On the Approximation Order of Splines on Spherical Triangulations

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Abstract. Bounds are provided on how well functions in Sobolev spaces on the sphere can be approximated by spherical splines, where a spherical spline of degree \(d\) is a \(C^d\) function whose pieces are the restrictions of homogeneous polynomials of degree \(d\) to the sphere. The bounds are expressed in terms of appropriate seminorms defined with the help of radial projection, and are obtained using appropriate quasi-interpolation operators.

§1. Introduction

In a series of papers [2–4], together with P. Alfeld we have developed a theory of spherical splines on triangulations of the sphere in \(\mathbb{R}^3\). Such splines closely resemble the classical piecewise polynomials on planar triangulations. Whereas splines in the plane are ordinary piecewise polynomials, spherical splines are formed by piecing together so-called spherical polynomials which are defined as the restrictions of homogeneous trivariate polynomials to the sphere. Since spherical polynomials are combinations of spherical harmonics, they are good candidates for constructing spaces of splines on the sphere.

In [4] we presented a variety of methods for fitting scattered data on the sphere using spherical splines. These included analogs of several classical bivariate methods, such as interpolation, least squares, and minimum-energy methods. That paper also presented the results of extensive numerical experimentation, and confirmed our expectations that spherical splines perform very well numerically. However, in contrast to the planar case, to date no theory has been developed to justify these experimental findings.

In this paper we fill this gap by providing bounds on the error of approximation of smooth functions by spherical splines. As in the planar case, the error bounds will be expressed in terms of the mesh size (the diameter of the largest triangle in a given triangulation), and also in terms of appropriate seminorms measuring the smoothness of the approximated functions. The error estimates will show that for spherical splines of sufficiently high degree \(d\), namely \(d \geq 3r + 2\), where \(r\) is the

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degree of global smoothness of the spline, the approximation order of these splines is the same as that of their planar counterparts.

To measure the norm of a spherical function (a function defined on the unit sphere $S$) and its smoothness, appropriate analogs of Sobolev smoothness classes on the sphere will be needed. Such Sobolev classes have been known for many years. However, it has not been clear, to us at least, how to define appropriate spherical analogs of Sobolev seminorms. In fact, this has been the main obstacle in our efforts to prove approximation rates for spherical splines. While it is not difficult to define seminorms that annihilate spherical harmonics (spherical polynomials) of a given degree on the whole sphere, for example written in terms of the Laplace-Beltrami operator, such seminorms are not adequate when working on a subdomain of the sphere. Sobolev-type seminorms are needed, among other things, to be able to establish Whitney-type theorems on the sphere, expressing how well a function can be approximated locally by polynomials (see Sect. 4 below). A theorem of this type was established recently in [13], but the error bound failed to provide an explicit dependence on the smoothness of the function being approximated. In our approach, we utilize the intimate connection between spherical polynomials and homogeneous polynomials, along with the fact that the spherical Sobolev spaces can be obtained as restrictions to the sphere of Sobolev spaces of homogeneous functions.

Given a nonnegative integer $d$, we write $\Pi_d$ for the space of trivariate homogeneous polynomials of degree $d$. We write $\Pi_d(\Omega)$ for the restriction of $\Pi_d$ to any subset $\Omega$ of the unit sphere $S$, and refer to it as the space of spherical polynomials of degree $d$. Similarly, we write $\Pi_d(H)$ for the restriction of $\Pi_d$ to any hyperplane $H$ in $\mathbb{R}^3$. This is just the usual space of bivariate polynomials. All of these polynomial spaces have the same dimension, namely $\binom{d+2}{2}$. Let $\Delta$ be a spherical triangulation consisting of a finite collection of spherical triangles (i.e., triangles whose edges are segments of great circles) whose union is $S$, and such that each pair of triangles in $\Delta$ are either disjoint or share a common vertex or an edge. Given integers $d$ and $r$, we define the space of spherical splines of degree $d$ and smoothness $r$ associated with $\Delta$ to be

$$S^r_d(\Delta) := \{s \in C^r(S) : s|_T \in \Pi_d(T), \ T \in \Delta\}; \quad (1.1)$$

where $s|_T$ denotes the restriction of $s$ to $T$.

Given a set $\Omega \subset S$, we define its diameter to be

$$\text{diam}(\Omega) := \sup\{\arccos(u \cdot v), \ u, v \in \Omega\}.$$ 

By $|\Delta|$, we denote the mesh size of $\Delta$, i.e., the diameter of the largest triangle in $\Delta$. We can now state the main result of the paper (to be proved in Sect. 5 below).

**Theorem 1.1.** Let $d \geq 3r+2$ and $1 \leq p \leq \infty$. Then there exists a spline $s \in S^r_d(\Delta)$ and a constant $C$, depending only on $d, p$, and the smallest angle in $\Delta$, such that

$$|f - s|_{k, p, S} \leq C|\Delta|^{d+1-k}|f|_{d+1, p, S},$$

where $k$ is some integer.
for all \( f \in W^{d+1,p}(S) \) and all \( 0 \leq k \leq d \) such that \( s \in W^{k,p}(S) \).

Here \( W^{t,p}(S) \) denote Sobolev spaces of spherical functions and \( | \cdot |_{t,p,S} \) are associated Sobolev-type seminorms to be introduced in Sect. 3.

\section*{§2. Radial Projection}

To get approximation results for spherical splines, we shall make use of known results for bivariate splines, along with a natural radial projection operator that we now define. Suppose \( \Omega \) is a subset of \( S \) with \( \text{diam}(\Omega) \leq 1 \). Then we define \( r_{\Omega} \) to be the center of a spherical cap of smallest possible radius containing \( \Omega \), and let \( T_{\Omega} \) be the tangent plane touching \( S \) at \( r_{\Omega} \).

We now define the radial projection from \( T_{\Omega} \) into \( S \) by

\[
R_{\Omega} \tilde{\omega} := \omega := \frac{\tilde{\omega}}{|\tilde{\omega}|} \in S, \quad \tilde{\omega} \in T_{\Omega}.
\]

Clearly, \( R_{\Omega} \) is one-to-one, and hence \( R_{\Omega}^{-1} \) is well defined. Let \( \bar{\Omega} \) be the image of \( \Omega \) under \( R_{\Omega}^{-1} \).

