Stable Local Nodal Bases for C^1 Bivariate Polynomial Splines

Oleg Davydov and Larry L. Schumaker

Abstract. We give a stable construction of local nodal bases for spaces of C^1 bivariate polynomial splines of degree $d \geq 5$ defined on arbitrary triangulations. The bases given here differ from recently constructed locally linearly independent bases, and in fact we show that stability and local linear independence cannot be achieved simultaneously.

§1. Introduction

Given a regular triangulation $\triangle$, let

$$\mathcal{S}_d^r(\triangle) := \{s \in C^1(\Omega) : \; s|_T \in \mathcal{P}_d \; \text{ for all triangles } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree d, and Ω is the union of the triangles in $\triangle$. In this paper we focus on the case $r = 1$ and $d \geq 5$. The main result of the paper is a construction of a basis $\mathcal{B} := \{B_i\}_{i=1}^n$ for $\mathcal{S}_d^1(\triangle)$ with the following properties:

P1) The basis $\mathcal{B}$ is *local* in the sense that for each $1 \leq i \leq n$, the support of B_i is contained in $\mathrm{star}(v_i)$ (see the end of this section) for some vertex v_i,

P2) The set $\mathcal{B}$ is *stable* in the sense that there exist constants K_1 and K_2 dependent only on the smallest angle $\theta_\triangle$ in $\triangle$ such that

$$K_1 \|c\|_\infty \; \leq \; \|\sum_{i=1}^n c_i B_i\|_\infty \; \leq \; K_2 \|c\|_\infty \qquad (1)_{\text{stab1}}$$

for all choices of the coefficient vector $c = (c_1, \ldots, c_n)$.

Bases for $\mathcal{S}_d^1(\triangle)$ satisfying property P1 were constructed in [14] using nodal techniques, but they fail to satisfy property P2 for triangulations with near

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singular vertices or near degenerate edges, even if the smallest angle in the triangulation is controlled.

For convenience, we recall the definitions of some of the terminology used above. Suppose v is a vertex of a triangulation which is connected to v_1, v_2, v_3 in counter-clockwise order. Then the edge $e := \langle v, v_2 \rangle$ is said to be near-degenerate at v (degenerate at v) provided that the edges $\langle v, v_1 \rangle$ and $\langle v, v_3 \rangle$ are near-collinear (collinear). The vertex v is called near-singular (singular) if there are exactly four near-degenerate (degenerate) edges attached to it. Given a vertex v of $\triangle$, $\mathrm{star}(v)$ is the set of triangles sharing v, and $\mathrm{star}^{\ell}(v)$ is defined recursively as the union of the stars of the vertices of $\mathrm{star}^{\ell-1}(v)$.

§2. Nodal determining sets and nodal bases

Suppose s is a spline in $\mathcal{S}_d^1(\triangle)$, and that v is a point in Ω. In this paper we are interested in certain *linear functionals* defined on $\mathcal{S}_d^1(\triangle)$ in terms of values and derivatives of s at points v in Ω. Such functionals are called nodal functionals. There are three types of nodal functionals of interest here:

1) the value $s(v)$,
2) the directional derivative $D_w^m s(v)$, where w is a given vector and m is a positive integer,
3) the mixed derivative $D_{w_1}^{m_1} D_{w_2}^{m_2} s(v)$ at a vertex v of $\triangle$, where w_1 and w_2 are two noncollinear vectors which point into a common triangle T of $\triangle$.

nodal **Definition 1.** *A collection $\mathcal{M} := \{\lambda_i\}_{i=1}^n$ of nodal functionals is called a* minimal nodal determining set *for $\mathcal{S}_d^1(\triangle)$ provided they form a basis for the dual space $(\mathcal{S}_d^1(\triangle))^*$. If $\mathcal{M}$ is such a set, then there exist unique splines $\mathcal{B} := \{B_i\}_{i=1}^n$ in $\mathcal{S}_d^1(\triangle)$ such that*

