Stable Local Nodal Bases for $C^1$ Bivariate Polynomial Splines

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Abstract. We give a stable construction of local nodal bases for spaces of $C^1$ bivariate polynomial splines of degree $d \geq 5$ defined on arbitrary triangulations. The bases given here differ from recently constructed locally linearly independent bases, and in fact we show that stability and local linear independence cannot be achieved simultaneously.

§1. Introduction

Given a regular triangulation $\triangle$, let

$$S^1_d(\triangle) := \{ s \in C^1(\Omega) : s|_T \in \mathcal{P}_d \text{ for all triangles } T \in \triangle \},$$

where $\mathcal{P}_d$ is the space of polynomials of degree $d$, and $\Omega$ is the union of the triangles in $\triangle$. In this paper we focus on the case $r = 1$ and $d \geq 5$. The main result of the paper is a construction of a basis $\mathcal{B} := \{ B_i \}_{i=1}^n$ for $S^1_d(\triangle)$ with the following properties:

P1) The basis $\mathcal{B}$ is local in the sense that for each $1 \leq i \leq n$, the support of $B_i$ is contained in $\text{star}(v_i)$ (see the end of this section) for some vertex $v_i$,

P2) The set $\mathcal{B}$ is stable in the sense that there exist constants $K_1$ and $K_2$ dependent only on the smallest angle $\theta_\triangle$ in $\triangle$ such that

$$K_1 \|c\|_\infty \leq \left\| \sum_{i=1}^n c_i B_i \right\|_\infty \leq K_2 \|c\|_\infty \quad (1)_{\text{stab}}$$

for all choices of the coefficient vector $c = (c_1, \ldots, c_n)$.

Bases for $S^1_d(\triangle)$ satisfying property P1 were constructed in [14] using nodal techniques, but they fail to satisfy property P2 for triangulations with near
singular vertices or near degenerate edges, even if the smallest angle in the triangulation is controlled.

For convenience, we recall the definitions of some of the terminology used above. Suppose \( v \) is a vertex of a triangulation which is connected to \( v_1, v_2, v_3 \) in counter-clockwise order. Then the edge \( e := \langle v, v_2 \rangle \) is said to be \textit{near-degenerate} at \( v \) (\textit{degenerate} at \( v \)) provided that the edges \( \langle v, v_1 \rangle \) and \( \langle v, v_3 \rangle \) are near-collinear (collinear). The vertex \( v \) is called \textit{near-singular} (\textit{singular}) if there are exactly four near-degenerate (degenerate) edges attached to it. Given a vertex \( v \) of \( \triangle \), \( \text{star}(v) \) is the set of triangles sharing \( v \), and \( \text{star}^t(v) \) is defined recursively as the union of the stars of the vertices of \( \text{star}^{t-1}(v) \).

\section{Nodal determining sets and nodal bases}

Suppose \( s \) is a spline in \( S_d^1(\triangle) \), and that \( v \) is a point in \( \Omega \). In this paper we are interested in certain \textit{linear functionals} defined on \( S_d^1(\triangle) \) in terms of values and derivatives of \( s \) at points \( v \) in \( \Omega \). Such functionals are called \textit{nodal functionals}. There are three types of nodal functionals of interest here:

1) the value \( s(v) \),
2) the directional derivative \( D^m_w s(v) \), where \( w \) is a given vector and \( m \) is a positive integer,
3) the mixed derivative \( D^m_{w_1} D^m_{w_2} s(v) \) at a vertex \( v \) of \( \triangle \), where \( w_1 \) and \( w_2 \) are two noncollinear vectors which point into a common triangle \( T \) of \( \triangle \).

\begin{definition}
A collection \( \mathcal{M} := \{ \lambda_i \}^n_{i=1} \) of nodal functionals is called a \textit{minimal nodal determining set} for \( S_d^1(\triangle) \) provided they form a basis for the dual space \( (S_d^1(\triangle))^* \). If \( \mathcal{M} \) is such a set, then there exist unique splines \( \mathcal{B} := \{ B_i \}^n_{i=1} \) in \( S_d^1(\triangle) \) such that

\[ \lambda_i B_{ij} = \delta_{ij}, \quad i, j = 1, \ldots, n. \tag{2}_{\text{dual}} \]

