Local Lagrange Interpolation with Bivariate Splines of Arbitrary Smoothness

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Abstract. We describe a method which can be used to interpolate function values at a set of arbitrarily scattered points in a planar domain using bivariate polynomial splines of any prescribed smoothness. The method starts with an arbitrary triangulation of the data points, and involves refining some of the triangles with Clough-Tocher splits. The construction of the interpolating splines requires some additional function values at selected points in the domain, but no derivatives are needed at any points. The interpolation method is local and stable, has linear complexity, and provides optimal order approximation of smooth functions.

§1. Introduction

Given a set of points $\mathcal{V} := \{\eta_i\}_{i=1}^n$ in the plane, our aim in this paper is to provide a constructive method for solving the following problem.

Problem 1.1. *Find a triangulation $\triangle$ whose set of vertices include $\mathcal{V}$, a space $\mathcal{S}$ of C^r splines defined on $\triangle$, and a set of additional points $\{\eta_i\}_{i=n+1}^N$ such that for every choice of the data $\{z_i\}_{i=1}^N$, there is a unique spline $s \in \mathcal{S}$ satisfying*

$$s(\eta_i) = z_i, \qquad i = 1, \ldots, N. \tag{1.1}$$

We call $P := \{\eta_i\}_{i=1}^N$ and $\mathcal{S}$ a Lagrange interpolation pair, *and refer to s as a* Lagrange interpolating spline.

We emphasize that the spline s solving Problem 1.1 must be uniquely determined from function values only, and no data on the derivatives of s are required.

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Although constructing Lagrange interpolation pairs sounds simple at first glance, it is in fact a complicated problem, especially since we want a method which

1) is local in the sense that the value of s at a given point η is only influenced by the data values z_i at points η_i which are near η,
2) is stable in the sense that small changes in the data z_i will result in a small change in s,
3) has linear complexity in the sense that the number of operations required to solve a problem with N data points should be $\mathcal{O}(N)$,
4) has optimal order approximation in the sense that if $z_i = f(\eta_i)$ for some smooth function f, then the interpolating spline s approximates f to the same order as is achievable with local polynomials on individual triangles.

To reach these goals, both $\mathcal{S}$ and P must be carefully chosen. For $r = 1$, Problem 1.1 was treated in a series of papers, see [9–11,13–16] and the survey paper [12] for further references. Recently [8], we have solved the problem for $r = 2$. The purpose of this paper is to treat the general case $r > 0$. Our approach here will differ from previous work in that we will not need to color triangulations, but instead process an initial triangulation to create a certain priority list of triangles. Our construction is based on the following steps:

1) Choose an initial triangulation $\triangle^{(0)}$ with vertices at the points of $\mathcal{V}$.
2) Classify the triangles of $\triangle^{(0)}$ and put them into an ordered priority list.
3) Define the triangulation $\triangle$ by applying the Clough-Tocher split to subdivide about half of the triangles of $\triangle^{(0)}$.
4) Define the space $\mathcal{S}$ over the triangulation $\triangle$ by enforcing certain individual smoothness conditions on appropriate classical superspline spaces.
5) Insert additional interpolation points into certain of the triangles so as to uniquely and locally define a spline $s \in \mathcal{S}$ on them.
6) Show that the smoothness conditions defining the space $\mathcal{S}$ uniquely determine $s \in \mathcal{S}$ on all of the remaining triangles.

The paper is organized as follows. In Section 2 we introduce some notation and describe the Bernstein-Bézier representation of splines. In Section 3 we discuss macro-elements needed for the construction, while in Section 4 we present a key algorithm for classifying the triangles of the initial triangulation $\triangle^{(0)}$. It also establishes a priority list for the triangles which permits a local construction. In Section 5 we state a result on constrained interpolation by polynomials that is needed for our construction. The proof is delayed until Section 8 where an explicit algorithm for choosing the points of P is given. The main result of the paper is in Section 6 where we introduce the Lagrange interpolating pair $P, \mathcal{S}$, and show that the corresponding interpolation process is a stable local method. Error bounds for the interpolant are given in Section 7. We conclude the paper with several remarks in Section 9.

§2. Preliminaries

Given a triangulation $\triangle$ and integers $0 \le r < d$, we write

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, \text{ all } T \in \triangle\}$$

for the usual space of polynomial splines of degree d and smoothness r, where $\mathcal{P}_d$ is the $\binom{d+2}{2}$ dimensional space of bivariate polynomials of degree d. Given $r \le \rho < d$, we also need the associated space of supersplines

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : s \in C^\rho(v), \text{ all vertices } v \text{ of } \triangle\}. \tag{2.1}$$

As usual, $s \in C^\rho(v)$ means that all polynomial pieces of s on triangles sharing the vertex v have common derivatives up to order ρ at v. In this case, we say that s possesses C^ρ super-smoothness at v.

Throughout the paper we make use of the well-known Bernstein-Bézier representation of splines. Given a triangle $T = \langle v_1, v_2, v_3 \rangle$ in $\triangle$ with vertices v_1, v_2, v_3, let $\mathcal{D}_{d,T} := \{\xi_{ijk}^T := (iv_1 + jv_2 + kv_3)/d\}_{i+j+k=d}$ be the associated set of domain points. Then for every spline s in $\mathcal{S}_d^0(\triangle)$,

$$s|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d, \tag{2.2}$$

where $B_{ijk}^d = \frac{d!}{i!j!k!}\, \lambda_1^i \lambda_2^j \lambda_3^k$ are the Bernstein basis polynomials of degree d associated with T. Here, $\lambda_\nu \in \mathcal{P}_1,\ \nu = 1, \ldots, 3$, are the barycentric coordinates of T. Thus, each spline in $\mathcal{S}_d^0(\triangle)$ is uniquely determined by its corresponding set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$, where $c_{\xi_{ijk}^T} := c_{ijk}^T,\ i+j+k = d,\ T \in \triangle$ and $\mathcal{D}_{d,\triangle} := \bigcup_{T \in \triangle} \mathcal{D}_{d,T}$.

Given $T := \langle v_1, v_2, v_3 \rangle$ and an integer $0 \le m < d$, let $D_m^T(v_1) := \{\xi_{ijk}^T : i \ge d - m\}$, and associated with the edge $e := \langle v_2, v_3 \rangle$, let $E_m^T(e) := \{\xi_{ijk}^T : i \le m\}$. We use the standard notation $D_m(v_1) := \bigcup\{D_m^T(v_1) : T \text{ has a vertex at } v_1\}$ for the disk of radius m around v_1. The analogous sets associated with other vertices and edges are defined similarly.

To describe smoothness conditions for splines, we recall some notation introduced in [2]. Suppose that $T := \langle v_1, v_2, v_3 \rangle$ and $\tilde{T} := \langle v_4, v_3, v_2 \rangle$ are two adjoining triangles from $\triangle$ which share the oriented edge $e := \langle v_2, v_3 \rangle$, and let

$$\begin{aligned} s|_T &:= \sum_{i+j+k=d} c_{ijk} B_{ijk}^d, \\ s|_{\tilde{T}} &:= \sum_{i+j+k=d} \tilde{c}_{ijk} \widetilde{B}_{ijk}^d, \end{aligned} \tag{2.3}$$

where B_{ijk}^d and $\widetilde{B}_{ijk}^d$ are the Bernstein polynomials of degree d on the triangles T and $\tilde{T}$, respectively. Given integers $0 \le n \le j \le d$, let $\tau_{j,e}^n$ be the linear functional defined on $\mathcal{S}_d^0(\triangle)$ by

$$\tau_{j,e}^n s := c_{n,d-j,j-n} - \sum_{\nu+\mu+\kappa=n} \tilde{c}_{\nu,\mu+j-n,\kappa+d-j} \widetilde{B}_{\nu\mu\kappa}^n(v_1). \tag{2.4}$$

These are called smoothness functionals of order n.

We note that a spline $s \in \mathcal{S}_d^0(\triangle)$ is C^r continuous across the edge e if and only if

$$\tau_{m,e}^n s = 0, \qquad n \le m \le d, \quad 0 \le n \le r. \tag{2.5}$$

In the sequel we shall make use of the $\tau_{j,e}^n$ to define certain superspline spaces by enforcing individual smoothness conditions. This is a standard procedure for eliminating undesired degrees of freedom from spline spaces.

§3. Macro-elements

Suppose $\triangle$ is a triangulation of a polygonal domain Ω, and that T is some triangle in $\triangle$. Our aim in this section is to describe two classes of macro-element spaces which can be used to extend a superspline $s \in \mathcal{S}_d^{r,\rho}(\triangle \backslash \{T\})$ to a spline $\tilde{s} \in \mathcal{S}_d^{r,\rho}(\triangle)$, where $\mathcal{S}_d^{r,\rho}(\triangle)$ is defined in (2.1). Here we are interested only in the cases where r, ρ and d take on the related values

$$(\rho, d) := \begin{cases} (3m+1, 6m+3), & r = 2m+1, \\ (3m, 6m+1), & r = 2m, \end{cases} \tag{3.1}$$

for $r \ge 1$, where $m = \lfloor r/2 \rfloor$.

Suppose $T := \langle v_1, v_2, v_3 \rangle$ and that $e_i := \langle v_i, v_{i+1} \rangle$ for $i = 1,2,3$, where we identify $v_4 = v_1$. Let T_{CT} be the Clough-Tocher split of T consisting of the three triangles $T_i := \langle v_T, v_i, v_{i+1} \rangle$ for $i = 1,2,3$, where v_T is the barycenter of T. Let $\tilde{e}_i := \langle v_i, v_T \rangle$ for $i = 1,2,3$.