$$\lambda_i B_j = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{2}$$ dual

We call $\mathcal{B}$ a nodal basis *for $\mathcal{S}_d^1(\triangle)$.*

In this paper we will concentrate on nodal functionals which involve derivatives D_e along edges $e := \langle v_1, v_2 \rangle$ of the triangulation $\triangle$, or perpendicular to such edges. Denoting the Cartesian coordinates of a point v by (v^x, v^y), we see that the derivative along the edge e is given by

$$D_e s(v) := \frac{(v_2^x - v_1^x) D_x s(v) + (v_2^y - v_1^y) D_y s(v)}{\sqrt{(v_2^x - v_1^x)^2 + (v_2^y - v_1^y)^2}},$$

while the derivative perpendicular to the edge e is given by

$$D_{e^\perp} s(v) := \frac{(v_2^y - v_1^y) D_x s(v) - (v_2^x - v_1^x) D_y s(v)}{\sqrt{(v_2^x - v_1^x)^2 + (v_2^y - v_1^y)^2}}.$$

Note that

$$D_{\langle v_1, v_2 \rangle} s(v) = -D_{\langle v_2, v_1 \rangle} s(v), \qquad D_{\langle v_1, v_2 \rangle^\perp} s(v) = -D_{\langle v_2, v_1 \rangle^\perp} s(v).$$

§3. Smoothness conditions between polynomial pieces

It is well-known how to describe smoothness between polynomials defined on adjoining triangles in terms of the Bernstein-Bézier coefficients of the two polynomials. Here we need similar conditions in terms of nodal information. Suppose $T = \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} = \langle v_1, v_2, \tilde{v}_3\rangle$ are two adjacent triangles with a common edge $e = \langle v_1, v_2\rangle$. We set $\theta_1 = \angle v_3 v_1 v_2$, $\theta_2 = \angle v_3 v_2 v_1$, $\tilde\theta_1 = \angle \tilde v_3 v_1 v_2$, $\tilde\theta_2 = \angle \tilde v_3 v_2 v_1$. Suppose

$$\begin{aligned} &v_1 < v_1^{e,0} < \cdots v_{d-5}^{e,0} < v_2 \\ &v_1 < v_1^{e,1} < \cdots v_{d-4}^{e,1} < v_2 \end{aligned} \qquad (3)_{\text{vi}}$$

are given points lying in the interior of the edge e.

nsmooth **Lemma 2.** *Let $p, \tilde p$ be polynomials of degree $d \ge 5$ defined on adjoining triangles T and $\widetilde T$ as above. Then p and $\tilde p$ join together with smoothness C^1 across the edge $e := \langle v_1, v_2\rangle$ if and only if the difference $g = p - \tilde p$ satisfies*

$$g(v_i) = D_e g(v_i) = D_{e^\perp} g(v_i) = D_e^2 g(v_i) = 0,, \quad i = 1, 2, \qquad (4)_{\text{nsv}}$$

$$\begin{aligned} g(v_i^{e,0}) &= 0, \quad i = 1, \ldots, d-5, \\ D_{e^\perp} g(v_i^{e,1}) &= 0, \quad i = 1, \ldots, d-4, \end{aligned} \qquad (5)_{\text{nsm1}}$$

and

$$\begin{aligned} \hat\sigma_1 D^2_{\langle v_1, v_2\rangle} p(v_1) &= \tilde\sigma_1 D_{\langle v_1, v_2\rangle} D_{\langle v_1, v_3\rangle} p(v_1) + \sigma_1 D_{\langle v_1, v_2\rangle} D_{\langle v_1, \tilde v_3\rangle} \tilde p(v_1), \\ \hat\sigma_2 D^2_{\langle v_2, v_1\rangle} p(v_2) &= \tilde\sigma_2 D_{\langle v_2, v_1\rangle} D_{\langle v_2, v_3\rangle} p(v_2) + \sigma_2 D_{\langle v_2, v_1\rangle} D_{\langle v_2, \tilde v_3\rangle} \tilde p(v_2), \end{aligned} \qquad (6)_{\text{nsm2}}$$

where $\sigma_i := \sin\theta_i$, $\tilde\sigma_i := \sin\tilde\theta_i$, $\hat\sigma_i := \sin(\theta_i + \tilde\theta_i)$, $i = 1, 2$.