We call \( \mathcal{B} \) a \textit{nodal basis} for \( S_d^1(\triangle) \).
\end{definition}

In this paper we will concentrate on nodal functionals which involve derivatives \( D_e \) along edges \( e := \langle v_1, v_2 \rangle \) of the triangulation \( \triangle \), or perpendicular to such edges. Denoting the Cartesian coordinates of a point \( v \) by \( (v^x, v^y) \), we see that the derivative along the edge \( e \) is given by

\[ D_e s(v) := \frac{(v^x_2 - v^x_1) D_x s(v) + (v^y_2 - v^y_1) D_y s(v)}{\sqrt{(v^x_2 - v^x_1)^2 + (v^y_2 - v^y_1)^2}}, \]

while the derivative perpendicular to the edge \( e \) is given by

\[ D_{e\perp} s(v) := \frac{(v^y_2 - v^y_1) D_x s(v) - (v^x_2 - v^x_1) D_y s(v)}{\sqrt{(v^x_2 - v^x_1)^2 + (v^y_2 - v^y_1)^2}}. \]

Note that

\[ D_{\langle v_1, v_2 \rangle} s(v) = -D_{\langle v_2, v_1 \rangle} s(v), \quad D_{\langle v_1, v_2 \rangle \perp} s(v) = -D_{\langle v_2, v_1 \rangle \perp} s(v). \]
§3. Smoothness conditions between polynomial pieces

It is well-known how to describe smoothness between polynomials defined on adjoining triangles in terms of the Bernstein-Bézier coefficients of the two polynomials. Here we need similar conditions in terms of nodal information. Suppose $T = \{v_1, v_2, v_3\}$ and $\tilde{T} = \{v_1, v_2, \tilde{v}_3\}$ are two adjacent triangles with a common edge $e = \{v_1, v_2\}$. We set $\theta_1 = \angle v_3 v_1 v_2$, $\theta_2 = \angle v_3 v_2 v_1$, $\tilde{\theta}_1 = \angle \tilde{v}_3 v_1 v_2$, $\tilde{\theta}_2 = \angle \tilde{v}_3 v_2 v_1$. Suppose

$$
v_1 < v_1^{c,0} < \cdots < v_{d-5}^{c,0} < v_2
$$

$$
v_1 < v_1^{c,1} < \cdots < v_{d-4}^{c,1} < v_2
$$

are given points lying in the interior of the edge $e$.

\textbf{Lemma 2.} Let $p, \tilde{p}$ be polynomials of degree $d \geq 5$ defined on adjoining triangles $T$ and $\tilde{T}$ as above. Then $p$ and $\tilde{p}$ join together with smoothness $C^1$ across the edge $e := \{v_1, v_2\}$ if and only if the difference $g = p - \tilde{p}$ satisfies

$$
g(v_i) = D_{c} g(v_i) = D_{c^{-1}} g(v_i) = D_{c^2} g(v_i) = 0, \quad i = 1, 2, \tag{4}_{\text{ev}}
$$

$$
g(v_i^{c,0}) = 0, \quad i = 1, \ldots, d - 5, \tag{5}_{\text{mm1}}
$$

$$
D_{c^{-1}} g(v_i^{c,1}) = 0, \quad i = 1, \ldots, d - 4, \tag{5}_{\text{mm2}}
$$

and

$$
\dot{\sigma}_1 D_{\{v_1,v_2\}}^2 p(v_1) = \tilde{\sigma}_1 D_{\{v_1,v_2\}} D_{\{v_1,v_3\}} p(v_1) + \sigma_1 D_{\{v_1,v_2\}} D_{\{v_1,\tilde{v}_3\}} \tilde{p}(v_1),
$$

