

as $\beta \to 0$, cf. Lemma 1.3.2 in [61]. Here $\mathcal{L}_s(H)$ denotes the space $Hom(H)$ given the strong operator topology. In this case one therefore expects that the solution to (3.5) should converge to solutions of the surface diffusion flow. We mention that this formal connection between (3.5) and the flows induced by (3.3) and (3.6), respectively, was first formulated in [19].

A rigorous study of these singular limits will be the topic of a separate paper. Here we shall establish, as a first step in the analysis, well-posedness of (3.5) and some global existence results. Hence, for the sake of simplicity, set $\alpha = \beta = 1$ and consider the evolution equation

$$V(t) = \Delta_{\Gamma(t)}(1-\Delta_{\Gamma(t)})^{-1}H(t), \; t>0, \quad \Gamma(0) = \Gamma_0. \tag{3.6}$$

Observe that

$$\Delta_{\Gamma(t)}(1-\Delta_{\Gamma(t)})^{-1}H(t) = div_{\Gamma(t)}[grad_{\Gamma(t)}(1-\Delta_{\Gamma(t)})^{-1}H(t)],$$

so that the flow (3.6) preserves the volume, cf. the remark following (3.2). Observe also that $\Delta_{\Gamma(t)}(1-\Delta_{\Gamma(t)})^{-1}$ is a non-positive self-adjoint operator in $L_2(\Gamma(t))$. Hence the flow (3.6) is area shrinking too, see (3.1).

This model has recently been derived by Cahn and Taylor [19] to model growth laws for morphological change for a class of problems where surface diffusion is the transport mechanism and the only driving force is the reduction of total surface free energy.

3.4. **The Mullins-Sekerka flow.** The Mullins-Sekerka model is a nonlocal evolution law in which $V$ is given by the jump across the interface of the normal derivative of a function being harmonic on either side and which equals the mean curvature of the moving interface. More precisely, let $\Omega$ be a bounded connected domain in $\mathbb{R}^{n+1}$ with a smooth boundary $\partial\Omega$. Suppose $\Gamma_0$ is a compact hypersurface being the boundary of an open set $\Omega_0^-$ which is compactly contained in $\Omega$. Additionally, we set $\Omega_0^+ := \Omega \setminus cl(\Omega_0^-)$. We are looking for a family $\Gamma := \{\Gamma(t)\,;\, t \geq 0\}$ of hypersurfaces, separating at each instant the domain $\Omega$ into the domain $\Omega^-(t)$ enclosed by $\Gamma(t)$ and $\Omega^+(t) := \Omega \setminus cl(\Omega^-(t))$, and a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \setminus \Gamma(t) \\ u = H & \text{on} \quad \Gamma(t) \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \\ V = [\partial_\nu u] & \text{on} \quad \Gamma(t) \\ \Gamma(0) = \Gamma_0, & \end{cases} \tag{3.7}$$

where $\partial_\nu u$ denotes the normal derivative of $u$ on $\partial\Omega$ and

$$[\partial_\nu u] := \partial_\nu u^+ - \partial_\nu u^-$$

stands for the jump of the normal derivative of $u$ across $\Gamma(t)$, with $u^\pm := u^\pm(\cdot, t)$ being the restriction of $u$ to $\Omega^\pm(t)$.

Let us now reduce system (3.7) to a single evolution equation of the form $V(t) = F(H(t))$. For this, fix $t$ and let $h \in h^{1+\alpha}(\Gamma(t))$ be given. It can be shown (cf. Lemma 2.2 in [35]) that there exists a unique solution $u_h \in h^{1+\alpha}(\Omega)$ satisfying

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \setminus \Gamma(t) \\ u = h & \text{on} \quad \Gamma(t) \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Hence, letting $F(h) := [\partial_\nu u_h]$, system (3.7) reduces to

$$V(t) = F(H(t)),\ t > 0, \qquad \Gamma(0) = \Gamma_0. \tag{3.8}$$

It is easy to see that $F$ is an operator of first order in the sense that the inclusion $F(h^{1+\alpha}(\Gamma(t))) \subset h^\alpha(\Gamma(t))$ holds true. In addition, we mention that $F$ is a nonlocal operator.