For later use we note that the diameters of \( \Omega \) and \( \bar{\Omega} \) are comparable. In particular, it is not hard to see that

\[
A_{1}^{-1} \text{diam}(\Omega) \leq \text{diam}(\bar{\Omega}) \leq A_{1} \text{diam}(\Omega),
\]

where \( A_{1} := 2\tan(1/2) \). Similarly, it is easy to see that there exists a constant \( 1 \leq A_{2} < \infty \) such that

\[
A_{2}^{-1} \rho_{\Omega} \leq \rho_{\bar{\Omega}} \leq A_{2} \rho_{\Omega},
\]

where \( \rho_{\Omega} \) is the diameter of the largest spherical cap contained in \( \Omega \) and \( \rho_{\bar{\Omega}} \) is the diameter of the largest disk contained in \( \bar{\Omega} \).

It is also important to observe that if \( T \) is a spherical triangle, then since great circles are mapped into straight lines under \( R_{\Omega}^{-1} \), any spherical triangulation lying in \( \Omega \) will be mapped into a planar triangulation in \( \bar{\Omega} \).

\section*{§3. Spherical Sobolev Spaces and Seminorms}

Throughout the remainder of the paper we fix \( 1 \leq p \leq \infty \). Suppose \( B \) is an open set in \( \mathbb{R}^{2} \) and that \( k \) is a nonnegative integer. Then the corresponding classical Sobolev space \( W^{k,p}(B) \) is just the space of functions on \( B \) whose derivatives up to order \( k \) belong to \( L_{p}(B) \), see e.g. [1]. A norm on \( W^{k,p}(B) \) can be defined as

\[
\| g \|_{W^{k,p}(B)} := \sum_{\gamma_{1} + \gamma_{2} \leq k} \| D_{\xi}^{\gamma_{1}} D_{\eta}^{\gamma_{2}} g \|_{L_{p}(B)},
\]

where \( D_{\xi}^{\gamma_{1}} D_{\eta}^{\gamma_{2}} = \partial^{\gamma_{1} + \gamma_{2}} / \partial \xi^{\gamma_{1}} \partial \eta^{\gamma_{2}} \).

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Our aim in this section is to define analogous spherical Sobolev spaces defined on open sets $\Omega \subset S$, and to construct corresponding seminorms which annihilate spherical polynomials. To get started, suppose that $\{(\Gamma_j, \phi_j)\}$ is an atlas for $\Omega$, i.e., a finite collection of charts $(\Gamma_j, \phi_j)$, where $\Gamma_j$ are open subsets of $\Omega$, covering $\Omega$, and where $\phi_j$ are infinitely differentiable mappings $\phi_j : \Gamma_j \rightarrow B_j$, $B_j$ an open subset of $\mathbb{R}^2$, whose inverses $\phi_j^{-1}$ are also infinitely differentiable. Also, let $\{\alpha_j\}$ be a partition of unity subordinated to the atlas $\{(\Gamma_j, \phi_j)\}$, i.e., a set of infinitely differentiable functions $\alpha_j$ on $\Omega$ vanishing outside the sets $\Gamma_j$, such that $\sum_j \alpha_j = 1$ on $\Omega$.

We now define spherical Sobolev spaces $W^{k,p}(\Omega)$ as follows:

$$W^{k,p}(\Omega) := \{ f : (\alpha_j f) \circ \phi_j^{-1} \in W^{k,p}(B_j), \text{ for all } j \}. \quad (3.2)$$

A norm on $W^{k,p}(\Omega)$ can be defined as

$$\|f\|_{W^{k,p}(\Omega)} := \sum_j \| (\alpha_j f) \circ \phi_j^{-1} \|_{W^{k,p}(B_j)}. \quad (3.3)$$

Then the Sobolev space $W^{k,p}(\Omega)$ is just the space of all functions $f$ defined on $\Omega$ for which $\|f\|_{W^{k,p}(\Omega)}$ is finite. It is well known [5,14] that this definition does not depend on the choice of the atlas and the partition of unity, in the sense that other choices will give rise to the same space with a norm that is equivalent to (3.3). For other ways to define a norm on $W^{k,p}(\Omega)$, see Remark 6.1 in Sect. 6.

We now turn to the problem of defining seminorms for the spaces $W^{k,p}(\Omega)$. In analogy with the Euclidean case where the Sobolev seminorms annihilate algebraic polynomials, we want to construct seminorms that annihilate spherical polynomials. When considering functions on the entire sphere $S$, i.e. when $\Omega = S$, then one possibility is to define a seminorm that annihilates all spherical polynomials of a given degree $d$ in terms of the well-known Laplace-Beltrami operator $\Delta^s$. Namely, the following operator, used extensively in [10,16], has $\Pi_d(S)$ as its null space:

$$\Delta_d := \begin{cases} (\Delta^s + \lambda_0)(\Delta^s + \lambda_2) \cdots (\Delta^s + \lambda_d), & d \text{ even,} \\ (\Delta^s + \lambda_1)(\Delta^s + \lambda_3) \cdots (\Delta^s + \lambda_d), & d \text{ odd,} \end{cases} \quad (3.4)$$

where $\lambda_k := k(k + 1), k \in \mathbb{Z}_+$. Unfortunately, $\Delta_d$ is not suitable for working with functions on domains $\Omega$ smaller than $S$ since in general its null space could be strictly larger than $\Pi_d(\Omega)$, i.e., $\Delta_d$ could annihilate other functions besides polynomials. For an example, see Remark 6.2.

Thus, in this paper we need to construct seminorms in a different way. Our approach will be to view each spherical function as a restriction of a trivariate homogeneous function to the sphere $S$. In particular, given any spherical function $f$ and any integer $n$, let $f_n$ be its homogeneous extension of degree $n$, defined by

$$f_n(u) := |u|^n f \left( \frac{u}{|u|} \right), \quad u \in \mathbb{R}^3 \setminus \{0\}, \quad (3.5)$$

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where \(|u|\) denotes the Euclidean norm of \(u\). If \(\Omega\) is such that \(\text{diam}(\Omega) \leq 1\) and \(\bar{\Omega}\) is the planar domain defined in Sect. 2, we will write

\[
\bar{f}_n := f_n|_{\bar{\Omega}},
\]

which is a bivariate function defined on \(\bar{\Omega}\).

We need several elementary facts concerning homogeneous extensions.