$$
\dot{\sigma}_2 D_{\{v_2,v_1\}}^2 p(v_2) = \tilde{\sigma}_2 D_{\{v_2,v_1\}} D_{\{v_2,v_3\}} p(v_2) + \sigma_2 D_{\{v_2,v_1\}} D_{\{v_2,\tilde{v}_3\}} \tilde{p}(v_2), \tag{6}_{\text{mm}}
$$

where $\sigma_i := \sin \theta_i$, $\tilde{\sigma}_i := \sin \tilde{\theta}_i$, $\hat{\sigma}_i := \sin(\theta_i + \tilde{\theta}_i)$, $i = 1, 2$.

\textbf{Proof:} We follow the method of proof of the main result in [14]. Concerning necessity, we first observe that if $p$ and $\tilde{p}$ join with $C^1$ continuity across $e$, then

$$
g(v) = D_w g(v) = 0, \quad \text{for all } v \in e, \tag{7}_{\text{w}}
$$

where $w$ is any unit vector noncollinear with the edge $e$. This implies (5) and the conditions on $g$, $D_c g$ and $D_{c^{-1}} g$ in (4). The conditions on the second derivatives are easily obtained by differentiating the identities (7) along the edge $e$ and using the fact that

$$
\dot{\sigma}_1 D_{\{v_1,v_2\}}^2 p(v_1) = \tilde{\sigma}_1 D_{\{v_1,v_3\}} D_{\{v_1,v_2\}} p(v_1) + \sigma_1 D_{\{v_1,v_2\}} D_{\{v_1,\tilde{v}_3\}} \tilde{p}(v_1),
$$

$$
\dot{\sigma}_2 D_{\{v_2,v_1\}}^2 p(v_2) = \tilde{\sigma}_2 D_{\{v_2,v_1\}} D_{\{v_2,v_3\}} p(v_2) + \sigma_2 D_{\{v_2,v_1\}} D_{\{v_2,\tilde{v}_3\}} \tilde{p}(v_2).
$$

To prove sufficiency, suppose that $p$ and $\tilde{p}$ satisfy (4)-(6). Then the univariate polynomial $g|_e$ is of degree at most $d$ and satisfies $d + 1$ homogeneous Hermite interpolation conditions on $e$. Therefore $g(v) \equiv 0$ for $v \in e$. 
This shows that $p$ and $\bar{p}$ join continuously. We now consider the cross-
derivative $q := D_{e^-} g|_c$ which is a univariate polynomial of degree at most
$d - 1$. By (4)-(5), $q$ has $d - 2$ zeros $v_1, v_1^{c_1}, \ldots, v_{d-4}^{c_1}, v_2$ on $e$. Moreover, by (6),
$D_{e^+} q(v_1) = D_{e^-} q(v_2) = 0$, as is easy to check by expressing $D_e D_{e^\pm} (p(v_1)$ as a
linear combination of $D_{e^2} p(v_1)$ and $D_{e^2} D_{(v_1, v_3)} p(v_1)$ and expressing $D_e D_{e^\pm} \bar{p}(v_1)$
in terms of $D_{e^2} \bar{p}(v_1)$ and $D_{e} D_{(v_1, v_3)} \bar{p}(v_1)$, and similarly for $v_2$. Therefore,
$q \equiv 0$, and we have shown that $p$ and $\bar{p}$ join with $C^1$-smoothness. \hfill \Box

For a different set of nodal smoothness conditions, see [5].