Assume now that there is a smooth solution $\Gamma$ to (3.8). Then we have

$$\frac{1}{n}\frac{d}{dt}A(t) = \int_{\Gamma(t)} HV\, d\sigma(t) = \int_{\Gamma(t)} u_H[\partial_\nu u_H]\, d\sigma(t) = -\int_\Omega |\nabla u_H|^2 dx \le 0,$$

and

$$\frac{d}{dt} vol(t) = \int_{\Gamma(t)} V\, d\sigma(t) = \int_{\Gamma(t)} [\partial_\nu u_H]\, d\sigma(t) = -\int_\Omega \Delta u_H dx = 0,$$

showing that (3.8) is volume preserving and area decreasing. System (3.7) was introduced by Mullins and Sekerka [59] to study solidification and liquiditation of materials of zero specific heat. This model is also closely related to the quasi-stationary two-phase Stefan problem with surface tension, cf. [36].

In the calculations above we always assumed existence of a smooth solution. In order to justify these arguments, let us first establish the following existence and uniqueness result for the flows induced by (3.3), (3.4), (3.6), and (3.8).

**Theorem 3.1.** *Assume that $0 < \beta < 1$ and let $\Gamma_0$ be a compact, closed, connected, embedded hypersurface in $\mathbb{R}^{n+1}$ of class $h^{1+\beta}$. Assume additionally in the cases of the surface diffusion flow (3.4), the intermediate surface diffusion flow (3.6), and the Mullins-Sekerka model (3.8) that $\Gamma_0$ is of class $h^{2+\beta}$.*

*a) Each of the flows induced by (3.3), (3.4), (3.6), and (3.8) has a unique classical solution $\Gamma = \{\Gamma(t)\ ;\ t \in [0,T)\}$, where $T := T(\Gamma_0)$ is the maximal existence time. Moreover, the mapping $[t \mapsto \Gamma(t)]$ is smooth on $(0,T)$ with respect to the $C^\infty$-topology and continuous on $[0,T)$ with respect to the $h^{1+\beta}$-topology in the case (3.3) and with respect to the $h^{2+\beta}$-topology in the cases*

*(3.4), (3.6), and (3.8), respectively.*
*b) Let* $\Sigma$ *be a smooth hypersurface and suppose that the initial data* $\Gamma_0$ *is a* $h^{1+\beta}$*-graph over* $\Sigma$ *in the case (3.3), and a* $h^{2+\beta}$*-graph over* $\Sigma$ *in the cases (3.4), (3.6), and (3.8), respectively. Then the mapping* $\varphi := [(t, \Gamma_0) \mapsto \Gamma(t)]$ *defines a smooth local semiflow on an open subset of* $h^{1+\beta}(\Sigma)$ *in the case (3.3), and on an open subset of* $h^{2+\beta}(\Sigma)$ *in the cases (3.4), (3.6), and (3.8), respectively.*

Let us explain the proof of Theorem 3.1 in the case of the intermediate surface diffusion flow (3.6). A proof of the assertions concerning the flows induced by (3.3), (3.4), and (3.8) can be found in [34, 35, 39, 40].

(i) We first parameterize an appropriate neighborhood of $\Gamma_0$. More precisely, given $a > 0$, there exists a smooth hypersurface $\Sigma$, with outer unit normal $\mu$, and $\rho_0 \in h^{2+\beta}(\Sigma)$ with $\|\rho_0\|_{C^1(\Sigma)} < a/2$ such that $id_\Gamma + \rho_0\mu$ is a diffeomorphism of class $C^{2+\beta}$, mapping $\Sigma$ onto $\Gamma_0$. For $a > 0$ small enough the mapping

$$X \,:\, \Sigma \times (-a, a) \to \mathbb{R}^{n+1}, \quad X(s, r) := s + r\mu(s)$$

is a smooth diffeomorphism onto its image $\mathcal{R} := im(X)$. We split the inverse of $X$ into $X^{-1} = (S, \Lambda)$, where

$$S \in C^\infty(\mathcal{R}, \Sigma) \quad \text{and} \quad \Lambda \in C^\infty(\mathcal{R}, (-a, a))$$

is the metric projection of $\mathcal{R}$ onto $\Sigma$ and the signed distance function with respect to $\Sigma$, respectively.
Let now $0 < \beta_0 < \beta < \alpha < 1$ be fixed and set

$$U := \{\rho \in h^{2+\beta_0}(\Sigma)\,;\, \|\rho\|_{C^1(\Sigma)} < a\}.$$

For $T > 0$ and $\rho \in C^1((0, T], U \cap C^\infty(\Sigma))$, define

$$\Phi_\rho \,:\, \mathcal{R} \times (0, T] \to \mathbb{R}, \quad (x, t) \mapsto \Lambda(x) - \rho(S(x), t).$$