Proof: We follow the method of proof of the main result in [14]. Concerning necessity, we first observe that if p and $\tilde p$ join with C^1 continuity across e, then

$$g(v) = D_w g(v) = 0, \qquad \text{for all } v \in e, \qquad (7)_{\text{ns}}$$

where w is any unit vector noncollinear with the edge e. This implies (5) and the conditions on g, $D_e g$ and $D_{e^\perp} g$ in (4). The conditions on the second derivatives are easily obtained by differentiating the identities (7) along the edge e and using the fact that

$$\begin{aligned} \hat\sigma_1 D_{\langle v_1, v_2\rangle} p(v_1) &= \tilde\sigma_1 D_{\langle v_1, v_3\rangle} p(v_1) + \sigma_1 D_{\langle v_1, \tilde v_3\rangle} p(v_1), \\ \hat\sigma_2 D_{\langle v_2, v_1\rangle} p(v_2) &= \tilde\sigma_2 D_{\langle v_2, v_3\rangle} p(v_2) + \sigma_2 D_{\langle v_2, \tilde v_3\rangle} p(v_2). \end{aligned}$$

To prove sufficiency, suppose that p and $\tilde p$ satisfy (4)–(6). Then the univariate polynomial $g|_e$ is of degree at most d and satisfies $d+1$ homogeneous Hermite interpolation conditions on e. Therefore $g(v) \equiv 0$ for $v \in e$.

This shows that p and $\tilde{p}$ join continuously. We now consider the cross-derivative $q := D_{e^\perp} g|_e$ which is a univariate polynomial of degree at most $d-1$. By (4)–(5), q has $d-2$ zeros $v_1, v_1^{e,1}, \ldots, v_{d-4}^{e,1}, v_2$ on e. Moreover, by (6), $D_e q(v_1) = D_e q(v_2) = 0$, as is easy to check by expressing $D_e D_{e^\perp} p(v_1)$ as a linear combination of $D_e^2 p(v_1)$ and $D_e D_{\langle v_1, v_3\rangle} p(v_1)$ and expressing $D_e D_{e^\perp} \tilde{p}(v_1)$ in terms of $D_e^2 \tilde{p}(v_1)$ and $D_e D_{\langle v_1, \tilde{v}_3\rangle} \tilde{p}(v_1)$, and similarly for v_2. Therefore, $q \equiv 0$, and we have shown that p and $\tilde{p}$ join with C^1-smoothness. □

For a different set of nodal smoothness conditions, see [5].

§4. Construction of a stable local nodal basis for $\mathcal{S}_d^1(\triangle)$

In this section we begin by defining a spanning set $\mathcal{N}_\Delta$ of nodal functionals for $(\mathcal{S}_d^1(\triangle))^*$. Then we choose an appropriate linearly independent subset $\mathcal{M}$ which forms a basis for $(\mathcal{S}_d^1(\triangle))^*$. This will involve analysing the linear dependencies between elements of $\mathcal{N}_\Delta$ (*i.e.*, the smoothness conditions). The corresponding nodal basis determined by the duality conditions (2) will be the desired stable local basis for $\mathcal{S}_d^1(\triangle)$. Given a triangle $T := \langle v_1, v_2, v_3\rangle$, let

$$v_{ijk} := \frac{iv_1 + jv_2 + kv_3}{d}, \qquad i + j + k = d.$$

Given an edge e of $\triangle$, let $v_i^{e,0}$ and $v_i^{e,1}$ be the points defined in (3). We define

$$\begin{aligned} \mathcal{C}^T &:= \{\lambda_{ijk}^T s = s(v_{ijk}^T) : \ i + j + k = d, \ \ 2 \le i, j, k\}, \\ \mathcal{E}(e) &:= \{\lambda_i^{e,0} s = s(v_i^{e,0}) : \ i = 1, \ldots, d-5\} \\ &\quad\ \cup \{\lambda_i^{e,1} s = |e| D_{e^\perp} s(v_i^{e,1}) : \ i = 1, \ldots, d-4\}, \end{aligned}$$

where $|e|$ denotes the length of e.