The following lemma, which follows from the results in [1], can be used to extend a superspline s defined on $\Omega \setminus T$ to a superspline $\tilde{s}$ defined on Ω in the case where T is in the interior of Ω, i.e., s and its derivatives are known along all three sides of T. The lemma shows that by requiring the extension $\tilde{s}|_T$ to satisfy some additional supersmoothness conditions inside of T_{CT}, we can insure that it is uniquely determined by the C^ρ super-smoothness at the vertices of T and the C^r smoothness conditions across the edges of T.

Lemma 3.1. *Given $r \ge 1$, let ρ, d be as in (3.1), and let*

$$\mu := \begin{cases} 5m+2, & r = 2m+1, \\ 5m+1, & r = 2m, \end{cases} \tag{3.2}$$

where $m = \lfloor r/2 \rfloor$. Let $\widehat{\mathcal{S}}_d^r(T_{CT})$ be the linear subspace of all splines g in $\mathcal{S}_d^{r,\rho}(T_{CT}) \cap C^\mu(v_T)$ that satisfy the following additional smoothness conditions:

$$\tau_{\rho+i+1,\tilde{e}_1}^{2m+1+i+j} g = 0, \qquad 1 \le j \le i, \quad 1 \le i \le r-m-1, \tag{3.3}$$

$$\tau_{\rho+i+1,\tilde{e}_2}^{2m+1+i+j} g = 0, \qquad 1 \le j \le i, \quad 1 \le i \le r-m-1, \tag{3.4}$$

$$\tau_{2r+i,\tilde{e}_1}^{\rho+i+j} g = 0, \qquad 1 \le j \le m-i+1, \quad 1 \le i \le m, \tag{3.5}$$

$$\tau_{2r+i,\tilde{e}_2}^{\rho+i+j} g = 0, \qquad 1 \le j \le m-i, \quad 1 \le i \le m-1. \tag{3.6}$$

Then for any spline s in $\mathcal{S}_d^{r,\rho}(\triangle \setminus \{T\})$, there is a unique spline $g \in \widehat{\mathcal{S}}_d^r(T_{CT})$ such that

$$\tilde{s} := \begin{cases} s, & \text{on } \Omega \setminus T, \\ g, & \text{on } T, \end{cases}$$

belongs to $\mathcal{S}_d^{r,\rho}(\triangle)$.

The following lemma is a consequence of the results in [17]. It can be used to extend a superspline s defined on $\Omega \setminus T$ to a superspline $\tilde{s}$ defined on Ω in the case where exactly one of the edges of T, say e_3, is on the boundary of Ω, i.e., s and its derivatives are known along exactly two sides of T. The lemma shows that by requiring the extension $\tilde{s}|_T$ to satisfy some additional supersmoothness conditions inside of T_{CT}, we can insure that it is uniquely determined by the C^ρ super-smoothness at the vertices of T, the C^r smoothness conditions across the edges of T, and certain additional smoothness conditions across the edges e_1, e_2 of T.

Lemma 3.2. *Let r, μ, ρ, d be as in Lemma 3.1, and let $\widetilde{\mathcal{S}}_d^r(T_{CT})$ be the linear subspace of all splines g in $\mathcal{S}_d^{r,\rho}(T_{CT}) \cap C^\mu(v_T)$ that satisfy the following additional smoothness conditions:*

$$\tau_{i,\tilde{e}_2}^k g = 0, \qquad i - r + m + 1 \le k \le 2i - 2r - 1, \quad \rho + 2 \le i \le 2r, \tag{3.7}$$

$$\tau_{i,\tilde{e}_2}^k g = 0, \qquad i - r + m + 1 \le k \le i, \quad 2r + 1 \le i \le d, \tag{3.8}$$

$$\tau_{\rho+i,\tilde{e}_3}^{2m+k+i} g = 0, \qquad 1 \le k \le r - m - i, \quad 1 \le i \le r - m - 1. \tag{3.9}$$

Then for any spline s in $\mathcal{S}_d^{r,\rho}(\triangle \setminus T)$, there is a spline $g \in \widetilde{\mathcal{S}}_d^r(T_{CT})$ such that

$$\tilde{s} := \begin{cases} s, & \text{on } \Omega \setminus T, \\ g, & \text{on } T, \end{cases}$$

belongs to $\mathcal{S}_d^{r,\rho}(\triangle)$. Moreover, g (and thus also $\tilde{s}$) is uniquely determined by the following special conditions across the edges e_1 and e_2 of T:

$$\tau_{d-m-i+j-1,e_1}^{r+j} \tilde{s} = 0, \quad 1 \le j \le \lfloor \frac{i - r + m + 1}{2} \rfloor, \ r - m + 1 \le i \le r - 1, \tag{3.10}$$

$$\tau_{d-\rho+j-1,e_1}^{i+j} \tilde{s} = 0, \quad 1 \le j \le \lfloor \frac{r - i + m + 1}{2} \rfloor, \ r \le i \le \rho - 1, \tag{3.11}$$

$$\tau_{d-\rho+i,e_2}^{r+j} \tilde{s} = 0, \quad 1 \le j \le \lfloor \frac{i - r + m}{2} \rfloor, \ r - m + 1 \le i \le r - 1, \tag{3.12}$$

$$\tau_{d-\rho+i,e_2}^{i+j} \tilde{s} = 0, \quad 1 \le j \le \lfloor \frac{r - i + m}{2} \rfloor, \ \ r \le i \le \rho - 1. \tag{3.13}$$

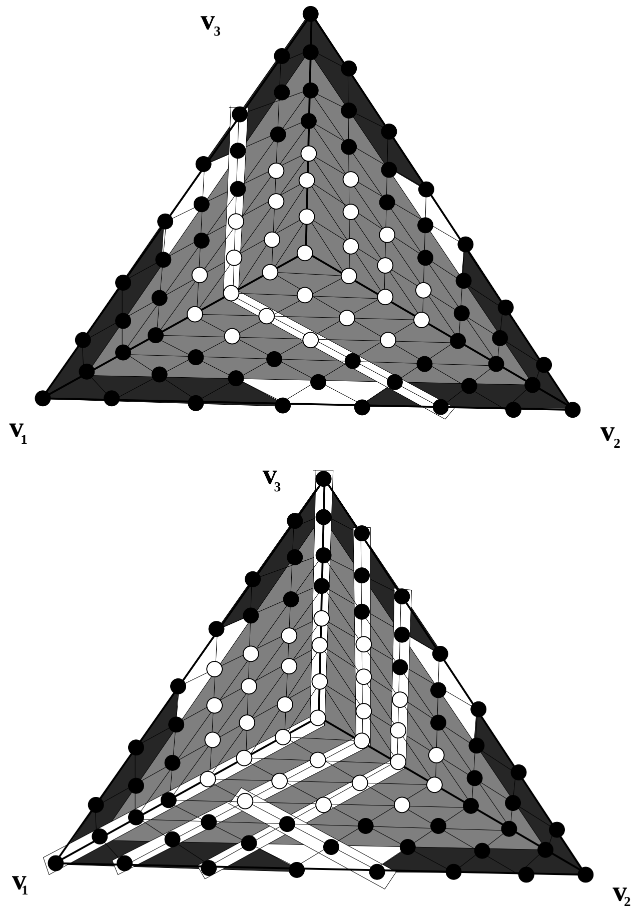

Fig. 1. The macro elements of Lemmas 3.1 and 3.2 for $r = 2$.

For $r = 1$ the macro element of Lemma 3.1 is the classical Clough-Tocher element, while the macro element of Lemma 3.2 is defined by one additional individual C^3 smoothness condition, namely $\tau^3_{3,\tilde{e}_2} g = 0$. As an aid to understanding these macro-elements spaces better, we now describe the cases $r = 2$ and $r = 3$ in more detail, and illustrate them in Figures 1 and 2. Each dot in these figures represents a BB-coefficient associated with the domain point located there. Black dots indicate BB-coefficients which are either determined from C^ρ super-smoothness at a vertex or from C^r smoothness across an edge. To help understand the super smoothness involved, we have used light grey to indicate the C^μ super-smoothness at v_T, and darker grey to indicate the C^ρ super-smoothness at the vertices of T. The white strips indicate the individual super-smoothness conditions used to eliminate undesired degrees of freedom.

For $r = 2$, we have $m = 1$ and $(\rho, \mu, d) = (3, 6, 7)$. Then in Lemma 3.1 there is one special condition corresponding to (3.5), namely $\tau^5_{5,\tilde{e}_1} g = 0$, see the white strip in the top triangle of Fig. 1. In Lemma 3.2 there are three conditions of type (3.8), namely $\tau^i_{i,\tilde{e}_2} g = 0$, $i = 5, 6, 7$. Moreover, in this case, there is one additional smoothness condition corresponding to (3.11), namely $\tau^3_{4,e_1} \tilde{s} = 0$. These

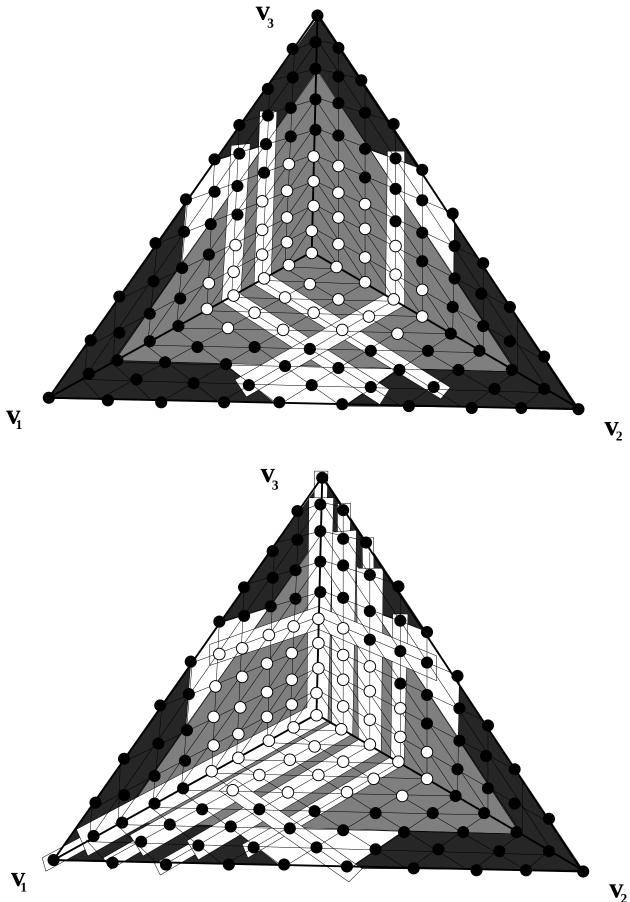

Fig. 2. The macro elements of Lemmas 3.1 and 3.2 for $r = 3$.

four special conditions are shown as white strips in the bottom triangle of Fig. 1.