At any instant $t \in (0, T]$, the zero-level set $\Gamma_{\rho(t)} := \Phi_\rho^{-1}(\cdot, t)(0)$ is a smooth compact connected hypersurface. The normal velocity of $\{\Gamma_{\rho(t)}\,;\, t \in (0, T]\}$ is then given by

$$V(s, t) = \frac{\partial_t \rho(s, t)}{|\nabla_x \Phi_\rho(x, t)|}\bigg|_{x = X(s, \rho(s,t))}, \qquad (s, t) \in \Sigma \times (0, T]. \tag{3.9}$$

Observe that

$$\Gamma_{\rho(t)} = \{x \in \mathbb{R}^{n+1}\,;\, x = X(s, \rho(s, t)),\; s \in \Sigma\}, \quad t \in (0, T]. \tag{3.10}$$

Hence, letting $\theta_{\rho(t)}(s) := X(s, \rho(s, t))$ for $s \in \Sigma$, we see that $\theta_{\rho(t)}$ is a diffeomorphism, mapping $\Sigma$ onto $\Gamma_{\rho(t)}$. We need some further notation. First let

$$L_\rho(s, t) := |\nabla_x \Phi_\rho(x, t)|\big|_{x = X(s, \rho(s,t))}, \quad (s, t) \in \Sigma \times (0, T].$$

Moreover, let $\theta_\rho^*\eta$ be the pull-back metric on $\Sigma$, where $\eta$ is the usual Euclidean metric. We denote by $\Delta_\rho$ and $H_\rho$ the Laplace-Beltrami operator and the mean curvature of $(\Sigma, \theta_\rho^*\eta)$, respectively. Then we have

$$H_\rho = \theta_\rho^* H_{\Gamma_\rho} \qquad \text{and} \qquad \Delta_\rho \theta_\rho^* = \theta_\rho^* \Delta_{\Gamma_\rho}, \quad \rho \in U, \tag{3.11}$$

where $\Delta_{\Gamma_\rho}$ and $H_{\Gamma_\rho}$ stands for the mean curvature and the Laplace-Beltrami of $(\Gamma_\rho, \eta)$, respectively. Finally, we set

$$G(\rho) := -L_\rho \theta_\rho^*(\Delta_{\Gamma_\rho}(1-\Delta_{\Gamma_\rho})^{-1} H_{\Gamma_\rho}), \quad \rho \in U$$

and we consider the evolution equation

$$\frac{d}{dt}\rho + G(\rho) = 0, \qquad \rho(0) = \rho_0, \tag{3.12}$$

where $\rho_0$ is determined by $\Gamma_0$. We set $W := h^{2+\beta}(\Sigma) \cap U$. A function $\rho$ : $[0,T) \to W$ is called a classical solution to (3.12) if

$$\rho \in C([0,T), W) \cap C^\infty((0,T) \times C^\infty(\Sigma))$$

and if $\rho$ satisfies (3.12) pointwise. By construction, the intermediate surface diffusion flow and (3.12) are equivalent in $\mathcal{R}$: If $\rho$ is a classical solution to (3.12) then $\Gamma := \{\Gamma_{\rho(t)}\,;\, t \in [0,T)\}$ is a classical solution to (3.6) such that $\Gamma_{\rho(t)} \subset \mathcal{R}$, $t \in [0,T)$. Conversely, if $\Gamma := \{\Gamma(t)\,; t \in [0,T)\}$ is a classical solution to (3.6) with $\Gamma(t) \subset \mathcal{R}$, then the above construction yields a classical solution to (3.12).

(ii) From (3.11) we easily deduce that

$$(1-\Delta_\rho)^{-1}\theta_\rho^* = \theta_\rho^*(1-\Delta_{\Gamma_\rho})^{-1}, \quad \rho \in U.$$

Hence we find $G(\rho) = -L_\rho \Delta_\rho (1-\Delta_\rho)^{-1} H_\rho$, implying that

$$G(\rho) = L_\rho H_\rho - L_\rho(1-\Delta_\rho)^{-1} H_\rho, \quad \rho \in U.$$