**Lemma 3.1.** Let \(\Omega\) be an open subset of \(S\) with \(\text{diam}(\Omega) \leq 1\). Let \(g \in L_p(\Omega)\). Then for any integer \(n\), the function \(\bar{g}_n\) defined as in (3.6) satisfies

\[
M_1 \|g\|_{L_p(\Omega)} \leq \|\bar{g}_n\|_{L_p(\bar{\Omega})} \leq M_2 \|g\|_{L_p(\Omega)},
\]

where

\[
M_1 := \begin{cases} 
\frac{m_{\bar{\Omega}}^{n+3/p}}{m^{n+3/p}}, & n + 3/p \geq 0, \\
\frac{M_{\bar{\Omega}}^{n+3/p}}{m^{n+3/p}}, & n + 3/p < 0,
\end{cases}
M_2 := \begin{cases} 
\frac{M_{\bar{\Omega}}^{n+3/p}}{m_{\bar{\Omega}}^{n+3/p}}, & n + 3/p \geq 0, \\
\frac{m^{n+3/p}}{M_{\bar{\Omega}}^{n+3/p}}, & n + 3/p < 0,
\end{cases}
\]

with

\[
m_{\bar{\Omega}} := \inf \{ |\bar{\omega}|, \bar{\omega} \in \bar{\Omega} \} \geq 1, \quad M_{\bar{\Omega}} := \sup \{ |\bar{\omega}|, \bar{\omega} \in \bar{\Omega} \} \leq (\cos 1/2)^{-1}.
\]

*Here the exponents in (3.8) are understood to be equal to \(n\) for \(p = \infty\).*

**Proof:** We prove (3.7) for \(p < \infty\). The case \(p = \infty\) is similar and simpler. Let \(\sigma\) and \(\bar{\sigma}\) denote the Lebesgue measures on \(S\) and \(T_{\bar{\Omega}}\), respectively. Using the substitution \(\omega \rightarrow \bar{\omega} := R_{\bar{\Omega}}^{-1} \omega \in \bar{\Omega}\), it follows by an elementary calculation that

\[
\int_{\Omega} |g(\omega)|^p d\sigma(\omega) = \int_{\bar{\Omega}} |g(R_{\bar{\Omega}} \bar{\omega})|^p |\bar{\omega}|^{-3} d\bar{\sigma}(\bar{\omega}).
\]

By the homogeneity of \(g_n\) and the identity \(|R_{\bar{\Omega}} \bar{\omega}| = |\omega| = 1, \bar{\omega} \in \bar{\Omega}\), we can write

\[
\bar{g}_n(\bar{\omega}) = g_n(\bar{\omega}) = g_n(|\bar{\omega}| R_{\bar{\Omega}} \bar{\omega}) = |\bar{\omega}|^n g_n(R_{\bar{\Omega}} \bar{\omega}) = |\bar{\omega}|^n g(R_{\bar{\Omega}} \bar{\omega}),
\]

and therefore

\[
\int_{\Omega} |g(\omega)|^p d\sigma(\omega) = \int_{\bar{\Omega}} |\bar{\omega}|^{-(np+3)} |\bar{g}_n(\bar{\omega})|^p d\bar{\sigma}(\bar{\omega}).
\]

Now (3.7) follows immediately using \(m_{\bar{\Omega}} \leq |\bar{\omega}| \leq M_{\bar{\Omega}}\). The bound \(m_{\bar{\Omega}} \geq 1\) is trivial, while \(M_{\bar{\Omega}} \leq (\cos 1/2)^{-1}\) is a consequence of \(\text{diam}(\Omega) \leq 1\). \(\square\)

Lemma 3.1 asserts that for any \(n, g \in L_p(\Omega)\) if and only if \(\bar{g}_n \in L_p(\bar{\Omega})\). The following lemma makes an analogous assertion for Sobolev spaces.
Lemma 3.2. Let $k, n \in \mathbb{Z}_+$, and suppose $f$ is a function defined on $\Omega$, with \( \text{diam}(\Omega) \leq 1 \). Then $f \in W^{k,p}(\Omega)$ if and only if $\overline{f}_n \in W^{k,p}(\overline{\Omega})$.

Proof: Let $\hat{f} : \overline{\Omega} \to \mathbb{R}$ be defined as $\hat{f}(\overline{\omega}) := f(R_\Omega \omega), \omega \in \overline{\Omega}$. It is well known that $f \in W^{k,p}(\Omega)$ if and only if $\hat{f} \in W^{k,p}(\overline{\Omega})$. This is because in the definition (3.2) of $W^{k,p}(\Omega)$, we can choose an atlas consisting of a single chart $(\Gamma, \phi)$, where $\Gamma = \Omega$ and $\phi : \Gamma \to B := \overline{\Omega}$, where $\phi(\omega) := R_\Omega^{-1} \omega = \overline{\omega} \in \overline{\Omega}$, for $\omega \in \Omega$. Clearly, since \( \text{diam}(\Omega) \leq 1 \), the mapping $\phi$ is a $C^\infty$-diffeomorphism of $\Omega$ onto $\overline{\Omega}$. Thus, in this case (3.2) expresses the fact that $f \in W^{k,p}(\Omega)$ if and only if $f \circ \phi^{-1} = \hat{f} \in W^{k,p}(\overline{\Omega})$.

Now note that $\overline{f}_n(\overline{\omega}) = |\overline{\omega}|^n \hat{f}(\overline{\omega}), \overline{\omega} \in \overline{\Omega}$. Since the functions $|\overline{\omega}|^n$ and $|\overline{\omega}|^{-n}$ are bounded infinitely differentiable functions whose derivatives are also bounded, using the Leibniz rule, we see that multiplying any function in $W^{k,p}(\overline{\Omega})$ by $|\overline{\omega}|^n$ and $|\overline{\omega}|^{-n}$ results in a new function in the same Sobolev space. We conclude that $\hat{f} \in W^{k,p}(\overline{\Omega})$ if and only if $\overline{f}_n \in W^{k,p}(\overline{\Omega})$, which combined with the above completes the proof. \( \square \)

The following lemma shows that the trivariate functions obtained as homogeneous extensions of spherical functions (belonging to a Sobolev space) are differentiable in some sense. As usual, for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$, we write $D^\alpha := D_x^\alpha D_y^\alpha D_z^\alpha$ and $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$.

Lemma 3.3. Let $f \in W^{k,p}(\Omega)$ for some $k \geq 1$ with $\text{diam}(\Omega) \leq 1$. Then $(D^\alpha f_{k-1})|_{\Omega} \in L_p(\Omega)$ for all multi-indices $\alpha$ with $|\alpha| = k$.

Proof: Let $g := (D^\alpha f_{k-1})|_{\Omega}$. Note that whenever a trivariate homogeneous function is differentiated, the derivative is also homogeneous. In fact, $D^\alpha f_{k-1}$ is homogeneous of degree $-1$ and hence, $g_{-1} = D^\alpha f_{k-1}$. It will be sufficient to show that $g_{-1} := g_{-1}|_{\Omega} \in L_p(\overline{\Omega})$ since then, by Lemma 3.1, also $g = g_{-1}|_{\Omega} \in L_p(\Omega)$.