Given a vertex v in $\triangle$, suppose the vertices connected to v are $v_1, \ldots, v_n$ in counterclockwise order (with v_1 a boundary vertex if v lies on the boundary), and let $T^{[i]} = \langle v, v_i, v_{i+1}\rangle$, $e_i = \langle v, v_i\rangle$, $\theta_i = \angle e_i e_{i+1}$, where if v is an interior vertex, we identify $v_{\ell+n} = v_\ell$, $e_{\ell+n} = e_\ell$. Denote by $|\mathrm{star}\,(v)|$ the area of $\mathrm{star}\,(v)$. Let

$$\begin{aligned} \mathcal{D}_1(v) &:= \{\lambda_{ij}^v s = |\mathrm{star}\,(v)|^{i+j} D_x^i D_y^j s(v) : \ 0 \le i + j \le 1\} \\ \mathcal{R}_2(v) &:= \{\lambda_{i,p}^v s = \tfrac{1}{\sin\theta_i \sin\theta_{i-1}} |\mathrm{star}\,(v)|^2 D_{e_i}^2 s(v) : \ i = 1, \ldots, n\} \\ &\quad\ \cup \{\lambda_{i,m}^v s = \tfrac{1}{\sin\theta_i} |\mathrm{star}\,(v)|^2 D_{e_i} D_{e_{i+1}} s(v) : \ i = 1, \ldots, n\} \end{aligned}$$

if v is an interior vertex, and

$$\begin{aligned} \mathcal{R}_2(v) &:= \{\lambda_{i,p}^v s = \tfrac{1}{\sin\theta_i \sin\theta_{i-1}} |\mathrm{star}\,(v)|^2 D_{e_i}^2 s(v) : \ i = 2, \ldots, n-1\} \\ &\quad\ \cup \{\lambda_{i,p}^v s = |\mathrm{star}\,(v)|^2 D_{e_i}^2 s(v) : \ i = 1, n\} \\ &\quad\ \cup \{\lambda_{i,m}^v s = \tfrac{1}{\sin\theta_i} |\mathrm{star}\,(v)|^2 D_{e_i} D_{e_{i+1}} s(v) : \ i = 1, \ldots, n-1\} \end{aligned}$$

if v is a boundary vertex. Let

$$\mathcal{N}_\triangle := \bigcup_{T\in\triangle} \mathcal{C}^T \cup \bigcup_{e\in\triangle} \mathcal{E}(e) \cup \bigcup_{v\in\triangle} \big[\mathcal{D}_1(v) \cup \mathcal{R}_2(v)\big].$$

The Markov inequality implies that for all $s \in \mathcal{S}_d^1(\triangle)$ and all $\lambda \in \mathcal{N}_\triangle$,

$$|\lambda s| \le K\|s\|_\infty, \tag{8}$$ stab2

for some constant depending only on d and the smallest angle $\theta_\triangle$ in $\triangle$.

NDelta **Lemma 3.** *The set $\mathcal{N}_\triangle$ is a spanning set for $(\mathcal{S}_d^1(\triangle))^*$. Moreover, the only linear dependencies between elements of $\mathcal{N}_\triangle$ are given by*

$$\lambda_{i,m}^v + \lambda_{i-1,m}^v = \sin(\theta_i + \theta_{i-1})\lambda_{i,p}^v \tag{9}$$ lsmoo

for every vertex v and every interior edge e_i attached to v.