§4. Construction of a stable local nodal basis for $S^1_d(\Delta)$

In this section we begin by defining a spanning set $\mathcal{N}_\Delta$ of nodal functionals
for $(S^1_d(\Delta))^*$. Then we choose an appropriate linearly independent subset
$\mathcal{M}$ which forms a basis for $(S^1_d(\Delta))^*$. This will involve analysing the linear
dependencies between elements of $\mathcal{N}_\Delta$ (i.e., the smoothness conditions). The
_corresponding nodal basis determined by the duality conditions (2) will be the
desired stable local basis for $S^1_d(\Delta)$. Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, let

$$v_{ijk} := \frac{iv_1 + jv_2 + kv_3}{d}, \quad i + j + k = d.$$ 

Given an edge $e$ of $\Delta$, let $v_i^{e,0}$ and $v_i^{e,1}$ be the points defined in (3). We define

$$\mathcal{C}^T := \{ \lambda_{ijk}^T s = s(v_{ijk}^T) : i + j + k = d, \ 2 \leq i, j, k \},$$

$$\mathcal{E}(e) := \{ \lambda_{i}^{e,0} s = s(v_{i}^{e,0}) : i = 1, \ldots, d - 5 \}$$

$$\cup \{ \lambda_{i}^{e,1} s = |e| D_{e^-} s(v_{i}^{e,1}) : i = 1, \ldots, d - 4 \},$$

where $|e|$ denotes the length of $e$.

Given a vertex $v$ in $\Delta$, suppose the vertices connected to $v$ are $v_1, \ldots, v_n$ in
clockwise order (with $v_1$ a boundary vertex if $v$ lies on the boundary),
and let $T_i = \langle v, v_i, v_{i+1} \rangle$, $e_i = \langle v, v_i \rangle$, $\theta_i = \angle e_i e_{i+1}$, where if $v$ is an interior
vertex, we identify $v_{i+n} = v_i$, $e_{i+n} = e_i$. Denote by $|\text{star}(v)|$ the area of
star $(v)$. Let

$$\mathcal{D}_1(v) := \{ \lambda_{i,j}^v s = |\text{star}(v)|^{i+j} D_x^i D_y^j s(v) : 0 \leq i + j \leq 1 \}$$

$$\mathcal{R}_2(v) := \{ \lambda_{i,p}^v s = \frac{1}{\sin \theta_i \sin \theta_{i-1}} |\text{star}(v)|^2 D_{e_i}^2 s(v) : i = 1, \ldots, n \}$$

$$\cup \{ \lambda_{i,m}^v s = \frac{1}{\sin \theta_i} |\text{star}(v)|^2 D_{e_i} D_{e_{i+1}} s(v) : i = 1, \ldots, n - 1 \}$$

if $v$ is an interior vertex, and

$$\mathcal{R}_2(v) := \{ \lambda_{i,p}^v s = \frac{1}{\sin \theta_i \sin \theta_{i-1}} |\text{star}(v)|^2 D_{e_i}^2 s(v) : i = 2, \ldots, n - 1 \}$$

$$\cup \{ \lambda_{i,m}^v s = \frac{1}{\sin \theta_i} |\text{star}(v)|^2 D_{e_i} D_{e_{i+1}} s(v) : i = 1, \ldots, n - 1 \}$$
if $v$ is a boundary vertex. Let
\[ \mathcal{N}_\Delta := \bigcup_{T \in \Delta} \mathcal{C}^T \cup \bigcup_{e \in \Delta} \mathcal{E}(e) \cup \bigcup_{v \in \Delta} [\mathcal{D}_1(v) \cup \mathcal{R}_2(v)]. \]

The Markov inequality implies that for all $s \in \mathcal{S}_d^1(\triangle)$ and all $\lambda \in \mathcal{N}_\Delta$,
\[ |\lambda s| \leq K\|s\|_\infty, \]
for some constant depending only on $d$ and the smallest angle $\theta_\Delta$ in $\triangle$.

**Lemma 3.** The set $\mathcal{N}_\Delta$ is a spanning set for $(\mathcal{S}_d^1(\triangle))^\ast$. Moreover, the only linear dependencies between elements of $\mathcal{N}_\Delta$ are given by
\[ \lambda_{i,m}^v + \lambda_{i-1,m}^v = \sin(\theta_i + \theta_{i-1})\lambda_{i,p}^v \]
for every vertex $v$ and every interior edge $e_i$ attached to $v$.