To do this, we assume without loss of generality that $r_\Omega$ is the north pole (i.e., $T_\Omega$ is the plane $z = 1$) so that we can choose a Cartesian coordinate system $(\xi, \eta)$ in $T_\Omega$ with $D_\xi = D_x$ and $D_\eta = D_y$. Let $D_r$ denote differentiation in the radial direction, i.e., for $|r| = 1$ and a trivariate function $h$, we have

$$D_r h(r) = \nabla h(r) \cdot r.$$ 

Since with $|\beta| = k-1$ the function $D^\beta f_{k-1}$ is homogeneous of degree zero, it follows that

$$D_r D^\beta f_{k-1} = 0. \quad (3.9)$$

Using $D_z = z^{-1}(D_r - xD_x - yD_y), z \neq 0$, and (3.9), we obtain

$$D_z D^\beta f_{k-1} = z^{-1}(D_r - xD_x - yD_y) D^\beta f_{k-1} = -z^{-1}(xD_x D^\beta f_{k-1} + yD_y D^\beta f_{k-1}).$$

Iterating this identity, we obtain the more general formula

$$D^\alpha_z D^\alpha_x D^\alpha_y f_{k-1} = (-z)^{-\alpha_3} \sum_{\ell=0}^{\alpha_3} \binom{\alpha_3}{\ell} x^\ell y^{\alpha_3-\ell} D^\alpha_x + \ell D^\alpha_y + \alpha_3-\ell f_{k-1}, \quad (3.10)$$

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which holds whenever \( \alpha_1 + \alpha_2 + \alpha_3 = k \).

Let \((x, y, z) \in \bar{\Omega} \). By our assumption on \( T_\Omega \), we have \( z = 1 \). Moreover,

\[
|x| \leq |(x, y, z)| \leq \sup \{ |\omega| : \omega \in \bar{\Omega} \} = M_\Omega,
\]

and similarly, \( |y| \leq M_\Omega \). Hence, it follows that \(|(\alpha) x^\ell y^{\alpha_3 - \ell}| \leq (|x| + |y|)^\alpha \leq (2M_\Omega)^\alpha \). We can now bound (3.10) as

\[
\|\bar{g} - 1\|_{L_p(\Omega)} = \|D_x^\alpha y^\alpha z^\alpha f_{k-1}\|_{L_p(\Omega)}
\]

\[
= \left| \sum_{\ell=0}^{\alpha_3} \binom{\alpha_3}{\ell} x^\ell y^{\alpha_3 - \ell} D_x^{\alpha_1 + \ell} D_y^{\alpha_2} f_{k-1} \right|_{L_p(\Omega)}
\]

\[
\leq (2M_\Omega)^\alpha \sum_{\gamma_1 + \gamma_2 = k} \|D_x^{\gamma_1} D_y^{\gamma_2} f_{k-1}\|_{L_p(\Omega)}
\]

\[
= (2M_\Omega)^\alpha \sum_{\gamma_1 + \gamma_2 = k} \|D_x^{\gamma_1} D_y^{\gamma_2} \bar{f}_{k-1}\|_{L_p(\Omega)}
\]

\[
\leq (2M_\Omega)^\alpha \|\bar{f}_{k-1}\|_{W^{k,p}(\bar{\Omega})} < \infty,
\]

where it is understood that the trivariate homogeneous functions involved in the above inequalities are first restricted to \( \bar{\Omega} \) before we take their \( L_p \) norm. The last inequality follows from Lemma 3.2 with \( n = k - 1 \) since \( f \in W^{k,p}(\Omega) \). \( \Box \)

The above lemmas motivate the following definition of a Sobolev-type seminorm on the sphere or any open subset \( \Omega \) of \( S \). For \( k \geq 0 \) and \( f \in W^{k,p}(\Omega) \), let

\[
|f|_{k,p,\Omega} := \sum_{|\alpha|=k} \|D_x^\alpha f_{k-1}\|_{L_p(\Omega)},
\]

where \( \|D_x^\alpha f_{k-1}\|_{L_p(\Omega)} \) should be understood as the \( L_p \)-norm of the restriction of the trivariate function \( D_x^\alpha f_{k-1} \) to \( \Omega \). For \( k = 0 \), the above seminorm reduces to the usual \( L_p \)-norm

\[
|f|_{0,p,\Omega} = \|f\|_{L_p(\Omega)}.
\]

One reason why the above seminorms make sense is that they are locally equivalent to the usual Sobolev seminorms of functions defined in a plane. The precise statement is as follows. Let \( \Omega \) be an open subset of \( S \) with \( \text{diam}(\Omega) \leq 1 \). Let \((\xi, \eta)\) be a local Cartesian system in the tangent plane \( T_\Omega \), i.e., \(|\xi| = |\eta| = 1\), \( \xi \cdot \eta = 0 \), and \( \xi \cdot r_\Omega = \eta \cdot r_\Omega = 0 \). Let \( |\cdot|_{k,p,\Omega} \) be the usual Sobolev seminorm on \( \bar{\Omega} \), i.e.,

\[
|g|_{k,p,\Omega} := \sum_{\gamma_1 + \gamma_2 = k} \|D_\xi^{\gamma_1} D_\eta^{\gamma_2} g\|_{L_p(\Omega)}, \quad g \in W^{k,p}(\bar{\Omega}).
\]

**Proposition 3.4.** Let \( \Omega \subset S \) with \( \text{diam}(\Omega) \leq 1 \). Then the seminorms \( |\cdot|_{k,p,\Omega} \) and \( |\cdot|_{k,p,\bar{\Omega}} \) are equivalent in the sense that there exist positive constants \( C_1 \) and \( C_2 \) depending only on \( k \) and \( p \) such that for every \( f \in W^{k,p}(\Omega) \),