It is known, see Lemma 3.1 in [38], that the mean curvature operator carries a quasi-linear structure in the sense that there exist functions

$$P \in C^\infty(U, \mathcal{L}(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma))) \quad \text{and} \quad Q \in C^\infty(U, h^{1+\beta_0}(\Sigma)) \tag{3.14}$$

such that

$$H_\rho = P(\rho)\rho + Q(\rho) \quad \text{for} \quad \rho \in h^{2+\alpha}(\Sigma) \cap U.$$

Additionally, given $\rho \in U$, the linear operator $[h \mapsto P(\rho)h]$ is a uniformly elliptic operator of second order. Since $L_\rho$ belongs to $h^{1+\beta_0}(\Sigma)$ and is strictly positive, it follows that $L_\rho P(\rho) \in \mathcal{H}(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma))$. Finally, let

$$F(\rho) := L_\rho[(1-\Delta_\rho)^{-1} H_\rho - Q(\rho)], \quad \rho \in U.$$

Then it is not difficult to verify that $F \in C^\infty(U, h^{1+\beta_0}(\Sigma))$, cf. the proof of Lemma 2.1 in [40]. We are now prepared to apply H. Amann's general theory of abstract quasilinear parabolic evolution equations. In order to verify the hypotheses of [5], let $E_0 := h^\alpha(\Sigma)$, $E_1 := h^{2+\alpha}(\Sigma)$, and let $E_\theta := (E_0, E_1)^0_{\theta,\infty}$, $\theta \in (0,1)$, be the continuous interpolation spaces between $E_0$ and $E_1$. Since the little Hölder spaces are stable under continuous interpolation, we find $0 < \theta_0 < \theta_1 < \theta_2 < 1$ such that

$$E_{\theta_0} = h^{1+\beta_0}(\Sigma), \quad E_{\theta_1} = h^{2+\beta_0}(\Sigma), \quad E_{\theta_2} = h^{2+\beta}(\Sigma).$$

The above considerations show that

$$[\rho \mapsto (L_\rho P(\rho), F(\rho))] \in C^\infty(U, \mathcal{H}(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma)) \times h^{1+\beta_0}(\Sigma)).$$

Hence Theorem 12.1 in [5] guarantees the existence of a unique solution $\rho \in C([0,T), W) \cap C^1((0,T), h^\alpha(\Sigma))$ to (3.12). An additional bootstrapping argument as in the proof of Theorem 1 in [35] shows that $\rho \in C^\infty((0,T), C^\infty(\Sigma))$. This completes the proof. ∎

Using Alexandrov's characterization of embedded surfaces of constant mean curvature, cf. [1], it is not difficult to verify that Euclidean spheres are the only embedded equilibria for each of the flows induced by (3.3), (3.4), (3.6), (3.8). Since these flows also preserve the volume and decrease the surface area, the isoperimetric inequality suggests that global smooth solutions should converge to spheres. In fact, in certain situations it is possible to rigorously justify this heuristic argument. The main idea here is to use the framework of Theorem 3.1 and some techniques from maximal regularity to construct a finite dimensional stable center manifold to each of the above flows. In a second step we then shall identify this stable manifold as the set of all equilibria. Therefore we conclude that if a solution starts close to this manifold, i.e., close to some sphere, it exists globally and converges to the manifold, i.e, to some sphere. The precise statement reads as follows:

**Theorem 3.2.** *Let $\mathcal{S}$ be a fixed Euclidean sphere and let $\mathcal{M}$ denote the set of all spheres which are sufficiently $C^{1+\beta}$-close to $\mathcal{S}$ in the case (3.3) and sufficiently $C^{2+\beta}$-close to $\mathcal{S}$ in the cases (3.4), (3.6), (3.8). Then $\mathcal{M}$ attracts at an exponential rate all solutions which are $C^{1+\beta}$-close in the case (3.3) and sufficiently $C^{2+\beta}$-close to $\mathcal{M}$ in the cases (3.4), (3.6), and (3.8). In particular, all solutions starting from such a neighborhood exist globally and converge exponentially fast to some sphere as $t \to \infty$. The convergence is in the $C^k$-topology for any fixed $k \in \mathbb{N}$.*