Proof: Let $s \in \mathcal{S}_d^1(\triangle)$. If $\lambda s = 0$ for all $\lambda \in \mathcal{N}_\triangle$, then on each triangle $T \in \Delta$ there are exactly $\binom{d+2}{2}$ homogeneous Hermite interpolation conditions on s, and it is easy to see that they force s to be zero. It follows that $\mathcal{N}_\triangle$ is a spanning set for $(\mathcal{S}_d^1(\triangle))^*$. The second statement follows immediately from Lemma 2. $\square$

A4 **Algorithm 4.** *(Construction of a stable local nodal basis for $\mathcal{S}_d^1(\triangle)$.) Let $\{B_i\}_{i=1}^n$ be the set of splines determined by the duality conditions (2) corresponding to the following set $\mathcal{M} := \{\lambda_i\}_{i=1}^n$ of nodal functionals:*

1) *For each triangle T, choose the $\binom{d-4}{2}$ nodal functionals λ_{ijk}^T in $\mathcal{C}^T$.*
2) *For each edge $e = \langle v_1, v_2\rangle$, choose the $2d-9$ nodal functionals $\lambda_i^{e,0}$ and $\lambda_i^{e,1}$ in $\mathcal{E}(e)$.*
3) *For each vertex v, choose the three nodal functionals λ_{ij}^v in $\mathcal{D}_1(v)$.*
4) *For each vertex v, choose the following nodal functionals in $\mathcal{R}_2(v)$:*
   a) *one of the functionals $\lambda_{i,m}^v$ corresponding to the first mixed derivative at v, and*
   b) *all functionals $\lambda_{i,p}^v$ corresponding to the pure second derivatives at v, with one exception: if v is a nonsingular interior vertex, the functional $\lambda_{i_0,p}^v$ is omitted, where i_0 is chosen such that*

$$|\sin(\theta_{i_0} + \theta_{i_0-1})| \ge |\sin(\theta_i + \theta_{i-1})|, \qquad \textit{for all } i = 1,\ldots,n. \tag{10}$$ maxangle

nbasis **Theorem 5.** *The set $\mathcal{M}$ of Algorithm 4 is a minimal nodal determining set for $\mathcal{S}_d^1(\triangle)$, and the nodal basis $\{B_1,\ldots,B_N\}$ for $\mathcal{S}_d^1(\triangle)$ defined in (2) is local and stable, i.e., it satisfies both conditions P1 and P2.*

Proof: The fact that $\mathcal{M}$ is a basis for $(\mathcal{S}_d^1(\triangle))^*$ follows easily from Lemma 3. To construct a typical basis spline B_j, we set $\lambda_i B_j = \delta_{ij}$ for all $i = 1,\ldots,n$. Then the remaining nodal values λB_j, $\lambda \in \mathcal{N}_\triangle \setminus \mathcal{M}$ are computed from the smoothness conditions (9). It is easy to see that the support of the resulting spline is at most the star of a vertex. This shows that P1 is satisfied.

It remains to show that the B_j form a stable basis. This follows from (8) by a standard argument [12], provided we can show that

$$\|B_j\|_\infty \le K, \qquad 1 \le j \le n, \tag{11}$$ P2a

where K is a constant depending only on d and the smallest angle $\theta_\triangle$ in $\triangle$. This clearly holds if

$$|\lambda B_j| \le K, \quad \text{for all } \lambda \in \mathcal{N}_\triangle,$$

for a similar constant K. By construction, $|\lambda B_j| \le 1$ for all $\lambda \in \mathcal{M}$. Since $\mathcal{N}_\triangle \setminus \mathcal{M} \subset \bigcup_v \mathcal{R}_2(v)$, let us take an arbitrary vertex v of $\triangle$ and notice that if $\lambda \in \mathcal{R}_2(v)$, then λB_j can be nonzero only if the corresponding λ_j lies in $\mathcal{R}_2(v)$. Therefore, it will be sufficient to show that $|\lambda B_j| \le K$ for all j such that $\lambda_j \in \mathcal{R}_2(v)$ and all $\lambda \in \mathcal{R}_2(v) \setminus \mathcal{M}$. We distinguish four cases.