**Proof:** Let $s \in \mathcal{S}_d^1(\triangle)$. If $\lambda s = 0$ for all $\lambda \in \mathcal{N}_\Delta$, then on each triangle $T \in \Delta$ there are exactly $\binom{d+2}{2}$ homogeneous Hermite interpolation conditions on $s$, and it is easy to see that they force $s$ to be zero. It follows that $\mathcal{N}_\Delta$ is a spanning set for $(\mathcal{S}_d^1(\triangle))^\ast$. The second statement follows immediately from Lemma 2. □

**Algorithm 4.** (Construction of a stable local nodal basis for $\mathcal{S}_d^1(\triangle)$.) Let $\{B_i\}_{i=1}^n$ be the set of splines determined by the duality conditions (2) corresponding to the following set $\mathcal{M} := \{\lambda_i\}_{i=1}^n$ of nodal functionals:

1) For each triangle $T$, choose the $\binom{d-1}{2}$ nodal functionals $\lambda_{ijk}^T$ on $\mathcal{C}^T$.

2) For each edge $e = \langle v_1, v_2 \rangle$, choose the $2d - 9$ nodal functionals $\lambda_i^{e,0}$ and $\lambda_i^{e,1}$ in $\mathcal{E}(e)$.

3) For each vertex $v$, choose the three nodal functionals $\lambda_i^v$ in $\mathcal{D}_1(v)$.

4) For each vertex $v$, choose the following nodal functionals in $\mathcal{R}_2(v)$:

   a) one of the functionals $\lambda_{i,m}^v$ corresponding to the first mixed derivative at $v$, and

   b) all functionals $\lambda_{i,p}^v$ corresponding to the pure second derivatives at $v$, with one exception: if $v$ is a nonsingular interior vertex, the functional $\lambda_{i_0,p}^v$ is omitted, where $i_0$ is chosen such that

\[ |\sin(\theta_{i_0} + \theta_{i_0-1})| \geq |\sin(\theta_i + \theta_{i-1})|, \quad \text{for all } i = 1, \ldots, n. \]

**Theorem 5.** The set $\mathcal{M}$ of Algorithm 4 is a minimal nodal determining set for $\mathcal{S}_d^1(\triangle)$, and the nodal basis $\{B_1, \ldots, B_N\}$ for $\mathcal{S}_d^1(\triangle)$ defined in (2) is local and stable, i.e., it satisfies both conditions P1 and P2.

**Proof:** The fact that $\mathcal{M}$ is a basis for $(\mathcal{S}_d^1(\triangle))^\ast$ follows easily from Lemma 3. To construct a typical basis spline $B_j$, we set $\lambda_i B_j = \delta_{ij}$ for all $i = 1, \ldots, n$. Then the remaining nodal values $\lambda B_j$, $\lambda \in \mathcal{N}_\Delta \setminus \mathcal{M}$ are computed from the
smoothness conditions (9). It is easy to see that the support of the resulting spline is at most the star of a vertex. This shows that P1 is satisfied.

It remains to show that the $B_j$ form a stable basis. This follows from (8) by a standard argument [12], provided we can show that

\[ \|B_j\|_\infty \leq K, \quad 1 \leq j \leq n, \]  

(11)${}_a$

where $K$ is a constant depending only on $d$ and the smallest angle $\theta_\Delta$ in $\triangle$. This clearly holds if

\[ |\lambda B_j| \leq K, \quad \text{for all } \lambda \in \mathcal{N}_\Delta, \]

for a similar constant $K$. By construction, $|\lambda B_j| \leq 1$ for all $\lambda \in \mathcal{M}$. Since $\mathcal{N}_\Delta \setminus \mathcal{M} \subset \bigcup_v \mathcal{R}_2(v)$, let us take an arbitrary vertex $v$ of $\triangle$ and notice that if $\lambda \in \mathcal{R}_2(v)$, then $\lambda B_j$ can be nonzero only if the corresponding $\lambda_j$ lies in $\mathcal{R}_2(v)$. Therefore, it will be sufficient to show that $|\lambda B_j| \leq K$ for all $j$ such that $\lambda_j \in \mathcal{R}_2(v)$ and all $\lambda \in \mathcal{R}_2(v) \setminus \mathcal{M}$. We distinguish four cases.