**Proof.** We provide a proof for the intermediate surface diffusion flow (3.6). The details for the cases (3.3.), (3.4) and (3.8) can be found in [38, 39, 40].
(i) For simplicity we assume that $\Sigma = \mathcal{S}$ is the Euclidean sphere of radius 1 with center at the origin. We first calculate the Frechét derivative of $G$ at 0. For this we note that the mapping $[\rho \mapsto G(\rho)] : h^{2+\alpha}(S) \cap U \to h^\alpha(S))$ is smooth. We obtain

$$\partial G(0)h = -\partial(L_\rho \Delta_\rho (1-\Delta_\rho)^{-1})|_{\rho=0}[h, H_0] - L_0\Delta_0(1-\Delta_0)^{-1}\partial H_\rho|_{\rho=0}h,$$

for $h \in h^{2+\alpha}(\Sigma)$. But $L_0 \equiv 1$ and $H_0 \equiv 1$, so that

$$L_\rho \Delta_\rho (1-\Delta_\rho)^{-1} H_0 = L_\rho (1-\Delta_\rho)^{-1}\Delta_\rho H_0 = 0, \quad \rho \in h^{2+\alpha}(S) \cap U.$$

Consequently,

$$\partial(L_\rho \Delta_\rho (1-\Delta_\rho)^{-1})|_{\rho=0}[h, H_0] = \frac{d}{d\varepsilon}(L_{\varepsilon h}\Delta_{\varepsilon h}(1-\Delta_{\varepsilon h})^{-1})H_0\big|_{\varepsilon=0} = 0.$$

The derivative of the mean curvature operator is given by

$$\partial H_\rho|_{\rho=0} = -\frac{1}{n}(n+\Delta_0),$$

see [38], Lemma 3.1. Hence we find

$$\partial G(0) = \frac{1}{n}\Delta_0(1-\Delta_0)^{-1}(n+\Delta_0).$$

ii) Next we locate the spectrum of $A := -\partial G(0)$. For this let $\{Y_k\,;1 \le k \le n+1\}$ be the spherical harmonics of degree 1 and set $Y_0 \equiv 1$. Then it is known that

$$N := span\{Y_0,\dots,Y_{n+1}\} = ker(\Delta_0(1-\Delta)^{-1}(n+\Delta_0)).$$

We conclude that 0 is an eigenvalue of $A$ of multiplicity $n+2$. Assume now that $\lambda \in \mathbb{C}^*$ and $h \in h^{2+\alpha}(\mathcal{S})$ satisfy $(\lambda + A)g = 0$. It follows that $h \in N^\perp$, where the orthogonal complement is taken with respect to the $L_2(\mathcal{S})$ inner product. Next, observe that there are positive constants $c_1$ and $c_2$ such that

$$(\Delta_0^{-1}g|g) \le -c_1(g|g), \qquad ((n+\Delta_0)g|g) \le -c_2(g|g) \tag{3.15}$$

for all $g \in h^{2+\alpha}(\mathcal{S}) \cap N^\perp$, where $(\cdot|\cdot)$ denotes the inner product in $L_2(\mathcal{S})$. Writing $w := (1-\Delta_0)^{-1/2}h$ we find

$$0 = ((\lambda + A)h|\Delta_0^{-1}h) = \lambda(h|\Delta_0^{-1}h) + \frac{1}{n}((n+\Delta_0)w|w).$$

(3.15) implies that $\lambda < 0$. Since $h^{2+\alpha}(\mathcal{S})$ is compactly embedded in $h^\alpha(\mathcal{S})$ we have that the spectrum of $A$ consists of a sequence $\{\mu_k\,;\,k \in \mathbb{N}\}$ of eigenvalues with $\mu_k < \mu_{k-1} < \cdots < \mu_1 < \mu_0$, where $\mu_0 = 0$ has multiplicity $(n+2)$.

(iii) In a next step we briefly sketch the construction of a locally invariant center manifold $\mathcal{M}^c$ over $N$. Let $Y_0 := |\mathcal{S}|^{-1}\mathbf{1}$ and let $Pg := \sum_{k=0}^{n+1}(g|Y_k)Y_k$ for $g \in h^r(\mathcal{S})$. Then $P$ is a continuous projection of $h^r(\mathcal{S})$ onto $N$ parallel to $ker(P)$, and it is easy to verify that $P$ commutes with $A$, that is, $PAg = APg = 0$ for every $g \in h^{2+\alpha}(\mathcal{S})$. Therefore, $N = im(P)$ and $ker(P)$ are complimentary subspaces of $h^{2+\alpha}(\mathcal{S})$ that reduce $A$. To simplify the notation we write $\pi^c = P$ and $\pi^s = (1-P)$, and we define $h_s^{2+\alpha}(\mathcal{S}) := \pi^s(h^{2+\alpha}(\mathcal{S}))$. It follows that $\sigma(\pi^c A) = \{0\}$ and $\sigma(\pi^s A) \subset (-\infty, 0)$. For this reason, $N$ and $h_s^{2+\alpha}(\mathcal{S})$ are called the *center subspace* and the *stable subspace* of $A$, respectively. We can now apply Theorem 4.1 in [65], see also [56] Theorem 9.2.2. These results imply that, given $m \in \mathbb{N}^*$, there exists an open neighborhood $U_0$ of 0 in $N$ and a mapping