Case 1: (v is a boundary vertex.) In this case, $\mathcal{R}_2(v) \setminus \mathcal{M} = \{\lambda^v_{i,m} : i = 1, \ldots, n-1,\ i \ne i_1\}$, where $\lambda^v_{i_1,m}$ is the functional included in $\mathcal{M}$ in step 4a) of Algorithm 4. Without loss of generality we assume that $i_1 = 1$. For any $s \in \mathcal{S}^1_d(\triangle)$, (9) implies

$$\begin{aligned}
\lambda^v_{2,m}s &= -\lambda^v_{1,m}s + \sigma_2\lambda^v_{2,p}s \\
\lambda^v_{3,m}s &= \lambda^v_{1,m}s - \sigma_2\lambda^v_{2,p}s + \sigma_3\lambda^v_{3,p}s \\
&\ \ \vdots \\
\lambda^v_{n-1,m}s &= (-1)^n\lambda^v_{1,m}s + \sum_{i=2}^{n-1}(-1)^{n+i-1}\sigma_i\lambda^v_{i,p}s,
\end{aligned}$$

where we set

$$\sigma_i := \sin(\theta_i + \theta_{i-1}).$$

If we take s to be the basis spline B_j corresponding to a $\lambda_j \in \mathcal{R}_2(v)$, then all but one of the values on the right-hand side of the expression for $\lambda^v_{i,m}B_j$ vanishes, and thus

$$|\lambda^v_{i,m}B_j| \le |\lambda_j B_j| = 1, \qquad i = 2, \ldots, n-1,$$

which proves the assertion.

Case 2: (v is an interior vertex with $n \ne 4$.) In this case, $\mathcal{R}_2(v) \setminus \mathcal{M} = \{\lambda^v_{i,m} :\ i = 1, \ldots, n,\ i \ne i_1\} \cup \{\lambda^v_{i_0,p}\}$. For $\lambda^v_{i,m}s$, $i = 1, \ldots, n$, $i \ne i_1$, the same calculation as in Case 1 applies: we start from $\lambda^v_{i_1,m}s$ and calculate $\lambda^v_{i,m}s$ consecutively counterclockwise until $\lambda^v_{i_0-1,m}s$, and then also clockwise until $\lambda^v_{i_0,m}s$. For $\lambda^v_{i_0,p}s$, we have by (9),

$$\lambda^v_{i_0,p}s = \sigma_{i_0}^{-1}(\lambda^v_{i_0,m}s + \lambda^v_{i_0-1,m}s). \tag{12}$$ lambda

Therefore, our claim will be established if we show that

$$|\sigma_{i_0}^{-1}| = |\sin^{-1}(\theta_{i_0} + \theta_{i_0-1})| \le K_3 \qquad \text{if } n \ne 4, \tag{13}$$ sigma

where K_3 is a constant dependent only on $\theta_\triangle$. This is obvious for $n = 3$. Assuming $n \ge 5$, we have $|\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi| \ge \theta_\triangle$. Hence,

$$|\theta_{i_0} + \theta_{i_0-1} - \pi| \ge \max\{|\theta_1 + \theta_2 - \pi|, |\theta_3 + \theta_4 - \pi|\} \ge \theta_\triangle/2,$$

and (13) follows.

Case 3: (v is a singular vertex.) In this case, $\mathcal{R}_2(v)\backslash\mathcal{M} = \{\lambda_{i,m}^v : i = 2, 3, 4\}$ (where we assume for simplicity that $i_1 = 1$). Since $\sigma_1 = \cdots = \sigma_4 = 0$ for a singular vertex, (9) now reduces to

$$\lambda_{i,m}^v s + \lambda_{i-1,m}^v s = 0, \qquad i = 1, 2, 3, 4.$$

Therefore,

$$\lambda_{i,m}^v s = (-1)^{i+1}\lambda_{1,m}^v s, \qquad i = 2, 3, 4,$$

and the assertion follows.