**Case 1:** ($v$ is a boundary vertex.) In this case, $\mathcal{R}_2(v) \setminus \mathcal{M} = \{\lambda_{i,m}^v : i = 1, \ldots, n - 1, \ i \neq i_1\}$, where $\lambda_{i,m}^v$ is the functional included in $\mathcal{M}$ in step 4a) of Algorithm 4. Without loss of generality we assume that $i_1 = 1$. For any $s \in \mathcal{S}_d^1(\Delta)$, (9) implies

\[ \lambda_{2,m}^v s = -\lambda_{1,m}^v s + \sigma_2 \lambda_{2,p}^v s, \]

\[ \lambda_{3,m}^v s = \lambda_{1,m}^v s - \sigma_2 \lambda_{2,p}^v s + \sigma_3 \lambda_{3,p}^v s, \]

\[ \vdots \]

\[ \lambda_{n-1,m}^v s = (-1)^n \lambda_{1,m}^v s + \sum_{i=2}^{n-1} (-1)^{n+i-1} \sigma_i \lambda_{i,p}^v s, \]

where we set

\[ \sigma_i := \sin(\theta_i + \theta_{i-1}). \]

If we take $s$ to be the basis spline $B_j$ corresponding to a $\lambda_j \in \mathcal{R}_2(v)$, then all but one of the values on the right-hand side of the expression for $\lambda_{i,m}^v B_j$ vanishes, and thus

\[ |\lambda_{i,m}^v B_j| \leq |\lambda_j B_j| = 1, \quad i = 2, \ldots, n - 1, \]

which proves the assertion.

**Case 2:** ($v$ is an interior vertex with $n \neq 4$.) In this case, $\mathcal{R}_2(v) \setminus \mathcal{M} = \{\lambda_{i,m}^v : \ i = 1, \ldots, n, \ i \neq i_1\} \cup \{\lambda_{i_0,p}^v\}$. For $\lambda_{i,m}^v s$, $i = 1, \ldots, n$, $i \neq i_1$, the same calculation as in Case 1 applies: we start from $\lambda_{i_1,m}^v s$ and calculate $\lambda_{i,m}^v s$ consecutively counterclockwise until $\lambda_{i_0-1,m}^v s$, and then also clockwise until $\lambda_{i_0,m}^v s$. For $\lambda_{i_0,p}^v s$, we have by (9),

\[ \lambda_{i_0,p}^v s = \sigma_{i_0}^{-1} (\lambda_{i_0,m}^v s + \lambda_{i_0-1,m}^v s). \]  

(12)${}_{\lambda}$
Therefore, our claim will be established if we show that
\[ |\sigma_i^{-1}| = |\sin^{-1}(\theta_{i+1} + \theta_{i-1})| \leq K_3 \quad \text{if} \quad n \neq 4, \tag{13}_{\text{sigma}} \]
where \( K_3 \) is a constant dependent only on \( \theta_\Delta \). This is obvious for \( n = 3 \).
Assuming \( n \geq 5 \), we have \( |\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi| \geq \theta_\Delta \). Hence,
\[ |\theta_i + \theta_{i-1} - \pi| \geq \max\{|\theta_1 + \theta_2 - \pi|, |\theta_3 + \theta_4 - \pi|\} \geq \theta_\Delta / 2, \]
and (13) follows.

**Case 3:** (\( v \) is a singular vertex.) In this case, \( \mathcal{R}_2(v) \setminus \mathcal{M} = \{\lambda_i \mid i = 2, 3, 4\} \)
(where we assume for simplicity that \( i_1 = 1 \)). Since \( \sigma_1 = \cdots = \sigma_4 = 0 \) for a
singular vertex, (9) now reduces to
\[ \lambda_i^{v} s + \lambda_{i-1}^{v} s = 0, \quad i = 1, 2, 3, 4. \]
Therefore,
\[ \lambda_i^{v} s = (-1)^{i+1} \lambda_{i-1}^{v} s, \quad i = 2, 3, 4, \]
and the assertion follows.