For $r = 3$, we have $m = 1$ and $(\rho, \mu, d) = (4, 7, 9)$. Then in Lemma 3.1 there are three individual smoothness conditions, namely $\tau^5_{6,\tilde{e}_1} g = 0$, $\tau^5_{6,\tilde{e}_2} g = 0$ and $\tau^6_{7,\tilde{e}_1} g = 0$. These are shown as white strips in the top triangle of Fig. 2. In Lemma 3.2 there are nine individual smoothness conditions, namely one condition of type (3.7), $\tau^5_{6,\tilde{e}_2} g = 0$, six conditions of type (3.8), $\tau^k_{i,\tilde{e}_2} g = 0$, $i = 7, 8, 9$, $k = i-1, i$, one condition of type (3.9), $\tau^4_{5,\tilde{e}_3} g = 0$, and one condition of type (3.11), $\tau^4_{5,e_1} \tilde{s} = 0$. These individual smoothness are shown as white strips in the bottom triangle of Fig. 2.

§4. Decomposing a Triangulation

Given a set of points $\mathcal{V}$ in $\mathbb{R}^2$, let $\triangle^{(0)}$ be a triangulation consisting of n_0 triangles with vertices at the points $\mathcal{V}$. We say that two triangles in $\triangle^{(0)}$ are neighbors provided they have a common edge. We say that they touch provided they have a common vertex. The key to our construction of a Lagrange interpolating pair giving a local and stable solution to Problem 1.1 is the following algorithm for separating the triangles of $\triangle^{(0)}$ into classes $\mathcal{T}_0, \ldots, \mathcal{T}_7$. The algorithm also creates an ordering (priority list) $T_1, \ldots, T_{n_0}$ of the triangles of $\triangle^{(0)}$.

Algorithm 4.1.

0) *Repeat until no longer possible: choose an unmarked triangle T that does not touch any marked triangle. Put T in $\mathcal{T}_0$ and mark T.*
1) *Repeat until no longer possible: choose an unmarked triangle T that touches some marked triangle at only one vertex of T. Put T in $\mathcal{T}_1$ and mark T.*
2) *Repeat until no longer possible: choose an unmarked triangle T that touches marked triangles at exactly two vertices of T, but is not a neighbor of any marked triangle. Put T in $\mathcal{T}_2$ and mark T.*
3) *Repeat until no longer possible: choose an unmarked triangle T that is a neighbor of exactly one marked triangle but does not touch any marked triangle at the opposing vertex. Put T in $\mathcal{T}_3$ and mark T.*
4) *Repeat until no longer possible: choose an unmarked triangle T that touches marked triangles at all three vertices of T, but has no marked triangle as a neighbor. Put T in $\mathcal{T}_4$ and mark T.*
5) *Repeat until no longer possible: choose an unmarked triangle T that is a neighbor of exactly one marked triangle and also touches a marked triangle at the vertex opposite the shared edge. Put T in $\mathcal{T}_5$ and mark T.*
6) *Repeat until no longer possible: choose an unmarked triangle T that is a neighbor of exactly two marked triangles. Put T in $\mathcal{T}_6$ and mark T.*
7) *Put all remaining triangles in $\mathcal{T}_7$.*

Algorithm 4.1 defines an ordering $T_1, \ldots, T_{n_0}$ of the triangles, where the triangles are listed in the order in which they are marked in the algorithm. The algorithm is easy to program, and is efficient enough to decompose very large triangulations (e.g. with $n_0 = \mathcal{O}(10^5)$) in a few seconds on a standard PC. Note that for a given triangulation, there may be many choices at each step, so obviously the decomposition is not unique. Fig. 3 shows an example of a triangulation which has been decomposed by this algorithm, where each triangle is labeled according to the class to which it belongs.

The next lemma establishes some simple properties of the decomposition. These properties will be used to prove the locality of the spline interpolation method described in Section 6.

Lemma 4.2. *Suppose $\mathcal{T}_0, \ldots, \mathcal{T}_7$ are the classes of triangles created by Algorithm 4.1. Then*

1) *No two triangles in the class $\mathcal{T}_0$ can touch each other.*
2) *Any two neighboring triangles must be in different classes.*
3) *If two triangles in the same class $\mathcal{T}_i$ touch at a vertex v, then they must also touch a triangle in one of the classes $\mathcal{T}_j$ with $0 \le j \le \min(3, i-1)$ at the same vertex v.*

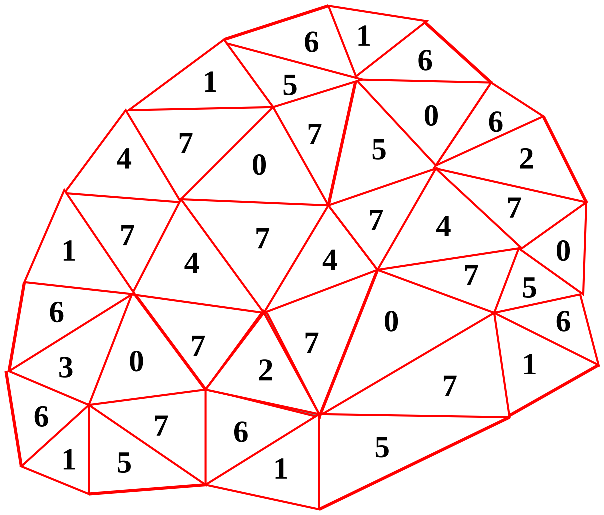

Fig. 3. A decomposed triangulation: triangles from the class $\mathcal{T}_i$ are labeled with i.

4) *If v is a vertex of a triangle $T \in \mathcal{T}_4$, then v must also be a vertex of a triangle $\tilde{T} \in \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$.*

Proof: The first assertion is obvious. We now establish 2). The claim is obvious for the classes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4$, since a triangle in one of these classes cannot have any marked neighbor at the time it is marked. To check 2) for the remaining classes, suppose that T and $\tilde{T}$ are neighboring triangles in the same class $\mathcal{T}_i$, and that T is marked before $\tilde{T}$. If $i = 3$, then before T was marked, $\tilde{T}$ did not share an edge with any marked triangle, and so $\tilde{T}$ would have been assigned to the class $\mathcal{T}_1$. If $i = 5$, then before T was marked, $\tilde{T}$ did not share an edge with any marked triangle, and so would have been assigned to the class $\mathcal{T}_4$. If $i = 6$, then before T was marked, $\tilde{T}$ would have shared only one edge with marked triangles, and so would have been assigned to the class $\mathcal{T}_5$. Clearly, two triangles in $\mathcal{T}_7$ cannot be neighbors, and we have established 2).

To establish 3), first note that after marking all triangles in classes $\mathcal{T}_0, \ldots, \mathcal{T}_3$, all vertices will belong to marked triangles. This establishes the claim for $i = 4, 5, 6, 7$. Now suppose two or more triangles in $\mathcal{T}_1$ touch at a vertex v, and let $T, \tilde{T}$ be the first two marked by Algorithm 4.1. If v is not a vertex of some triangle in $\mathcal{T}_0$, then before T was marked, $\tilde{T}$ would not have touched any marked triangle, and so would have been put in class $\mathcal{T}_0$. A similar argument shows that 3) holds for the class $\mathcal{T}_2$. The statement is trivially true for $\mathcal{T}_3$, since two triangles in $\mathcal{T}_3$ can touch only if they are both neighbors of some triangle in $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$.

We now establish 4). Assume that v is a vertex of $T \in \mathcal{T}_4$ and that $\tilde{T}$ is the first triangle chosen by Algorithm 4.1 that contains v. Then $\tilde{T} \in \mathcal{T}_j$ with $0 \le j \le 3$. We claim that $\tilde{T}$ cannot be in $\mathcal{T}_3$, since if it were, then before $\tilde{T}$ was marked in the algorithm, T would have had only two marked vertices, and hence would have been assigned to class $\mathcal{T}_2$. $\square$

§5. Interpolation with B-Polynomials

In this section we discuss interpolation with bivariate polynomials where certain of the B-coefficients are set in advance. Given $1 \le r$, let ρ and d be as in (3.1), and suppose $T := \langle v_1, v_2, v_3 \rangle$ is a triangle. Let

$$A_T^i := \begin{cases} \emptyset, & i = 0, \\ D_\rho^T(v_1), & i = 1, \\ D_\rho^T(v_1) \cup D_\rho^T(v_2), & i = 2, \\ D_\rho^T(v_1) \cup D_\rho^T(v_2) \cup E_r^T(\langle v_1, v_2 \rangle), & i = 3, \\ D_\rho^T(v_1) \cup D_\rho^T(v_2) \cup D_\rho^T(v_3), & i = 4, \\ D_\rho^T(v_1) \cup D_\rho^T(v_2) \cup D_\rho^T(v_3) \cup E_r^T(\langle v_2, v_3 \rangle), & i = 5. \end{cases} \tag{5.1}$$

Fig. 8 shows examples of the sets A_T^i for $i = 0, \ldots, 5$ for $(r, \rho, d) = (1, 1, 3)$ and $(r, \rho, d) = (2, 3, 7)$. Domain points in the sets A_T^i are marked with open circles in the figure.