$$\gamma \in C^m(U_0, h_s^{2+\alpha}(\mathcal{S})) \quad \text{with} \quad \gamma(0)=0,\ \partial\gamma(0)=0$$

such that $\mathcal{M}^c := \mathrm{graph}(\gamma)$ is a locally invariant manifold for the semiflow generated by the quasilinear evolution equation (3.12). $\mathcal{M}^c$ is an $(n+2)$-dimensional submanifold of $h^{2+\alpha}(\mathcal{S})$. Moreover, $\mathcal{M}^c$ attracts solutions of (3.12) that start in a sufficiently small $h^{2+\beta}(\mathcal{S})$-neighborhood $W_0 \subset W$ of 0 at an exponential rate, and $\mathcal{M}^c$ contains all small equilibria of (3.12), see [65] Theorems 4.1 and 5.8.

iv) Step (iii) shows that $\mathcal{M}^c$ contains all small equilibria of (3.12). We show that $\mathcal{M}^c$ in fact coincides with $\mathcal{M}$ near 0. Suppose that $\mathcal{S}'$ is a sphere which is sufficiently close to $\mathcal{S}$. Let $(z_1, \dots, z_{n+1})$ be the coordinates of its center and let $r$ be its radius. Recall that $\mathcal{S} \subset \mathbb{R}^{n+1}$ is the unit sphere centered at the origin and let $z_0 := 1 - r$. If $\rho$ measures the distance of $\mathcal{S}$ to $\mathcal{S}'$ in normal direction with respect to $\mathcal{S}$, then it can be verified that

$$(1 + z_0)^2 = \sum_{k=1}^{n+1} \big((1+\rho)Y_k - z_k\big)^2.$$

Here we used that the spherical harmonics $Y_k$, $k = 1, \cdots, n+1$, are given as the restrictions of the harmonic coordinate functions $[x \mapsto x_k]$. Let $Y_0 := \mathbf{1}$. Solving the above identify for $\rho$, we obtain that $\mathcal{S}'$ can be parameterized over $\mathcal{S}$ by the distance function

$$\rho(z) = \sum_{k=1}^{n+1} z_k Y_k - 1 + \sqrt{\big(\sum_{k=1}^{n+1} z_k Y_k\big)^2 + (1+z_0)^2 - \sum_{k=1}^{n+1} z_k^2}, \tag{3.16}$$

where $z := (z_0, \dots, z_{n+1}) \in \mathbb{R}^{n+2}$. If $O$ is a sufficiently small neighborhood of 0 in $\mathbb{R}^{n+2}$, then it is clear that any sphere $\mathcal{S}'$ which is close to $\mathcal{S}$ can be characterized by (3.16) with $z \in O$. The mapping $[z \mapsto \rho(z)] : O \to h^{2+\alpha}(\mathcal{S})$ is well-defined and smooth. Let $\mathcal{M} := \{\rho(z); z \in U_0\}$. We conclude that $\mathcal{M} \subset \mathcal{M}^c$, since $\mathcal{M}$ consists of spheres, which are the equilibria of the intermediate surface diffusion flow. We intend to show that $\mathcal{M} = \mathcal{M}^c$. This follows, for instance, if we can verify that $\pi^c(\mathcal{M})$ is an open neighborhood of 0 in $N$. In order to show this we investigate the mapping

$$F : O \to N, \qquad F(z) := \pi^c \rho(z).$$

It is easy to see that the partial derivatives of $F$ with respect to $z_j$ at $0 \in O$ are given by $\partial_{z_0} F(0) = \mathbf{1}$ and $\partial_{z_k} F(0) = Y_k$ for $1 \le k \le n+1$. We conclude that the Fréchet derivative $\partial F(0)$ of $F$ at 0 is given by

$$\partial F(0) h = \sum_{k=0}^{n+1} h_k Y_k, \qquad h \in \mathbb{R}^{n+2}. \tag{3.17}$$