**Case 4:** (\( v \) is a nonsingular interior vertex with \( n = 4 \).) We proceed as in
Case 2, but calculate \( \lambda_i^{v} s \) differently. At first glance it may seem that (10)
does not guarantee stability since \( |\sigma_i| \) may be arbitrary small (in the case
of near-singularity), while \( \lambda_i^{v} s \) is to be computed from (12). However, the
complete system of equations (9) for \( \mathcal{R}_2(v) \) is
\[ \sigma_i \lambda_i^{v} s = \lambda_i^{v} s + \lambda_{i-1}^{v} s, \quad i = 1, 2, 3, 4. \]
Taking the sum with alternating signs, we get
\[ \sum_{i=1}^{4} (-1)^i \sigma_i \lambda_i^{v} s = 0, \]
and hence
\[ |\lambda_i^{v} s| \leq \sum_{i \neq i_0} \frac{|\sigma_i|}{|\sigma_{i_0}|} |\lambda_i^{v} s| \leq 1 \]
for every \( s = B_j \), with \( \lambda_j \in \mathcal{R}_2(v) \). This completes the proof of (11), and the
theorem has been established.

\[ \blacksquare \]

§5. Stability vs. LLI

We recall (cf. \([2, 4, 6, 8, 9]\)) that a set \( \mathcal{B} \) of basis splines in \( \mathcal{S}_d^1(\Delta) \) is called locally
linearly independent (LLI) provided that for every \( T \in \Delta \), the splines \( \{B_i : i \in \Sigma_T\} \) are linearly independent on \( T \), where
\[ \Sigma_T := \{i : T \subset \text{supp } B_i\}. \tag{14}_{\text{sigma}} \]
A star-supported LLI nodal basis was constructed for \( \mathcal{S}_d^1(\Delta) \) in \([4]\). We now
establish the following surprising result.
Theorem 6. For $d \geq 5$, it is impossible to construct a basis for $S^1_d(\Delta)$ which satisfies both conditions P2 and (14) simultaneously.

Proof: Suppose $\{B_1, \ldots, B_n\}$ is a locally linearly independent basis for $S^1_d(\Delta)$ on a triangulation $\Delta$ which contains an interior near-singular vertex. Suppose $v$ is connected to $v_1, \ldots, v_4$ in counterclockwise order, and let $e_i$ be the edge $\langle v, v_i \rangle$, $T_i$ the triangle $\langle v_i, v_{i+1}, v \rangle$, and $\theta_i$ the angle between $e_i$ and $e_{i+1}$. Suppose that none of $e_i$ is degenerate at $v$. For each $1 \leq j \leq 4$, let $s_j$ be the unique spline in $S^1_d(\Delta)$ such that

$$\lambda_{i,m}^v s_j = \delta_{ij}, \quad i, j = 1, 2, 3, 4,$$

$$\lambda s_j = 0, \quad \text{for all } \lambda \in N_\Delta \setminus R_2(v).$$

Clearly,

$$s_j = \sum_{i \in I_j} c_i^{[j]} B_i,$$

where $I_j := \{i : \text{ supp } B_i \subset \text{ supp } s_j\}$. We now consider the spline

$$\hat{s} = -s_1 + s_2 - s_3 + s_4 = -\sum_{i \in I_1} c_i^{[1]} B_i + \sum_{i \in I_2} c_i^{[2]} B_i - \sum_{i \in I_3} c_i^{[3]} B_i + \sum_{i \in I_4} c_i^{[4]} B_i$$

$$= \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} a_i B_i,$$

Using the smoothness conditions (9), it is easy to see that

$$\lambda_{i,p}^v \hat{s} = 0, \quad \lambda_{i,m}^v \hat{s} = (-1)^i, \quad i = 1, 2, 3, 4,$$

$$\lambda \hat{s} = 0, \quad \text{for all } \lambda \in N_\Delta \setminus R_2(v).$$