The following theorem follows from a more general result to be established below, see Theorem 8.2. Let

$$\alpha_i := \begin{cases} \kappa, & i = 0, \\ \kappa - a, & i = 1, \\ \kappa - 2a, & i = 2, \\ \kappa - 2a - b, & i = 3, \\ \kappa - 3a, & i = 4, \\ \kappa - 3a - b, & i = 5, \end{cases} \tag{5.2}$$

where $\kappa := \binom{d+2}{2}$, $a := \binom{\rho+2}{2}$, and $b := \binom{r+1}{2}$. The following table lists the values of α_i for selected values of r, ρ, d.

r	ρ	d	α_0	α_1	α_2	α_3	α_4	α_5
1	1	3	10	7	4	3	1	0
2	3	7	36	26	16	13	6	3
3	4	9	55	40	25	19	10	4

Theorem 5.1. *Given $0 \le i \le 5$, let p be a polynomial of degree d whose B-coefficients corresponding to domain points in the set A_T^i are given. Then we can explicitly choose a set of α_i points P_T^i in T such that the α_i remaining B-coefficients of p are uniquely determined by the values $p(\xi)$ for $\xi \in P_T^i$, where the points in the sets P_T^i can be defined in terms of the barycentric coordinates of T, so that the relative position of the points is the same for every triangle T, independent of the size and shape of T.*

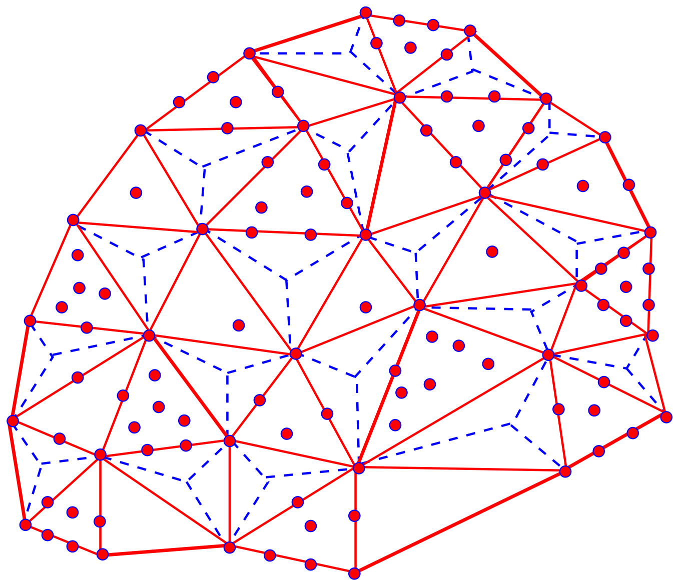

Fig. 4. The set P for Example 6.2, where $r = 1$ and $(\rho, \mu, d) = (1, 2, 3)$.

§6. Construction of a Lagrange Interpolation Pair

Given $r \geq 1$, let $m = \lfloor r/2 \rfloor$ and let (ρ, μ, d) be as in (3.1)–(3.2). We now define a spline space $\mathcal{S}$ of smoothness r and degree d and a corresponding point set P so that $P, \mathcal{S}$ form a Lagrange interpolation pair, and thus can be used to solve Problem 1.1. We also show that the corresponding interpolant is local and stable.

Let $\triangle^{(0)}$ be a triangulation with vertices $\mathcal{V}$ that has been decomposed into classes $\mathcal{T}_0, \ldots, \mathcal{T}_7$ by Algorithm 4.1, and suppose that $T_1, \ldots, T_{n_0}$ is the ordering of the triangles induced by the algorithm. For each $T \in \mathcal{T}_i$ with $0 \leq i \leq 5$, let $A_T := A_T^i$ be the set defined in (5.1), where by Lemma 4.2, we may assume that if $T := \langle v_1, v_2, v_3 \rangle$, then v_1, v_2, v_3 are numbered so that the vertices and edges appearing in the definition (5.1) of A_T^i are those vertices and edges of T which are shared by a triangle in a lower class. For each $T \in \mathcal{T}_i$, let $P_T := P_T^i$ be the corresponding set defined in Theorem 5.1.

Definition 6.1. *Let $\triangle$ be the triangulation obtained from $\triangle^{(0)}$ by applying the Clough-Tocher split with center v_T to each triangle T in the classes $\mathcal{T}_6$ and $\mathcal{T}_7$. Let $\mathcal{S}$ be the subspace of splines s in $\mathcal{S}_d^{r,\rho}(\triangle)$ such that*

1) *For each triangle $T \in \mathcal{T}_7$, $s \in C^\mu(v_T)$ and s satisfies the additional smoothness conditions (3.3)–(3.6) of Lemma 3.1.*

2) *For each triangle $T \in \mathcal{T}_6$, $s \in C^\mu(v_T)$ and s satisfies the additional smoothness conditions (3.7)–(3.9) and (3.10)–(3.13) of Lemma 3.2, where e_1 and e_2 are the two edges of T shared by triangles which appear earlier than T in the list $T_1, \ldots, T_{n_0}$.*

Set

$$P := \bigcup_{j=1}^{n_0} P_{T_j} = \bigcup_{i=0}^{5} \bigcup_{T \in \mathcal{T}_i} P_T. \tag{6.1}$$

Clearly,

$$\#P = \sum_{i=0}^{5} \alpha_i N_i, \tag{6.2}$$

where N_i is the number of triangles in class $\mathcal{T}_i$ and α_i is as in (5.2). We give two examples to illustrate the choice of Lagrange interpolation points.

Example 6.2. Let $\triangle^{(0)}$ be the triangulation of 41 triangles which has been decomposed as shown in Fig. 3, and let $r = 1$.

Discussion: In this case $N_0 = 5$, $N_1 = 6$, $N_2 = 2$, $N_3 = 1$, $N_4 = 4$, $N_5 = 5$, $N_6 = 7$, and $N_7 = 11$. The Clough-Tocher split is applied to 18 triangles of $\triangle^{(0)}$, and therefore the resulting triangulation $\triangle$ consists of 77 triangles. Here $\rho = 1$, $d = 3$, and using the values of α_i listed in the table in Sect. 5, it follows from (6.2) that the cardinality of P is 107. One explicit choice of P is shown in Fig. 4, where the points in P are shown as dots, and the Clough-Tocher splits are indicated by the dotted lines. □

Example 6.3. Let $\triangle^{(0)}$ be as in Example 6.2, and let $r = 2$.

Discussion: Here $\rho = 3$, $d = 7$, and using the values of α_i listed in the table in Sect. 5, it follows from (6.2) that the cardinality of P is 420. The dots in Fig. 5 show one explicit choice of an interpolation set P. The example can be compared with Example 14 in [8], where a different set P was constructed, see also Fig. 9 below. □

Following standard terminology, we say that an interpolation method based on a Lagrange interpolation pair $P, \mathcal{S}$ is local provided there is an integer ℓ such that for every triangle T, the B-coefficients of $s|_T$ depend only on the values z_η at the points $P \cap \operatorname{star}^\ell(T)$. Here $\operatorname{star}^0(T) := T$, and for $i \geq 1$, $\operatorname{star}^i(T)$ is the union of the set of all triangles which touch a triangle in $\operatorname{star}^{i-1}(T)$. Moreover, we say that the method is stable provided there exists a constant C depending only on the smallest angle in $\triangle^{(0)}$ such that the B-coefficients of the spline s interpolating data z_η as in (1.1) are bounded by $C \max_{\eta \in P} |z_\eta|$.

We are now ready to prove the main result of the paper, namely that the P and $\mathcal{S}$ of Definition 6.1 form a Lagrange interpolating pair. At the same time we show that the corresponding interpolation method is local and stable.

Theorem 6.4. *Let $\mathcal{S}$ and P be as in Definition 6.1. Then given any real numbers $\{z_\eta\}_{\eta \in P}$, there exists a unique $s \in \mathcal{S}$ such that $s(\eta) = z_\eta$ for all $\eta \in P$. Moreover, the computation of the B-coefficients of s is a local and stable process. In particular, for every domain point $\xi \in \mathcal{D}_{d,\triangle}$, there exists a triangle $T \in \triangle^{(0)}$ and a set*

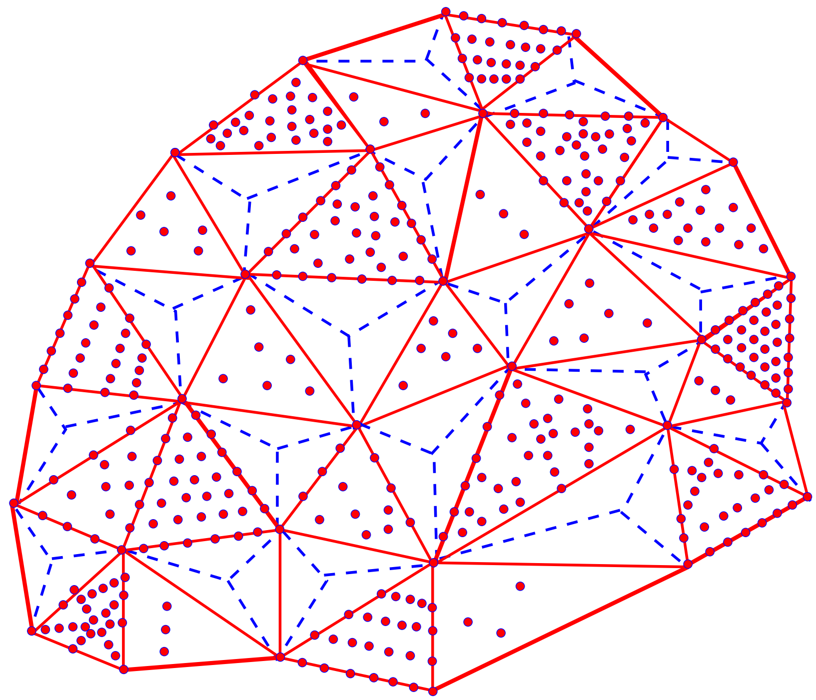

Fig. 5. The set P for Example 6.3, i.e. $m = 1$, $r = 2$ and $(\rho, \mu, d) = (3, 6, 7)$.