Since the set $\{Y_k\}$ is a basis of $N$, we conclude that $\partial F(0) \in L(\mathbb{R}^{n+2}, N)$ is an isomorphism. Consequently, the Inverse Function Theorem implies that $F$ is a smooth diffeomorphism from $O$ onto its image $V := \mathrm{im}(F)$, provided $O$ is small enough. Therefore, $\pi^c(\mathcal{M})$ is an open neighborhood of 0 in $N$ which can be assumed to coincide with the open neighborhood $U_0$ constructed in step (iii).

v) It follows from step (iv) that the reduced flow of (3.12) on $\mathcal{M}^c$ consists exactly of equilibria. Therefore, 0 is a stable equilibrium for the reduced flow and we conclude that 0 is also stable for the evolution equation (3.12), see Theorem 3.3 in [64]. In particular, there exists a neighborhood $W_0$ of 0 in $h^{2+\beta}(\mathcal{S})$ such that solutions of (3.12) exist globally and converge to $\mathcal{M}^c$ exponentially fast for every initial value $\rho_0 \in W_0$.

(vi) As in [38] Theorem 6.5 and Proposition 6.6, one shows the following result. Given $k \in \mathbb{N}$ and $\omega \in (0, -\mu_1)$ there exists a neighborhood $W_0 = W_0(k, \omega)$ of 0 in $h^{2+\beta}(\mathcal{S})$ with the following property: Given $\rho_0 \in W_0$, the solution $\rho(\cdot, \rho_0)$ of (3.12) exists globally and there exist $c = c(k, \omega) > 0$, $T = T(k, \omega) > 0$, and a unique $z_0 = z_0(\rho_0) \in U_0$ such that

$$\|\big(\pi^c\rho(t,\rho_0), \pi^s\rho(t,\rho_0)\big) - \big(z_0, \gamma(z_0)\big)\|_{C^k} \leq ce^{-\omega t}\|\pi^s\rho_0 - \gamma(\pi^c\rho_0)\|_{h^{2+\beta}}$$

for $t > T$. According to step (iv), $(z_0, \gamma(z_0))$ is a sphere and the proof is now complete. ∎

**Remarks 3.3. a)** By Theorem 3.1 the averaged mean curvature flow (3.3) generates a smooth semiflow on an open subset of $h^{1+\beta}(\Sigma)$. Moreover, Theorem 3.2 ensures that solutions starting sufficiently $C^{1+\beta}$-close to a sphere exist globally. Since in every $C^{1+\beta}$-neighborhood we also find non-convex surfaces, we get global solutions to (3.3) emerging from non-convex initial data. Of course, these global in time solutions are obtained as small perturbations of equilibria. However, it seems that this result is not reachable by the techniques in [42, 43, 49, 50].

**b)** It was shown in [18] by formal asymptotics that the surface diffusion flow is the singular limit of the zero level set of solutions to the Cahn-Hilliard equation with a concentration dependent mobility. Analytically, the surface diffusion flow was investigated for curves in two space dimensions by [11, 28, 44].

**c)** The results for the intermediate surface diffusion flow (3.6) obtained in Theorem 3.1 and Theorem 3.2 are new. The only other analytic results we are aware of are contained in [28]. As for the surface diffusion flow, the methods in [28] seem restricted to curves in $\mathbb{R}^2$.

**d)** In contrast to the Mullins-Sekerka model (3.7), equations (3.3), (3.4), and (3.6) make perfectly sense for immersed hypersurfaces. In fact, our methods are general and flexible enough to extend Theorem 3.1 to the case of compact closed immersed orientable hypersurfaces, see e.g. [40]. Observe that there are immersed surfaces of constant mean curvature which are not Euclidean spheres, cf. [70]. Consequently, the flows induced by (3.3), (3.4), and (3.6) admit equilibria which are not spheres. The stability properties of these non-embedded equilibria are not known. We also refer to the numerical simulations in [40] which show, for instance, that a four-leaf rose evolves in such a way as to approach a limiting configuration which is a triply covered immersed circle.

**e)** It is well-known that the maximum principle prevents embedded hypersurfaces from developing self-intersections under the mean curvature flow. This is no longer true for the averaged mean curvature flow and the surface diffusion flow. It is shown in [58] that both flows can drive embedded hypersurfaces to a self-intersection in finite time. This behavior was conjectured for the averaged mean curvature flow in [42]. Moreover, it was suggested in [28] and later proved [44] that the surface diffusion flow can drive a dumbbell curve of appropriate shape to a self-intersection. The methods in [44] seem restricted to curves in $\mathbb{R}^2$.