\[
C_1 |f|_{k,p,\Omega} \leq |\bar{f}_{k-1}|_{k,p,\Omega} \leq C_2 |f|_{k,p,\Omega}.
\]
**Proof:** Suppose \( f \in W^ {k, p}(\Omega) \). Then
\[
|f|_{k, p, \Omega} = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p}(\Omega)
\leq M_1^{-1} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p}(\Omega)
\leq M_1^{-1} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = k} (2M_\Omega)^{\alpha_3} \sum_{\gamma_1 + \gamma_2 = k} \|D_x^{\gamma_1} D_y^{\gamma_2} f_{k-1}\|_{L^p}(\Omega)
\leq M_1^{-1} (2M_\Omega)^k \left( \frac{k+2}{2} \right) |f_{k-1}|_{k, p, \Omega},
\]
where above, in the first inequality, we used Lemma 3.1 with \( n = -1 \), and in the second inequality, we employed (3.11). This proves the left-hand inequality in (3.14) with \( C_1 = M_1(2M_\Omega)^{-k \left( \frac{k+2}{2} \right)^{-1}} > 0 \). On the other hand,
\[
|f|_{k, p, \Omega} = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p}(\Omega)
\geq M_2^{-1} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p}(\Omega)
\geq M_2^{-1} \sum_{\gamma_1 + \gamma_2 = k} \|D_x^{\gamma_1} D_y^{\gamma_2} f_{k-1}\|_{L^p}(\Omega)
= M_2^{-1} |f_{k-1}|_{k, p, \Omega},
\]
where in the first inequality, we used (3.7). This gives the right-hand inequality in (3.14) with \( C_2 = M_2 \). \( \square \)

The main motivation behind definition (3.12) of Sobolev seminorms for spherical functions is the requirement that these seminorms annihilate spherical polynomials. We are now ready to prove this fact.

**Proposition 3.5.** Suppose \( \Omega \) is an open connected subset of \( S \). For all \( f \in W^ {k, p}(\Omega) \) with \( k \geq 1 \), \( |f|_{k, p, \Omega} = 0 \) if and only if \( f \in \Pi_{k-1}(\Omega) \).

**Proof:** Clearly, \( |f|_{k, p, \Omega} = 0 \) if and only if \( |f|_{k, p, \Omega'} = 0 \), for all \( \Omega' \subset \Omega \) such that \( \text{diam}(\Omega') \leq 1 \). By Proposition 3.4 (applied to \( \Omega' \)), \( |f|_{k, p, \Omega'} = 0 \) if and only if \( |\tilde{f}_{k-1}|_{k, p, \Omega'} = 0 \). Since \( \Omega' \) is a planar region, it is well known that \( |\tilde{f}_{k-1}|_{k, p, \Omega'} = 0 \) if and only if \( \tilde{f}_{k-1} \in \Pi_{k-1}(\Omega') \), i.e., \( \tilde{f}_{k-1} \) is an ordinary bivariate polynomial of total degree at most \( k-1 \) on every open subset of \( \Omega'. \) Since \( \Omega \) is connected, this is equivalent to \( f_{k-1} \) being a trivariate homogeneous polynomial of degree \( k-1 \) on \( \Omega \). This in turn is equivalent to \( f \in \Pi_{k-1}(\Omega) \) since \( \Pi_{k-1}(\Omega) \) is precisely the space of trivariate homogeneous polynomials of degree \( k-1 \) restricted to \( \Omega \), see e.g., [15]. \( \square \)

Next, we show that the Sobolev norm of \( f_n = f_n|_{\Omega} \) does not depend in an essential way on the degree \( n \) of the homogeneous extension of \( f \) that is used to define \( f_n \).
Lemma 3.6. Let $\Omega \subset S$ with $\text{diam}(\Omega) \leq 1$. Suppose $f \in \mathcal{W}^{k,p}(\Omega)$ and let $\tilde{f}_m$ and $\tilde{f}_n$ be two homogeneous extensions of $f$ restricted to $\tilde{\Omega}$. Then

$$\|\tilde{f}_m\|_{k,p,\tilde{\Omega}} \leq C_3 \|\tilde{f}_n\|_{k,p,\tilde{\Omega}},$$

(3.15)

for some constant $C_3$ depending only on $k$, $m$, and $n$.

Proof: Note that $\tilde{f}_m = g\tilde{f}_n$, where $g(u) := \|u\|^{m-n}$, $u \in T_\Omega$, i.e., $g$ is the restriction of the trivariate function $\| \cdot \|^{m-n}$ to $T_\Omega$. It is not difficult to see that $g$ is infinitely differentiable, and hence all of its partial derivatives are bounded on $\tilde{\Omega}$, since $\tilde{\Omega}$ is bounded. Let $(\xi, \eta)$ be a Cartesian coordinate system in $T_\Omega$, and let

$$K := \sup\{\|D^\gamma g\|_{\infty,\tilde{\Omega}}, \; |\gamma| \leq k\} < \infty,$$

where $D^\gamma = D^\xi_1 D^\eta_2$. By the Leibnitz rule, we obtain

$$\|\tilde{f}_m\|_{k,p,\tilde{\Omega}} = \sum_{|\alpha| \leq k} \|D^\alpha (g\tilde{f}_n)\|_{L_p(\Omega)} \leq K \sum_{|\alpha| \leq k} \sum_{|\beta| \leq \alpha} \|D^\beta \tilde{f}_n\|_{L_p(\Omega)}$$

$$= K \sum_{|\beta| \leq k} \# \{\alpha : |\alpha| \leq k, \; \alpha \geq \beta\} \|D^\beta \tilde{f}_n\|_{L_p(\Omega)}$$

$$\leq K \binom{k+2}{2} \|\tilde{f}_n\|_{k,p,\tilde{\Omega}},$$

which shows (3.15) with $C_3 = K \binom{k+2}{2}$. Now since $\text{diam}(\Omega) \leq 1$, by (2.1) the diameter of a smallest disk $B$ in $T_\Omega$ containing $\tilde{\Omega}$ and centered at $r_\Omega$ is at most $A_1$ which implies that the constant $K$ is bounded by

$$\sup\{\|D^\gamma g\|_{\infty,B}, \; |\gamma| \leq k\} < \infty,$$

which thus depends only on $m - n$ and $k$. \Box

§4. Local Approximation by Spherical Polynomials

As in the planar case, cf. [7,12], the key to getting error bounds for approximation of smooth functions by spherical splines is to first investigate how well such functions can be approximated locally by polynomials. As both a guide to the type of result to be expected and as a tool in proving that result, we begin by recalling what is known in the bivariate setting.

Let $\tilde{\Omega} \subset \mathbb{R}^2$ be a set whose interior is a connected set and such that $\Omega$ is the union of a finite number of non-degenerate planar triangles, which are either disjoint or share a common edge or a vertex.
Proposition 4.1. Let \( g \in W^{d+1,p}(\bar{\Omega}) \) with \( d \geq 0 \). Then there exists a polynomial \( q \in \Pi_d(\bar{\Omega}) \) such that for every \( 0 \leq k \leq d \),
\[
|g - q|_{k,p,\Omega} \leq C_4 \text{diam}(\bar{\Omega})^{d+1-k} |g|_{d+1,p,\Omega},
\]
where \( C_4 \) is a constant depending only on \( d, p \), the minimum angle of the triangulation of \( \bar{\Omega} \), and the Lipschitz constant of the boundary \( \partial \bar{\Omega} \) of \( \bar{\Omega} \). If \( \bar{\Omega} \) is convex, then \( C_4 \) can be chosen to be independent of the Lipschitz constant of \( \partial \bar{\Omega} \).