$\Gamma_\xi \subset P \cap \mathrm{star}^5(T)$ such that c_ξ depends only on the values of $\{z_\eta\}_{\eta\in\Gamma_\xi}$. In addition, there exists a constant C depending only on the smallest angle in $\triangle^{(0)}$ such that

$$|c_\xi| \le C \max_{\eta\in\Gamma_\xi} |z_\eta|, \tag{6.3}$$

for all $\xi \in \mathcal{D}_{d,\triangle}$.

Proof: Given $\{z_\eta\}_{\eta\in P}$, we show how to uniquely compute the coefficients of s, one triangle at a time, where we go through the triangles $T_1, \ldots, T_{n_0}$ of $\triangle^{(0)}$ in the order defined by Algorithm 4.1. Let $\mathcal{T}_0, \ldots, \mathcal{T}_7$ be the classes of triangles created by the algorithm. We say that a vertex of $\triangle$ is a type-k vertex if it is a vertex of a triangle in $\mathcal{T}_k$, but not a vertex of any triangle in $\mathcal{T}_j$ with $0 \le j < k$. Note that by part 3) of Lemma 4.2, every vertex of $\triangle^{(0)}$ must be of type 0,1,2 or 3.

We first consider triangle $T := T_1$. By Theorem 5.1, the B-coefficients of s restricted to this triangle are uniquely determined by the interpolation conditions at the $\alpha_0 = \binom{d+2}{2}$ points in P_T. In this case $\Gamma_\xi = P_T \subset \mathrm{star}^0(T)$, and (6.3) holds with $C_0 := \|M_0^{-1}\|$, where $M_0 := \{B_\xi^T(\eta)\}_{\xi\in\mathcal{D}_{d,T},\eta\in P_T}$ is the matrix corresponding to interpolation at the points of P_T. Similarly, s is uniquely defined on each of the other triangles in the class $\mathcal{T}_0$, since by Lemma 4.2, two triangles in class $\mathcal{T}_0$ cannot touch each other. At this point in the process, s is uniquely determined on all of the triangles of class $\mathcal{T}_0$. Moreover, since $s \in C^\rho(u)$ for every vertex u of $\triangle^{(0)}$, it follows that all of the B-coefficients c_ξ of s corresponding to domain points ξ in the disks $D_\rho(u)$ are also uniquely determined for all type-0 vertices u of $\triangle^{(0)}$. For these ξ, we have $\Gamma_\xi \subset \text{star}(T)$, and (6.3) holds with a constant $\tilde{C}_0$ depending on C_0 and the smallest angle in $\triangle^{(0)}$.

Now suppose we have completed the computation of c_ξ for all domain points in the triangles $T_1, \ldots, T_{i-1}$, and let $T := T_i \in \mathcal{T}_1$. Then by the definition of $\mathcal{T}_1$, there must be a vertex u of T where T touches at least one of the triangles in $\{T_1, \ldots, T_{i-1}\}$, and does not touch any of these triangles anywhere else. Let T_u be the first such triangle, which by the ordering must be in $\mathcal{T}_0 \cup \mathcal{T}_1$. Statement 3 of Lemma 4.2, implies $T_u \in \mathcal{T}_0$, and thus u must be a type-0 vertex. But then we already know the coefficients c_ξ of s corresponding to domain points $\xi \in A_T := D_\rho^T(u)$. The coefficients corresponding to the remaining domain points in $\mathcal{D}_{d,T}$ are uniquely determined by interpolation at the points of P_T, and are computed by solving a linear system of α_1 equations, where α_1 is as in (5.2). This system can be written in the form $M_1 x = y$, where x is the vector with components $\{c_\eta\}_{\eta \in \mathcal{D}_{d,T} \setminus A_T}$ in lexicographical order, and y is the vector with components $\{z_\eta - \sum_{\xi \in A_T} c_\xi\, B_\xi^d(\eta)\}_{\eta \in P_T}$ in the same order, and where $M_1 := (B_\xi^d(\eta))_{\xi \in \mathcal{D}_{d,T} \setminus A_T, \eta \in P_T}$. It follows that for all $\xi \in \mathcal{D}_{d,T}$, $\Gamma_\xi \subset T \cup T_u \subset \text{star}^1(T)$, and (6.3) holds with $C_1 := \|M_1^{-1}\|(1 + \tilde{C}_0 \|M_0^{-1}\|)$. We emphasize here that T can touch other triangles $\tilde{T}$ in class $\mathcal{T}_1$ at u, but even if it does, the coefficients of s do not depend on $\{z_\eta\}_{\eta \in \tilde{T}}$ since u is a type-0 vertex.

Suppose we have now completed all triangles in $\mathcal{T}_1$. Then for every type-1 vertex v of $\triangle$, we can use the smoothness condition $s \in C^\rho(v)$ to determine the B-coefficients c_ξ of s corresponding to domain points $\xi \in D_\rho(v)$. For these ξ, we have $\Gamma_\xi \subset \text{star}^2(T)$ and (6.3) holds with a constant $\tilde{C}_1$ depending on C_1 and the smallest angle in $\triangle^{(0)}$.

Next, suppose we have completed all triangles $T_1, \ldots, T_{i-1}$, and let $T := T_i \in \mathcal{T}_2$. Then there are two vertices u, v of T where T touches triangles in $\{T_1, \ldots, T_{i-1}\}$, and T does not touch any of these triangles anywhere else. Let T_u and T_v be the first triangles in the ordering which touch T at u and v, respectively. Statement 3 of Lemma 4.2 implies T_u and T_v are in $\mathcal{T}_0 \cup \mathcal{T}_1$, i.e., u and v are vertices of type 0 or 1. But then the B-coefficients of s corresponding to domain points in $A_T := D_\rho^T(u) \cup D_\rho^T(v)$ are already uniquely determined. The coefficients of s corresponding to the remaining domain points in $\mathcal{D}_{d,T}$ are now uniquely determined from the α_2 interpolation conditions at points $\eta \in P_T$. This involves solving a system with matrix $M_2 := (B_\xi^d(\eta))_{\xi \in \mathcal{D}_{d,T} \setminus A_T, \eta \in P_T}$. For these coefficients we also have $\Gamma_\xi \subset \text{star}^2(T)$ and (6.3) holds with a constant C_2 which now also depends on $\|M_2^{-1}\|$. Once we have completed all triangles in $\mathcal{T}_2$, then for all type-2 vertices w, we can use the smoothness condition $s \in C^\rho(w)$ to determine the B-coefficients of s corresponding to domain points $\xi \in D_\rho(w)$. For these ξ, we have $\Gamma_\xi \subset \text{star}^3(T)$ and (6.3) holds with a constant $\tilde{C}_2$ depending on C_2 and the smallest angle in $\triangle^{(0)}$.

Suppose we have completed all triangles $T_1, \ldots, T_{i-1}$, and let $T := T_i \in \mathcal{T}_3$. Then T shares an edge $e := \langle u, v \rangle$ with some triangle in $\{T_1, \ldots, T_{i-1}\}$ and does not touch any triangles in $\{T_1, \ldots, T_{i-1}\}$ except at points on this edge. Let $\tilde{T}$ be the first triangle sharing e. By statement 2 of Lemma 4.2, $\tilde{T} \in \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$, and thus u and v must be vertices of type 0,1, or 2. (In fact, an argument similar to the proof of statement 4) in Lemma 4.2 shows that u and v must be vertices of type 0 or 1, see the triangle labeled with a 3 in Fig. 6). This means that the coefficients c_ξ corresponding to $\xi \in D_\rho^T(u) \cup D_\rho^T(v)$ are already uniquely determined. Now using the C^r smoothness across e, we can uniquely determine the coefficients of s corresponding to the set $E_r^T(e)$ from coefficients of $s|_{\tilde{T}}$. It follows that for these ξ, $\Gamma_\xi \subset \operatorname{star}^2(T)$ and (6.3) holds. We then uniquely determine the coefficients corresponding to the remaining domain points in $\mathcal{D}_{d,T}$ by solving the linear system corresponding to interpolation at the α_3 points of P_T, and we again have $\Gamma_\xi \subset \operatorname{star}^2(T)$. For these coefficients (6.3) holds with a constant C_3 now depending also on $\|M_3^{-1}\|$, where $M_3 := (B_\xi^d(\eta))_{\xi \in \mathcal{D}_{d,T} \setminus A_T, \eta \in P_T}$ with $A_T := D_\rho^T(u) \cup D_\rho^T(v) \cup E_r^T(e)$. Once we have completed all triangles in $\mathcal{T}_3$, then for all type-3 vertices w we can use the smoothness condition $s \in C^\rho(w)$ to determine the B-coefficients of s corresponding to domain points $\xi \in D_\rho(w)$. For these ξ, we have $\Gamma_\xi \subset \operatorname{star}^3(T)$ and (6.3) holds with a constant $\tilde{C}_3$ depending on C_3 and the smallest angle in $\triangle$. At this point we have uniquely determined all coefficients corresponding to domain points in $D_\rho(u)$ for all vertices u of $\triangle^{(0)}$.