Case 4: (v is a nonsingular interior vertex with $n = 4$.) We proceed as in Case 2, but calculate $\lambda_{i_0,p}^v s$ differently. At first glance it may seem that (10) does not guarantee stability since $|\sigma_{i_0}|$ may be arbitrary small (in the case of near-singularity), while $\lambda_{i_0,p}^v s$ is to be computed from (12). However, the complete system of equations (9) for $\mathcal{R}_2(v)$ is

$$\sigma_i \lambda_{i,p}^v s = \lambda_{i,m}^v s + \lambda_{i-1,m}^v s, \quad i = 1, 2, 3, 4.$$

Taking the sum with alternating signs, we get

$$\sum_{i=1}^{4} (-1)^i \sigma_i \lambda_{i,p}^v s = 0,$$

and hence

$$|\lambda_{i_0,p}^v s| \le \sum_{i \ne i_0} \frac{|\sigma_i|}{|\sigma_{i_0}|} |\lambda_{i,p}^v s| \le 1$$

for every $s = B_j$, with $\lambda_j \in \mathcal{R}_2(v)$. This completes the proof of (11), and the theorem has been established. □

§5. Stability vs. LLI

We recall (cf. [2,4,6,8,9]) that a set $\mathcal{B}$ of basis splines in $\mathcal{S}_d^1(\triangle)$ is called locally linearly independent (LLI) provided that for every $T \in \triangle$, the splines $\{B_i : i \in \Sigma_T\}$ are linearly independent on T, where

$$\Sigma_T := \{i : T \subset \operatorname{supp} B_i\}. \tag{14}$$ Sigma

A star-supported LLI nodal basis was constructed for $\mathcal{S}_d^1(\triangle)$ in [4]. We now establish the following surprising result.

nostable2 **Theorem 6.** *For $d \geq 5$, it is impossible to construct a basis for $\mathcal{S}_d^1(\triangle)$ which satisfies both conditions P2 and (14) simultaneously.*

Proof: Suppose $\{B_1, \dots, B_n\}$ is a locally linearly independent basis for $\mathcal{S}_d^1(\triangle)$ on a triangulation $\triangle$ which contains an interior near-singular vertex. Suppose v is connected to $v_1, \dots, v_4$ in counterclockwise order, and let e_i be the edge $\langle v, v_i \rangle$, T_i the triangle $\langle v_i, v_{i+1}, v \rangle$, and θ_i the angle between e_i and e_{i+1}. Suppose that none of e_i is degenerate at v. For each $1 \leq j \leq 4$, let s_j be the unique spline in $\mathcal{S}_d^1(\triangle)$ such that

$$\begin{aligned} \lambda_{i,m}^v s_j &= \delta_{ij}, \qquad i, j = 1, 2, 3, 4, \\ \lambda s_j &= 0, \qquad \text{for all } \lambda \in \mathcal{N}_\triangle \setminus \mathcal{R}_2(v) \end{aligned}$$

Clearly,

$$\operatorname{supp} s_j = T_{j-1} \cup T_j \cup T_{j+1},$$

and we can write (see [2,9])

$$s_j = \sum_{i \in I_j} c_i^{[j]} B_i,$$

where $I_j := \{i : \operatorname{supp} B_i \subset \operatorname{supp} s_j\}$. We now consider the spline

$$\begin{aligned} \hat{s} = -s_1 + s_2 - s_3 + s_4 &= -\sum_{i \in I_1} c_i^{[1]} B_i + \sum_{i \in I_2} c_i^{[2]} B_i - \sum_{i \in I_3} c_i^{[3]} B_i + \sum_{i \in I_4} c_i^{[4]} B_i \\ &= \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} a_i B_i. \end{aligned}$$

Using the smoothness conditions (9), it is easy to see that

$$\begin{aligned} &\lambda_{i,p}^v \hat{s} = 0, \quad \lambda_{i,m}^v \hat{s} = (-1)^i, \qquad i = 1, 2, 3, 4, \\ &\lambda \hat{s} = 0, \qquad \text{for all } \lambda \in \mathcal{N}_\triangle \setminus \mathcal{R}_2(v). \end{aligned}$$

and thus $\|\hat{s}\|_\infty \leq K_4$, where K_4 depends only on d. If the basis $\{B_1, \dots, B_n\}$ satisfies P2, we get