and thus $\|\hat{s}\|_\infty \leq K_4$, where $K_4$ depends only on $d$. If the basis $\{B_1, \ldots, B_n\}$ satisfies P2, we get

$$\|a\|_\infty \leq K_1^{-1} \|\hat{s}\|_\infty \leq K_4/K_1.$$ 

Moreover, since $\lambda_{2,m}^v B_i \neq 0$ only if $T_1 \cup T_2 \cup T_3 \subset \text{ supp } B_i$, we have

$$1 = \lambda_{2,m}^v \hat{s} = \sum_{i \in I_2} a_i \lambda_{2,m}^v B_i \leq \#\hat{I}_2 \|a\|_\infty \max_i |\lambda_{2,m}^v B_i|,$$

with $\hat{I}_2 := \{i : \text{ supp } B_i = T_1 \cup T_2 \cup T_3\}$. Clearly, $\#\hat{I}_2 < 3^{(d+2)/2}$, and hence there exists $i_0 \in \hat{I}_2$ such that

$$|\lambda_{2,m}^v B_{i_0}| \geq K_5 > 0,$$

where $K_5$ depends only on $\theta_\Delta$. However, $\lambda_{1,m}^v B_{i_0} = 0$, so that by (9) we have

$$|\lambda_{2,p}^v B_{i_0}| = \frac{1}{|\sin(\theta_1 + \theta_2)|} |\lambda_{2,m}^v B_{i_0}| \geq \frac{K_5}{|\sin(\theta_1 + \theta_2)|},$$

which is unbounded as $\theta_1 + \theta_2 \to \pi$. In view of the Markov inequality, it follows that $\|B_{i_0}\|_\infty$ is unbounded. But then the basis $\{B_1, \ldots, B_n\}$ cannot be stable. □
§6. Remarks

**Remark 1.** Stable local bases are important for both theoretical and practical purposes. For example, it can be shown (see [12]) that if a spline space has such a basis, then it has full approximation power. Applications where stable bases are useful include data fitting and the numerical solution of boundary-value problems.

**Remark 2.** For $d \geq 5$, stable local bases for certain superspline subspaces of $S^1_d(\Delta)$, can be constructed using classical finite elements, see [15]. However, it is also important to have such bases for the full spaces $S^1_d(\Delta)$, since in contrast to supersplines, they are nested, *i.e.*, $S^1_d(\Delta_1) \subseteq S^1_d(\Delta_2)$ whenever $\Delta_2$ is a refinement of $\Delta_1$. This is important for multiresolution applications, see [3,13].

**Remark 3.** Algorithm 4 is a modification of the algorithm used in [14] to construct a star-supported basis for $S^1_d(\Delta)$. The only change is in the choice of nodal functionals in step 4b) where $i_0$ was taken to be any index such that $e_{i_0}$ is nondegenerate at $v$. To get stability, we have to choose $i_0$ more carefully. The basis constructed in Algorithm 4 is not locally linearly independent. To get an LLI basis, step 4) has to be modified in a different way, see [4].

**Remark 4.** Star-supported bases were constructed for general spline spaces $S^r_d(\Delta)$ for $d \geq 4r+1$ in [1], and for $d \geq 3r+2$ in [10,11]. The constructions were based on Bernstein-Bézier techniques, and are not stable for triangulations that contain near-degenerate edges and/or near-singular vertices.

**Remark 5.** In [7] we use Bernstein-Bézier techniques to construct stable local bases for general spline spaces $S^r_d(\Delta)$ and their superspline subspaces for all $d \geq 3r+2$. In a related work [6], we also used Bernstein-Bézier techniques to construct locally-linearly independent bases for the same range of spline spaces and superspline spaces. For more on LLI spaces, including applications to almost interpolation, see [2,4,6,8,9].

**Remark 6.** Following the arguments in [7], it is easy to show that a natural renorming of our stable bases is $L_p$-stable for all $p \in [1, \infty]$.

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