Now suppose we have completed the triangles $T_1, \ldots, T_{i-1}$, and let $T := T_i \in \mathcal{T}_4$. Then $T := \langle u, v, w \rangle$ can touch other triangles in $\{T_1, \ldots, T_{i-1}\}$ only at its vertices, and must touch at least one such triangle at each vertex. Let T_u, T_v, T_w be the first triangles touching at u, v, w, respectively. By statement 4 of Lemma 4.2, these vertices must be of type 0,1, or 2, and thus the coefficients c_ξ for $\xi \in D_\rho^T(u)$ are already uniquely determined. The coefficients c_ξ corresponding to the remaining domain points in $\mathcal{D}_{d,T}$ are then uniquely determined from the interpolation conditions at the points of P_T. It follows that for all $\xi \in \mathcal{D}_{d,T}$, $\Gamma_\xi \subset \operatorname{star}^3(T)$ and (6.3) holds with a constant C_4 that now also depends on $\|M_4^{-1}\|$, where $M_4 := (B_\xi^d(\eta))_{\xi \in \mathcal{D}_{d,T} \setminus A_T, \eta \in P_T}$ with $A_T := D_\rho^T(u) \cup D_\rho^T(v) \cup D_\rho^T(w)$.

Once we have finished with all triangles in $\mathcal{T}_4$, we can deal with the triangles in $\mathcal{T}_5$ in the same way, except now if $T := T_i \in \mathcal{T}_5$, then it shares some edge e with a completed triangle $\tilde{T}$ which must be in $\mathcal{T}_0, \ldots, \mathcal{T}_4$. Then the coefficients c_ξ corresponding to $\xi \in E_r^T(e)$ are uniquely determined from coefficients of $s|_{\tilde{T}}$, and we have $\Gamma_\xi \subset \operatorname{star}^4(T)$, and (6.3) holds for all $\xi \in \mathcal{D}_{d,T}$ with a constant C_5 that now also depends on $\|M_5^{-1}\|$, where $M_5 := (B_\xi^d(\eta))_{\xi \in \mathcal{D}_{d,T} \setminus A_T, \eta \in P_T}$ with $A_T := D_\rho^T(u) \cup D_\rho^T(v) \cup D_\rho^T(w) \cup E_r^T(e)$.

To complete the proof, we note that if $T \in \mathcal{T}_6$, then by Lemma 3.2 the coefficients c_ξ of s corresponding to $\xi \in \mathcal{D}_{d,T_{CT}}$ are uniquely determined from coefficients in neighboring completed triangles lying in $\mathcal{T}_0 \cup \cdots \cup \mathcal{T}_5$. It follows that for these ξ, $\Gamma_\xi \subset \operatorname{star}^4(T)$, and (6.3) holds with a constant C_6 depending on C_5 and the smallest angle in $\triangle^{(0)}$. Similarly, if $T \in \mathcal{T}_7$, then by Lemma 3.1 the coefficients c_ξ of s corresponding to $\xi \in \mathcal{D}_{d,T_{CT}}$ are uniquely determined from coefficients in neighboring completed triangles lying in $\mathcal{T}_0 \cup \cdots \cup \mathcal{T}_6$. It follows that for these ξ, $\Gamma_\xi \subset \operatorname{star}^5(T)$, and (6.3) holds with a constant C_7 depending on C_6 and the smallest angle in $\triangle^{(0)}$. $\square$

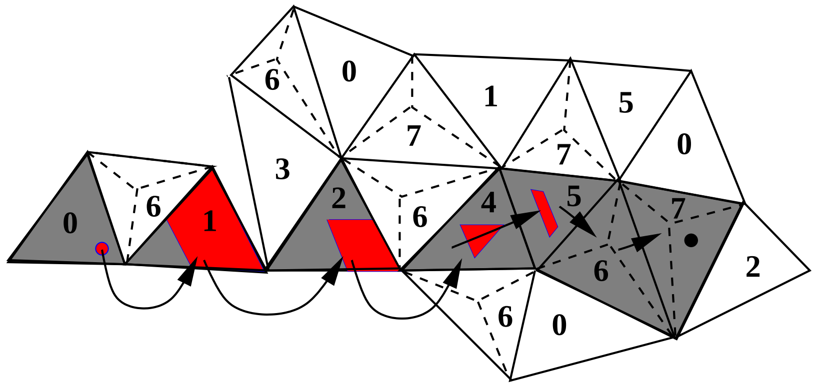

Fig. 6. An example where Γ_ξ is contained in $\mathrm{star}^5(T)$ but not in $\mathrm{star}^4(T)$.

The proof of Theorem 6.4 shows that the dimension of $\mathcal{S}$ is equal to the cardinality of the set P, which is given by the formula in (6.2). The theorem shows that in the worst case, a coefficient c_ξ with ξ in triangle T of $\triangle^{(0)}$ depends only on data in a set Γ_ξ which is contained in $\mathrm{star}^5(T)$. In Fig. 6 we give an example to show that this worst case can occur. The numbers in the triangles indicate the classes to which they belong. Suppose ξ is the point marked with a black dot in the triangle in class $\mathcal{T}_7$ on the far right. We claim that the value of c_ξ depends on the value of z_η, where η is a point in P (marked with a circle) lying in the triangle on the far left. The arrows indicate the direction of propagation, and the triangles in the chain of influence are shown in grey. The points in P in these triangles are symbolized with a darker shade of grey.

As is clear from the proof of Theorem 6.4, the worst case of $\mathrm{star}^5(T)$ only appears in very particular constellations, and for most ξ, the set Γ_ξ is much smaller. For instance, it is easy to see that in Examples 6.2 and 6.3 the worst case is $\mathrm{star}^3(T)$. We illustrate this in Fig. 7, where the longest chain of influence is the set of triangles colored grey.

§7. Bounds on the Error of Interpolation

Given a set $\mathcal{V}$ of points in a planar domain Ω, let $\triangle^{(0)}$ be some initial triangulation of Ω with vertices at the points of $\mathcal{V}$. Suppose P and $\mathcal{S}$ form a Lagrange interpolation pair as in Definition 6.1, where $\mathcal{S}$ is defined on the refined triangulation $\triangle$ obtained from the triangulation $\triangle^{(0)}$ by splitting certain triangles as in the definition. Then for every $f \in C(\Omega)$, there is a unique spline $\mathcal{I}f \in \mathcal{S}$ such that

$$\mathcal{I}f(\eta) = f(\eta), \qquad \eta \in P.$$

Clearly, this defines a linear projector $\mathcal{I}$ mapping $C(\Omega)$ onto $\mathcal{S}$. We now give an error bound for $f - \mathcal{I}f$ and its partial derivatives $D_x^\alpha D_y^\beta$ in the infinity norm. Similar results hold for general p-norms.

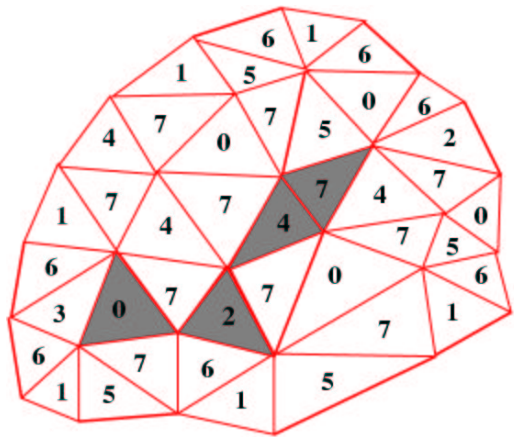

Fig. 7. A maximal chain for Examples 6.2 and 6.3.

Theorem 7.1. *Suppose f lies in the Sobolev space $W_\infty^{m+1}(\Omega)$ for some $0 \le m \le d$. Then*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_\Omega \le K h^{m+1-\alpha-\beta} |f|_{m+1,\Omega}, \tag{7.1}$$

for $0 \le \alpha + \beta \le m$. Here $|\cdot|_{m+1,\Omega}$ is the usual Sobolev semi-norm, and h is the mesh size of $\triangle^{(0)}$. The constant K depends only on the smallest angle in $\triangle^{(0)}$.

Proof: Fix $0 \le m \le d$, and let $f \in W_\infty^{m+1}(\Omega)$. Let T be some triangle in $\triangle$, and let $\Omega_T := \text{star}^5(\tilde{T})$, where $\tilde{T} \in \triangle^{(0)}$ with $T \subseteq \tilde{T}$. Note that $T = \tilde{T}$ if and only if $T \in \mathcal{T}_i$, $i = 0, \ldots, 5$. Then it is well known, cf. Lemma 4.6 in [6] or Lemma 4.3.8 in [3], that there exists a polynomial $q := q_f \in \mathcal{P}_m$ such that

$$\|D_x^\alpha D_y^\beta (f - q)\|_{\Omega_T} \le K_0 \; |\Omega_T|^{m+1-\alpha-\beta} |f|_{m+1,\Omega_T}, \tag{7.2}$$

for $0 \le \alpha + \beta \le m$, where K_0 is an absolute constant and $|\Omega_T|$ is the diameter of Ω_T. Now

$$|\Omega_T| \le 11 \; h, \tag{7.3}$$

implies

$$\|D_x^\alpha D_y^\beta (f - q)\|_{\Omega_T} \le K_1 \; h^{m+1-\alpha-\beta} |f|_{m+1,\Omega_T}, \tag{7.4}$$

with $K_1 := 11^{m+1} K_0$. Since $\mathcal{I}q = q$, we have

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_T \le \|D_x^\alpha D_y^\beta (f - q)\|_T + \|D_x^\alpha D_y^\beta \mathcal{I}(f - q)\|_T.$$

In view of (7.4), it suffices to estimate the second term. By the Markov inequality [20],

$$\|D_x^\alpha D_y^\beta \mathcal{I}(f - q)\|_T \le K_2 \; h^{-(\alpha+\beta)} \|\mathcal{I}(f - q)\|_T, \tag{7.5}$$

where again K_2 depends on the smallest angle in T.