Proposition 4.1 has been proved in various degrees of generality by several authors, see e.g., Lemma (4.3.8) in [6] and Theorem 3.1.5 in [8]. In the above form, the result was essentially proved in [12], Lemma 4.6, the only difference being that here the Sobolev seminorm \( | \cdot |_{k,p,\Omega} \) differs from that in [12], but is equivalent to it. The proof in [12] provides an explicit construction of an optimal polynomial \( q \).

We now prove a spherical analog of Proposition 4.1. Suppose that \( \Omega \subset S \) is a connected set that is the union of a finite number of non-degenerate spherical triangles which are either disjoint or share a common edge or a vertex. We define the Lipschitz constant of the boundary \( \partial \Omega \) of \( \Omega \) to be the minimum exterior angle of \( \partial \Omega \). Also, from now on we shall assume that \( \Omega \) is such that \( \text{diam}(\Omega) \leq 1 \).

In addition to (2.1) and (2.2), this condition implies that the minimum angles of the triangulations of \( \Omega \) and \( \bar{\Omega} \) are of comparable size, and the same is true of the Lipschitz constants of \( \partial \Omega \) and \( \partial \bar{\Omega} \). A rigorous proof of these facts is elementary.

Theorem 4.2. Suppose \( \Omega \) is an open subset of \( S \) with \( \text{diam}(\Omega) \leq 1 \). Let \( f \in W^{d+1,p}(\Omega) \) with \( d \geq 0 \). Then there exists a spherical polynomial \( s \in \Pi_d(S) \) such that for every \( 0 \leq k \leq d \),
\[
|f - s|_{k,p,\Omega} \leq C_5 \text{diam}(\Omega)^{d+1-k} |f|_{d+1,p,\Omega}.
\]

Here \( C_5 \) is a constant that depends only on \( d, p \), the smallest angle in the triangulation of \( \Omega \), and the Lipschitz constant of \( \partial \Omega \). If \( \Omega \) is such that the cone \( \bar{\Omega} := \{ u \in \mathbb{R}^3 \setminus \{0\}, u/|u| \in \Omega \} \) is convex, then \( C_5 \) can be taken to be independent of the Lipschitz constant of \( \partial \bar{\Omega} \).

Proof: As before, let \( \bar{\Omega} \) be the planar region which is the radial projection of \( \Omega \) onto the tangent plane \( T_{\bar{\Omega}} \). Since \( f \in W^{d+1,p}(\Omega) \), Lemma 3.2 implies \( \bar{f}_d \in W^{d+1,p}(\bar{\Omega}) \). But then Proposition 4.1 guarantees the existence of a bivariate polynomial \( \bar{s} \in \Pi_d(\bar{\Omega}) \) satisfying
\[
|\bar{f}_d - \bar{s}|_{\ell,p,\Omega} \leq C_4 \text{diam}(\bar{\Omega})^{d+1-\ell} |\bar{f}_d|_{d+1,p,\Omega}
\]
\[
\leq C_4 A_1^{d+1} C_2 \text{diam}(\bar{\Omega})^{d+1-\ell} |f|_{d+1,p,\Omega},
\]
for every \( \ell = 0, \ldots, k \), where \( C_4, A_1, \) and \( C_2 \) are the constants in (4.1), (2.1), and (3.14), respectively.

We are now able to define a polynomial \( s \in \Pi_d(\Omega) \) which we claim will satisfy (4.2). Indeed, since the polynomial \( \bar{s} \) can be thought of as the restriction to \( \bar{\Omega} \) of a
unique trivariate polynomial that is homogeneous of degree $d$, it follows that there exists a unique spherical polynomial $s \in \Pi_d(\Omega)$ whose homogeneous extension of degree $d$ restricted to $\Omega$ is equal to $\bar{s}$, i.e., such that $\bar{s} = s_d|_\Omega$. This is the desired polynomial.

Using Proposition 3.4 and Lemma 3.6, in conjunction with the previous inequality, it follows that

$$|f - s|_{k,p,\Omega} \leq C_1^{-1} \|\bar{f}_{k-1} - \bar{s}_{k-1}\|_{k,p,\hat{\Omega}} \leq C_1^{-1} \|\bar{f}_{k-1} - \bar{s}_{k-1}\|_{k,p,\hat{\Omega}}$$

$$\leq C_1^{-1} C_3 \|\bar{f}_d - \bar{s}_d\|_{k,p,\hat{\Omega}} = C_1^{-1} C_3 \sum_{t=0}^k |\bar{f}_d - \bar{s}_d|_{t,p,\hat{\Omega}}$$

$$\leq C_1^{-1} C_3 C_4 A_1^{d+1} C_2 \sum_{t=0}^k \text{diam}(\hat{\Omega})^{d+1-t} |f|_{d+1,p,\hat{\Omega}}$$

$$= C_1^{-1} C_3 C_4 A_1^{d+1} C_2 \left( \sum_{t=0}^k \text{diam}(\hat{\Omega})^t \right) \text{diam}(\Omega)^{d+1-k} |f|_{d+1,p,\hat{\Omega}}$$

$$\leq C_5 \text{diam}(\Omega)^{d+1-k} |f|_{d+1,p,\hat{\Omega}},$$

where $C_1$ and $C_3$ are the constants in (3.14) and (3.15), respectively, and $C_5 := C_1^{-1} C_3 C_4 A_1^{d+1} C_2 (d + 1)$. Here we have used the fact $\text{diam}(\hat{\Omega}) \leq 1$ implies $\sum_{t=0}^k \text{diam}(\hat{\Omega})^t \leq k + 1 \leq d + 1$.

The dependence of $C_5$ on the Lipschitz constant of $\partial \hat{\Omega}$ enters via the constant $C_4$, i.e., via the Lipschitz constant of $\partial \hat{\Omega}$. However, the convexity of $\hat{\Omega}$ is equivalent with the convexity of $\Omega$, in which case $C_4$ can be chosen to be independent of the Lipschitz constant of $\partial \hat{\Omega}$ and hence also independent of the Lipschitz constant of $\partial \Omega$. $\square$

We close this section with several results on spherical polynomials. Our first result is a spherical analog of a Markov-type inequality for spherical polynomials.