$$\|a\|_\infty \leq K_1^{-1} \|\hat{s}\|_\infty \leq K_4 / K_1.$$

Moreover, since $\lambda_{2,m}^v B_i \neq 0$ only if $T_1 \cup T_2 \cup T_3 \subset \operatorname{supp} B_i$, we have

$$1 = \lambda_{2,m}^v \hat{s} = \sum_{i \in \tilde{I}_2} a_i \lambda_{2,m}^v B_i \leq \#\tilde{I}_2 \|a\|_\infty \max_i |\lambda_{2,m}^v B_i|,$$

with $\tilde{I}_2 := \{i : \operatorname{supp} B_i = T_1 \cup T_2 \cup T_3\}$. Clearly, $\#\tilde{I}_2 < 3\binom{d+2}{2}$, and hence there exists $i_0 \in \tilde{I}_2$ such that

$$|\lambda_{2,m}^v B_{i_0}| \geq K_5 > 0,$$

where K_5 depends only on $\theta_\triangle$. However, $\lambda_{1,m}^v B_{i_0} = 0$, so that by (9) we have

$$|\lambda_{2,p}^v B_{i_0}| = \frac{1}{|\sin(\theta_1 + \theta_2)|} |\lambda_{2,m}^v B_{i_0}| \geq \frac{K_5}{|\sin(\theta_1 + \theta_2)|},$$

which is unbounded as $\theta_1 + \theta_2 \to \pi$. In view of the Markov inequality, it follows that $\|B_{i_0}\|_\infty$ is unbounded. But then the basis $\{B_1, \dots, B_n\}$ cannot be stable. $\square$

§6. Remarks

approx **Remark 1.** Stable local bases are important for both theoretical and practical purposes. For example, it can be shown (see [12]) that if a spline space has such a basis, then it has full approximation power. Applications where stable bases are useful include data fitting and the numerical solution of boundary-value problems.

nested **Remark 2.** For $d \geq 5$, stable local bases for certain superspline subspaces of $\mathcal{S}_d^1(\triangle)$, can be constructed using classical finite elements, see [15]. However, it is also important to have such bases for the full spaces $\mathcal{S}_d^1(\triangle)$, since in contrast to supersplines, they are nested, *i.e.*, $\mathcal{S}_d^1(\triangle_1) \subseteq \mathcal{S}_d^1(\triangle_2)$ whenever $\triangle_2$ is a refinement of $\triangle_1$. This is important for multiresolution applications, see [3,13].

star **Remark 3.** Algorithm 4 is a modification of the algorithm used in [14] to construct a star-supported basis for $\mathcal{S}_d^1(\triangle)$. The only change is in the choice of nodal functionals in step 4b) where i_0 was taken to be any index such that e_{i_0} is nondegenerate at v. To get stability, we have to choose i_0 more carefully. The basis constructed in Algorithm 4 is not locally linearly independent. To get an LLI basis, step 4) has to be modified in a different way, see [4].

srd **Remark 4.** Star-supported bases were constructed for general spline spaces $\mathcal{S}_d^r(\triangle)$ for $d \geq 4r+1$ in [1], and for $d \geq 3r+2$ in [10,11]. The constructions were based on Bernstein-Bézier techniques, and are not stable for triangulations that contain near-degenerate edges and/or near-singular vertices.

general **Remark 5.** In [7] we use Bernstein-Bézier techniques to construct stable local bases for general spline spaces $\mathcal{S}_d^r(\triangle)$ and their superspline subspaces for all $d \geq 3r+2$. In a related work [6], we also used Bernstein-Bézier techniques to construct locally-linearly indepependent bases for the same range of spline spaces and superspline spaces. For more on LLI spaces, including applications to almost interpolation, see [2,4,6,8,9].

Lp **Remark 6.** Following the arguments in [7], it is easy to show that a natural renorming of our stable bases is L_p-stable for all $p \in [1, \infty]$.

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Oleg Davydov
Mathematisches Institut
Justus-Liebig-Universität
D-35392 Giessen, Germany
oleg.davydov@math.uni-giessen.de

Larry L. Schumaker
Department of Mathematics
Nashville, TN 37240
s@mars.cas.vanderbilt.edu