Now suppose $\mathcal{I}(f - q)|_T = \sum_{\xi \in \mathcal{D}_{d,T}} c_\xi B_\xi^d$. Since the Bernstein basis polynomials form a partition of unity, this implies

$$\|\mathcal{I}(f - q)\|_T \le \max_{\xi \in \mathcal{D}_{d,T}} |c_\xi|.$$

But by Theorem 6.4,

$$|c_\xi| \le C \max_{\eta \in P\cap\Omega_T} |(f-q)(\eta)| \le C\,\|f-q\|_{\Omega_T}, \qquad \xi \in \mathcal{D}_{d,T},$$

where C is a constant depending only on the smallest angle in $\triangle^{(0)}$. Combining the above inequalities leads immediately to

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_T \le K_3 h^{m+1-\alpha-\beta} |f|_{m+1,\Omega_T}, \tag{7.6}$$

and taking the maximum over all T in $\triangle$ gives (7.1) with a constant $K > 0$ that depends only on the smallest angle in $\triangle^{(0)}$. □

§8. Constrained Interpolation with B-Polynomials

In this section we discuss interpolation with bivariate polynomials where certain of the B-coefficients are set in advance. The results here provide a proof of Theorem 5.1. First we state a simple lemma.

Lemma 8.1. *Let $p = wq$, where $p \in \mathcal{P}_d$, $q \in \mathcal{P}_{d-1}$, and $w \in \mathcal{P}_1$ is non-constant. Let $u \ne 0$ be an arbitrary vector which does not point in the direction of the line $W := \{(x,y) : w(x,y) = 0\}$, and let D_u be the associated directional derivative. Let v be some point in $\mathbb{R}^2$ and let $m \ge 0$.*

1) *If $w(v) = 0$, then $D_u^i p(v) = 0$ for $i = 0,\ldots,m$ implies $D_u^i q(v) = 0$ for $i = 0,\ldots,m-1$.*
2) *If $w(v) \ne 0$, then $D_u^i p(v) = 0$ for $i = 0,\ldots,m$ implies $D_u^i q(v) = 0$ for $i = 0,\ldots,m$.*

Proof: For any v, $D_u w(v) \ne 0$ and $D_u^j w(v) = 0$ for all $j \ge 2$. But then $D_u^i p(v) = w(v) D_u^i q(v) + i D_u w(v) D_u^{i-1} q(v)$ for all $i \ge 1$. This immediately implies 1). Statement 2) follows by a simple induction. □

We are now ready to discuss the interpolation problem of interest here. Suppose $T := \langle v_1, v_2, v_3\rangle$ and define $e_i := \langle v_i, v_{i+1}\rangle$ for $i = 1,2,3$, where we identify $v_4 = v_1$. In order to state a general result, we define $D_{\rho_i}^T(v_i)$ to be empty if $\rho_i = -1$. Similarly, we take the set $E_{r_i}^T(e_i)$ to be empty if $r_i = -1$.

Theorem 8.2. *Let $r_i, \rho_i \ge -1$ for $i = 1,2,3$. Suppose we are given the B-coefficients c_ξ of a polynomial p corresponding to domain points ξ in the set*

$$\Gamma := \mathcal{D}_{d,T} \cap \bigcup_{i=1}^{3} \left[D_{\rho_i}^T(v_i) \cup E_{r_i}^T(e_i) \right],$$

and $\mathcal{D}_{d,T} \setminus \Gamma \ne \emptyset$. Then we can explicitly choose $n = \binom{d+2}{2} - \#\Gamma$ points $t_1,\ldots,t_n$ in the interior of T such that the remaining B-coefficients of p are uniquely determined by the values $\{p(t_i)\}_{i=1}^n$.

Proof: It is enough to show how to choose n points $t_1,\ldots,t_n$ in the interior of T such that if we set the B-coefficients c_ξ of p to zero for all $\xi \in \Gamma$, then $p(t_i) = 0$ for

$i = 1, \ldots, n$ implies $p \equiv 0$. We proceed by induction on d. The result is trivial for polynomials of degree $d = 0$. We now establish the result for polynomials of degree d, assuming it holds for polynomials of degree $d-1$. There are three cases.

Case 1: Some $r_i \geq 0$. Without loss of generality we may assume that $r_1 \geq r_2, r_3$. Suppose we set the B-coefficients c_ξ of p to zero for all $\xi \in \Gamma$. This implies that for $i = 1, 2, 3$, all derivatives of p at the vertex v_i vanish up to order ρ_i, and the cross derivatives of p associated with the edge e_i also vanish identically for $i = 0, \ldots, r_i$. Since $r_1 \geq 0$, this implies that p vanishes identically on the edge e_1. It follows from Bezout's theorem that $p = wq$, where q is a polynomial of degree $d-1$ and $w \neq 0$ is a linear polynomial which vanishes on e_1. By Lemma 8.1, the derivatives of q up to order ρ_3 vanish at v_3 and the derivatives of q up to order $\rho_i - 1$ vanish at v_i for $i = 1, 2$. Moreover, the cross derivatives of q associated with the edge e_1 vanish up to order $r_1 - 1$, while those associated with the edges e_2 and e_3 vanish up to order r_2 and r_3, respectively. It follows that all B-coefficients of q corresponding to domain points in $\widetilde{\Gamma} := \mathcal{D}_{d-1,T} \cap \left[D^T_{\rho_1-1}(v_1) \cup D^T_{\rho_2-1}(v_2) \cup D^T_{\rho_3}(v_3) \cup E^T_{r_1-1}(e_1) \cup E^T_{r_2}(e_2) \cup E^T_{r_3}(e_3)\right]$ are zero. Note that $\#\widetilde{\Gamma} = \#\Gamma - d - 1$. Now by the inductive hypothesis, there exist $\binom{d+1}{2} - \#\widetilde{\Gamma} = n$ points in the interior of T such that $q(t_i) = 0$ for $i = 1, \ldots, n$ implies $q \equiv 0$. But $p(t_i) = 0$ for $i = 1, \ldots, n$ implies $q(t_i) = 0$ for $i = 1, \ldots, n$, and we conclude that $p \equiv 0$.

Case 2: $r_1 = r_2 = r_3 = -1$ and some $\rho_i \geq 0$. Without loss of generality, we may assume $\rho_1 \geq \rho_2, \rho_3$. The assumption on the r_i implies $\rho_i + \rho_{i+1} < d - 1$ for $i = 1, 2, 3$. Let $\widetilde{\Gamma} := \mathcal{D}_{d-1,T} \cap \left[D^T_{\rho_1-1}(v_1) \cup D^T_{\rho_2}(v_2) \cup D^T_{\rho_3}(v_2)\right]$. Note that $\#\widetilde{\Gamma} = \#\Gamma - d + \rho_1$. By the inductive hypothesis, we can choose $m = \binom{d+1}{2} - \#\widetilde{\Gamma} = n - d + \rho_1$ points $t_1, \ldots, t_m$ in T, so that if q is a polynomial of degree $d-1$ whose B-coefficients corresponding to domain points in $\widetilde{\Gamma}$ vanish, then $q(t_i) = 0$ for $i = 1, \ldots, m$ implies q is identically zero. Now choose any line W passing through v_1 that does not pass through v_2 or v_3, or any of the points $t_1, \ldots, t_m$, and choose any $d - \rho_1$ points $t_{m+1}, \ldots, t_n$ on W in the interior of T. Suppose $p \in \mathcal{P}_d$ is such that its B-coefficients corresponding to domain points in Γ vanish and $p(t_i) = 0$ for $i = 1, \ldots, n$. To complete the proof, it suffices to show that $p \equiv 0$. Clearly, $g := p|_W$ is a univariate polynomial of degree d with the property $D^i_{\bar{w}} g(v_1) = 0$ for $i = 0, \ldots, \rho_1$, where $\bar{w} \neq 0$ is an arbitrary vector which points in direction of W. Since $g(t_i) = 0$ for $i = m+1, \ldots, n$, p vanishes on W, and by Bezout's theorem, we can write $p = wq$ where $q \in \mathcal{P}_{d-1}$ and $w \neq 0$ is a linear polynomial that vanishes identically on the line W. By Lemma 8.1, the derivatives of q up to order $\rho_1 - 1$ vanish at v_1 while the derivatives up to order ρ_i vanish at v_i for $i = 1, 2$. This implies that the B-coefficients of q corresponding to domain points in $\widetilde{\Gamma}$ vanish. Since $p(t_i) = 0$ implies $q(t_i) = 0$ for $i = 1, \ldots, m$, it follows that q and thus also p must be identically zero.