**Proposition 4.3.** Let $\Omega \subset S$ with $\text{diam}(\Omega) \leq 1$. Then there exists a constant $M$ depending on $d$ such that for all $s \in \Pi_d(\Omega)$ and all $0 \leq k \leq d$,

$$|s|_{k,p,\Omega} \leq M \rho_\Omega^{-k} \|s\|_{L_p(\Omega)},$$

where $\rho_\Omega$ is the diameter of the largest spherical cap contained in $\Omega$.

**Proof:** As in the previous theorem, we will utilize the connection between spherical polynomials and bivariate polynomials defined on $T_\Omega$. The following Markov inequality for bivariate polynomials is well known (see e.g., [12], Lemma 4.2):

$$|\tilde{s}_d|_{k,p,\Omega} \leq M \rho_\Omega^{-k} \|\tilde{s}_d\|_{L_p(\Omega)},$$

$$\|f\|_{L_p(\Omega)} \leq C_1^{-1} \|\bar{f}_d - \bar{s}_d\|_{k,p,\hat{\Omega}}.$$
for some constant $\tilde{M}$ depending on $d$. Thus, as in the proof of Theorem 4.2, we conclude that
\[
|s|_{k,p,\Omega} \leq C_1^{-1}|\tilde{s}|_{k-1,\Omega} \leq C_1^{-1}||\tilde{s}_{k-1}||_{k,p,\Omega} \leq C_1^{-1}C_3||\tilde{s}_d||_{k,p,\Omega}
\]
\[
\leq C_1^{-1}C_3\tilde{M} \sum_{l=0}^{k} \rho_{\Omega}^{-l}||\tilde{s}_d||_{p,\Omega}
\]
\[
= C_1^{-1}C_3\tilde{M} \left( \sum_{l=0}^{k} \rho_{\Omega}^l \right) \rho_{\Omega}^{-k}||\tilde{s}_d||_{p,\Omega}
\]
\[
\leq C_1^{-1}C_3\tilde{M} A_2^{d+1}(d+1)\rho_{\Omega}^{-k}||s||_{p,\Omega},
\]
where in the second line we used (4.4). For the last step we used (2.2) to get $\rho_{\Omega} \leq A_2 \rho_{\Omega} \leq A_2 \text{diam}(\Omega) \leq A_2$ which in turn implies $\sum_{l=0}^{k} \rho_{\Omega}^l \leq A_2^{d+1}(k+1) \leq A_2^{d+1}(d+1)$. We have shown that (4.3) holds with $M := C_1^{-1}C_3\tilde{M} A_2^{d+1}(d+1)$. 

Our next result compares the sizes of the $L_p$ and $L_\infty$ norms of a polynomial on a triangle $T$.

**Lemma 4.4.** Let $T$ be a spherical triangle such that $\text{diam}(T) \leq 1$. Then there exists a constant $M_3$ such that
\[
A_T^{-1/p}||s||_{L_p(T)} \leq ||s||_{L_\infty(T)} \leq M_3 A_T^{-1/p}||s||_{L_p(T)}, \quad \forall s \in \Pi_d(T),
\]
where $A_T$ is the area of $T$. The constant $M_3$ depends only on $d$, $p$, and the minimum angle of $T$.

**Proof:** The first inequality is an elementary consequence of Hölder’s inequality. As for the second inequality, let $\tilde{s}_d := s_d|_{\tilde{T}}$. We already noted that $\tilde{s}_d$ is an ordinary bivariate polynomial, i.e., $\tilde{s}_d \in \Pi_d(\tilde{T})$. The planar analog of the second inequality in (4.5) is (see the proof of Lemma 4.1 in [12])
\[
||\tilde{s}_d||_{L_\infty(\tilde{T})} \leq \tilde{M}_3 A_T^{-1/p}||\tilde{s}_d||_{L_p(\tilde{T})}, \quad \forall \tilde{s}_d \in \Pi_d(\tilde{T}),
\]
where $\tilde{M}_3$ depends only on $d$, $p$, and the minimum angle in $\tilde{T}$. Using this fact and Lemma 3.1 (with $n = d$), we obtain
\[
||s||_{L_\infty(T)} \leq M_1^{-1}||\tilde{s}_d||_{L_\infty(T)} \leq M_1^{-1}M_3 A_T^{-1/p}||\tilde{s}_d||_{L_p(\tilde{T})} \leq M_1^{-1}M_2 \tilde{M}_3 A_T^{-1/p}||s||_{L_p(T)},
\]
where $M_1$ and $M_2$ depend only on $d$ and $p$. Here we used the elementary fact that $A_\tilde{T} \geq A_T$. To finish the proof, it is enough to note that since $\text{diam}(T) \leq 1$, the minimum angle of $\tilde{T}$ is bounded from below by an absolute constant multiplied by the minimum angle of $T$. 

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§5. Approximation Order of Spherical Splines

In this section we prove Theorem 1.1 by giving bounds on how well smooth functions can be approximated by spherical splines. Our results are direct analogs of similar results for bivariate splines on planar triangulations, see e.g., [7,12] and references therein.

Let $\Delta$ be a spherical triangulation of $S$. Since we are interested in approximation order for small values of the mesh size $|\Delta|$, without loss of generality we assume throughout the remainder of the paper that $|\Delta| \leq 1$. Let $d, r$ be integers such that $d \geq 3r + 2$, and let $S^r_d(\Delta)$ be a space of spherical splines defined in (1.1). We write $N$ for the dimension of $S^r_d(\Delta)$. Given any vertex $v$ of $\Delta$, we write $\text{star}^0(v) := \{v\}$, and for all $\ell \geq 1$ inductively define $\text{star}^\ell(v)$ to be the union of all triangles in $\Delta$ sharing a vertex with some vertex of $\text{star}^{\ell-1}(v)$. In [3] we gave explicit formulae for $N$, along with a construction of a set of basis functions $\{B_i\}_{i=1}^N$ which are locally supported in the sense that for each $i$, there is a vertex $v_i$ of the triangulation $\Delta$ such that $\text{supp}(B_i) \subseteq \text{star}(v_i)$. This basis is, however, not adequate for our purposes here since it is not stable in the presence of near-singular vertices (cf. the discussion in [9,12]).