**f)** It is known that the averaged mean curvature flow preserves strict convexity, cf. Theorem 1.3 in [50] and Theorem 4.1 in [42]. In contrast, the surface diffusion flow and the Mullins-Sekerka flow do not share this property, see [57, 45].

**g)** The Mullins-Sekerka model arises as a singular limit of the Cahn-Hilliard equation. This was formally derived in [62] and rigorously proved in [2] under the assumption that there exist classical solutions to (3.7). The first existence and uniqueness results for classical solutions to the Mullins-Sekerka model were obtained in [34, 35] and independently for initial data in $C^{3+\beta}$ in [21]. It should be mentioned that even existence of weak solutions to (3.7) in higher space dimensions was not established previously. In two dimensions, existence of global weak solutions for initial curves that are small perturbations of circles was shown in [20]. In [22] the authors prove existence of classical solutions starting from small analytic perturbation of a circle. However, the methods of these papers seem restricted to the plane setting. The Mullins-Sekerka model was also analyzed in [27] for strip-like domains in $\mathbb{R}^2$. Observe that Theorem 3.1 guarantees local existence for arbitrarily large initial data. Finally, the Mullins-Sekerka model can also be obtained as an asymptotic limit of some phase field models [16, 17, 68].

**h)** The Mullins-Sekerka model (3.7) describes solidification and liquiditation phenomena of two materials which are separated by a connected interface. Of course, in applications the situation is considerably more involved. In particular, one usually has to deal with multi-component processes. The Mullins-Sekerka model has also been proposed to account for aging or Ostwald ripening in phase transitions. In general, the kinetics of a first order phase transition is characterized by a first stage where small droplets of a new phase are created out of the old phase, e.g., solid formation in an undercooled liquid. The first stage, called nucleation, yields a large number of small particles. During the next stage the nuclei grow rapidly at the expense of the old phase. When the phase regions are formed, the mass of the new phase is close to equilibrium and the amount of undercooling is small, but large surface area is present. At the next stage, the configuration of phase regions is coarsened, and the geometric shape of the phase regions become simpler and simpler, eventually tending to regions of minimum surface area with given volume. The driving force of this process comes from the need to decrease the interfacial energy. There have been considerable efforts in finding a theory which describes Ostwald ripening, and the Mullins-Sekerka model is a prominent candidate.
Since the mechanism of the Mullins-Sekerka flow shows a distinct nonlocal feature, the corresponding mathematical formulation leads to a strongly coupled system of nonlinear evolution equations of third order. This multi-component system will be the topic of the forthcoming paper [3].

**i)** The construction of center manifolds for finite dimensional dynamical system is well-known. Its extension to quasilinear and fully nonlinear infinite-dimensional semiflows (e.g. for $\varphi$) is considerably more involved. To overcome the technical difficulties involved with this situation we strongly rely on the theory of maximal regularity, see [25, 56, 65].

**j)** Observe that the averaged mean curvature flow induces a nonlinear and nonlocal operator of second order, meaning that the principal part of its linearization is a second order elliptic operator. The same is true for the intermediate surface diffusion flow. In this sense the surface diffusion flow is of forth order and the Mullins-Sekerka model is of third order, where in the latter case the principle part of the linearization is represented by an elliptic pseudo-differential operator of third order. It is worthwhile to note that there are also first order evolution equations driven by mean curvature, which conserve the volume and decrease the area. Let us mention the Stokes problem with surface tension

$$\begin{cases} -\Delta v + \nabla p = 0 & \text{in} \quad \Omega(t) \\ div\, v = 0 & \text{in} \quad \Omega(t) \\ T(v,p)\nu(t) = -H(t)\nu(t) & \text{on} \quad \Gamma(t) \\ V = \partial_\nu u & \text{on} \quad \Gamma(t) \\ \Gamma(0) = \Gamma_0, & \end{cases}$$

where $v$, $p$, and $T$ stand for the velocity field, the pressure field, and the stress tensor of a liquid drop $\Omega(t)$ with boundary $\Gamma(t)$ moving freely under the influence of an exterior field and of surface tension. For a detailed study of this problem we refer to [46, 63].

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