Case 3: $r_i = \rho_i = -1$ for $i = 1, 2, 3$. In this case Γ is empty and there are no constraints on the coefficients of p. By the inductive hypothesis, we can choose $m = \binom{d+1}{2} = n - d - 1$ points $t_1, \ldots, t_m$ in the interior T such that $q(t_i) = 0$ for $i = 1, \ldots, m$ implies q is identically zero. Now choose any line W that does not pass through any of the points $t_1, \ldots, t_m$, and choose any $d+1$ points $t_{m+1}, \ldots, t_n$ on W

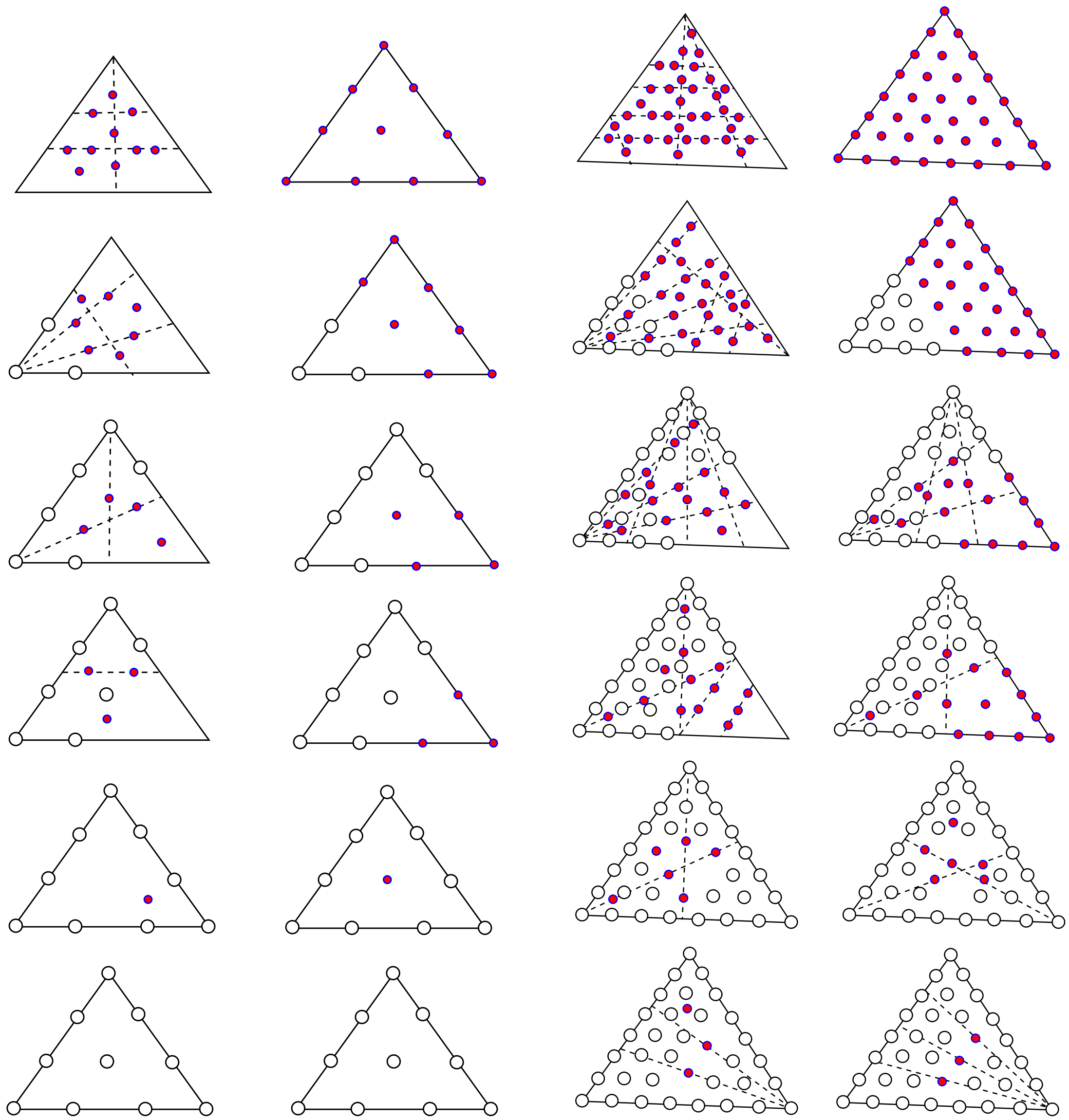

Fig. 8. Typical point sets in Theorem 8.2 for $d = 3$ and $d = 7$.

in the interior of T. Suppose $p \in \mathcal{P}_d$ is such that $p(t_i) = 0$ for $i = 1, \ldots, n$. Clearly, $g := p|_W$ is a univariate polynomial of degree d. Since $g(t_i) = 0$ for $i = m+1, \ldots, n$, p vanishes on W, and by Bezout's theorem, we can write $p = wq$ where $q \in \mathcal{P}_{d-1}$ and $w \neq 0$ is a linear polynomial that vanishes identically on the line W. Since $p(t_i) = 0$ implies $q(t_i) = 0$ for $i = 1, \ldots, m$, it follows that q and thus also p must be identically zero. $\square$

We note that the points $t_1, \ldots, t_n$ of Theorem 8.2 can be defined in terms of barycentric coordinates, which means that their relative positions are the same in every triangle, regardless of its size or shape. Moreover, a slight modification of the arguments given in the proof of Theorem 8.2 show that if the set $\bigcup_{i=1}^{3} E_0^T(e_i) \setminus \Gamma$ is non-empty, then the points from this set (lying on the boundary edges of

T) can be chosen as a subset of $\{t_1, \ldots, t_n\}$. In particular, this guarantees that we can interpolate at the given set of points $\mathcal{V}$ (see Problem 1.1). Examples of point constellations obtained from the inductive process described in the proof of Theorem 8.2 as well as for this modification are shown in Fig. 8. In this figure, the interpolation points are shown as small filled circles, while B-coefficients which are set in advance are shown as larger open circles. The two columns to the left show the configurations for cubic polyomials, while the two columns to the right show examples of chosen points for polyomials of degree seven. This figure shows all point configurations P_T, $T \in \mathcal{T}_i$, $i = 0, \ldots, 5$ needed to construct sets P which can be used with the C^1 and C^2 spline spaces $\mathcal{S}$ described in Theorem 5.1. The example also shows that the interpolation points do not necessarily have to be located at domain points.

§9. Remarks

Remark 9.1. Lagrange interpolation with C^1 splines was investigated in [9–11,13–16] using splines of various degrees on either triangulations or triangulated quadrangulations. All of these results depended on certain colorings of the triangulations or quadrangulations, and in [4,16] the approximation order was established with the help of weak interpolation techniques rather than the direct approach used here. Lagrange interpolation with C^2 splines was treated recently in [8], and was also based on a certain coloring algorithm. The idea of extending local polynomial pieces to splines using smoothness conditions has been used before in scattered data fitting, for example in [5] in the bivariate case and in [19] for the trivariate case.

Remark 9.2. For given smoothness r, it is possible to modify our construction by using polynomials of degree $1 \le m < d$ for triangles of class $\mathcal{T}_0$. In this case we would need only $\binom{m+2}{2}$ interpolation points in each such triangle. We then apply degree-raising to turn each such polynomial into one of degree d, and proceed as before. For example, using linear polynomials, we would not need to introduce any additional interpolation points in the triangles of class $\mathcal{T}_0$. However, using polynomials of degree m, we can only expect order $m + 1$ approximation rather than order $d + 1$.

Remark 9.3. Local Lagrange interpolation methods are useful for the construction and reconstruction of surfaces and for scattered data fitting problems. A major advantage is that they do not require knowing or approximating values of derivatives, i.e. only Lagrange data is needed. One way to use them would be in a two-stage process, where in the first stage one constructs a C^0 linear spline based on a very fine triangulation, which in turn is interpolated by our C^r method on a coarser triangulation. See [10,14,16] where numerical results based on this idea are discussed.

Remark 9.4. The Bernstein-Bézier representation used here is not only a theoretical tool, but is also of practical importance, since all of the computations needed to construct a spline can be done directly with the Bernstein-Bézier-coefficients. In particular, there is no need to construct basis functions.

Remark 9.5. As noted above, the matrices which appear in the various linear systems arising in the computation of our interpolating spline do not depend on the size or shape of triangles in the triangulation. This means that there are only a small number of fixed matrices which can be precomputed and inverted once and for all.

Remark 9.6. Given a Lagrange interpolation pair $P, \mathcal{S}$, it is clear that for each $\xi \in P$, there exists a unique spline L_ξ such that

$$L_\xi(\eta) = \delta_{\xi,\eta}, \qquad \eta \in P.$$

These are the fundamental splines or cardinal splines associated with P. Following the arguments in the proof of Theorem 6.4, it can be seen that for all $\xi \in P \cap T$, the support of L_ξ is contained in $\mathrm{star}^5(T)$, see Fig. 6.

Remark 9.7. Suppose P and $\mathcal{S}$ are as in Theorem 6.4. Then clearly P describes a nodal minimal determining set for $\mathcal{S}$ in the sense that setting the values z_η for $\eta \in P$ uniquely determines a spline in $\mathcal{S}$. If $P \subseteq \mathcal{D}_{d,\triangle}$, then the set P is also a minimal determining set in the classical sense that if we set all B-coefficients of s corresponding to the points of P, then s is uniquely determined. This means that for each point $\xi \in P$, there exists a unique spline $\tilde{L}_\xi$ such that $c_\xi = 1$ and $c_\eta = 0$ for all $\eta \in P \setminus \{\xi\}$. These basis functions are different from the basis functions in Remark 9.6, and in general have smaller supports. In fact, it can be seen from similar (but much simpler) arguments that the maximal support of $\tilde{L}_\xi$ is contained in star^2 (where we measure the supports as in Theorem 6.4). For C^1 cubic splines and C^2 splines of degree seven, it is possible to choose $P \subseteq \mathcal{D}_{d,\triangle}$, which follows from the constructions given in [8,10,16].

Remark 9.8. In [10,15] the concept of separable quadrangulations and triangulations was introduced. If we apply the algorithm from [15] to construct a separable triangulation $\triangle_{sep}$ from $\triangle^{(0)}$ first, and then apply Algorithm 4.1, we obtain only five classes of triangles, namely $\mathcal{T}_i,\ i \in \{0,4,5,6,7\}$. In this case the support of the fundamental splines L_ξ from Remark 9.6 is contained in $\mathrm{star}^3(T)$, where $T \in \triangle_{sep}$.

Remark 9.9. It has been conjectured [18] that Theorem 8.2 holds if we choose $\{t_1,\dots,t_n\} = \mathcal{D}_{d,T} \setminus \Gamma$. This has been verified for all $d \le 7$, see [8].

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