The construction of stable bases is a delicate process, even in the case of bivariate splines, see [9] and references therein. Fortunately, as pointed out in [9], Remark 13.13, the construction presented there for the bivariate analog of $S^r_d(\Delta)$ also carries over to the spherical spline space $S^r_d(\Delta)$. We now briefly outline this construction. The key is to work with the Bernstein-Bézier representation of splines in $S^r_d(\Delta)$, see [2–4]. First we introduce the set of domain points

$$
D := \bigcup_{T:=(u,v,w) \in \Delta} \left\{ \xi_T^\nu_{\mu \kappa} = \frac{\nu u + \mu v + \kappa w}{d} \right\}_{\nu + \mu + \kappa = d}.
$$

It is well known [4] that each spline in $S^0_d(\Delta)$ is uniquely determined by associating one Bézier coefficient with each domain point.

A set $\mathcal{M} := \{\xi_i\}_{i=1}^N \subset D$ is called a minimal determining set for $S^r_d(\Delta)$ if the values of the coefficients of $s \in S^r_d(\Delta)$ associated with domain points in $\mathcal{M}$ uniquely determine all of the coefficients of $s$. Given a minimal determining set, we can construct a basis $\{B_i\}_{i=1}^N$ for $S^r_d(\Delta)$ by requiring

$$
\mu_j B_i = \delta_{ij}, \quad 1 \leq i, j \leq N,
$$

where for all $1 \leq j \leq N$, $\mu_j$ is the linear functional which picks the coefficient associated with the domain point $\xi_j$. In particular, $B_i$ has the property that the coefficient associated with $\xi_i$ is 1 while the coefficients associated with all other $\xi_j$ in $\mathcal{M}$ are zero. The remaining coefficients of $B_i$ are computed using smoothness conditions.

For any given spline space $S^r_d(\Delta)$, there are many possible choices for a minimal determining set $\mathcal{M}$. To get a stable local basis requires choosing $\mathcal{M}$ carefully

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to insure the local support, and to insure that for each $B_i$ all of the calculated coefficients remain bounded. The choice of $\mathcal{M}$ presented in [9] leads to a basis with the following properties, where for each $i$, $\Omega_i := \text{supp}(B_i)$ and $T_i$ is the triangle in which $\xi_i$ lies.

**Proposition 5.1.** Let $\{B_i\}_{i=1}^N$ be the basis for $S^r_d(\Delta)$ corresponding to the minimal determining set $\mathcal{M}$ described in [9]. Then there exist constants $K_1, \ldots, K_7$ depending only on $d, p$, and the minimal angle in $\Delta$ such that for each $1 \leq i \leq N$,

1) there exists a vertex $v_i \in \Delta$ such that $\Omega_i \subseteq \text{star}^3(v_i)$,

2) $\|B_i\|_{L_\infty(S)} \leq K_1$,

3) $|\mu_is| \leq K_2\|s\|_{L_\infty(T_i)}$, for all $s \in S^r_d(\Delta)$,

4) $|\mu_is| \leq K_3A_T^{-1/p}\|s\|_{L_p(T_i)}$, for all $s \in S^r_d(\Delta)$,

and for every $T \in \Delta$,

5) $\|B_i\|_{L_p(T)} \leq K_4A_T^{1/p}$,

6) $|I_T| \leq K_5$, where $I_T := \{i : T \subset \Omega_i\}$,

7) $|B_i|_{k, \infty, T} \leq K_6\rho_T^{-k}$, for all $0 \leq k \leq d$,

8) $|B_i|_{k, p, T} \leq K_7\rho_T^{-k}A_T^{1/p}$, for all $0 \leq k \leq d$.

**Proof:** Properties 1) and 2) follow directly from the construction of the basis. Property 3) follows from the stability of the spherical Bernstein basis (see Remark 6.3) since restricted to $T_i$, $s$ is just a spherical polynomial. Combining 3) and (4.5), we get 4) with $K_3 := K_2M_3$. Combining 2) with (4.5) leads to 5) with $K_4 := K_1$. Property 6) follows from 1) as in the bivariate case (cf. Lemma 3.1 of [12]). Combining 2) with the Markov inequality (4.3) gives 7) with $K_6 := MK_1$. Combining 5) with the Markov inequality (4.3) gives 8) with $K_7 := MK_4$. □

We are now ready to construct a quasi-interpolation operator $Q : L_p(S) \rightarrow S^r_d(\Delta)$ which will produce the desired approximation results. To this end, we first use the Hahn-Banach theorem to extend the linear functionals $\mu_i$ appearing in (5.1) to all of $L_p(S)$. We will continue to use the same symbol for these extensions. By the theorem and by 4) of Proposition 5.1, we know that for every $f \in L_p(T_i)$,

$$|\mu_if| \leq K_3A_T^{-1/p}\|f\|_{L_p(T_i)}, \quad 1 \leq i \leq N. \quad (5.2)$$

This inequality immediately implies that for each $i$, the carrier of the extended functional $\mu_i$ is contained in $T_i$, i.e., if $f \equiv 0$ on $T_i$, then $\mu_if = 0$.

**Proposition 5.2.** For each $f \in L_p(\Omega)$, let

$$Qf := \sum_{i=1}^N (\mu_if)B_i. \quad (5.3)$$

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Then \( Qg = g \) for all \( g \in \Pi_d(S) \). Moreover, there exists a constant \( K_8 \) depending only on \( d, p \), and the smallest angle in \( \Delta \) such that for each triangle \( T \in \Delta \),

\[
|Qf|_{k,p,T} \leq K_8 \rho_T^{-k} \|f\|_{L_p(\Omega_T)},
\]

where \( I_T := \{i : T \subset \Omega_i\} \) and \( \Omega_T := \bigcup_{i \in I_T} \Omega_i \).

**Proof:** The fact that \( Q \) reproduces polynomials in \( \Pi_d(S) \) is clear from (5.1). To prove (5.4), by (5.2) we have

\[
|Qf|_{k,p,T} = \left| \sum_{i \in I_T} (\mu_i f) B_i \right|_{k,p,T} \\
\leq \sum_{i \in I_T} |\mu_i f||B_i|_{k,p,T} \leq K_3 \sum_{i \in I_T} A_{T_i}^{-1/p} \|f\|_{L_p(T_i)} |B_i|_{k,p,T} \\
\leq K_3 A^{-1/p} \|f\|_{L_p(\Omega_T)} \sum_{i \in I_T} |B_i|_{k,p,T} \\
\leq K_3 A^{-1/p} \|f\|_{L_p(\Omega_T)} \# I_T K_7 \rho_T^{-k} A_1^{1/p},
\]

where \( A := \min\{A_T: i \in I_T\} \) and where we have inserted \( 8 \) from Proposition 5.1 in the last inequality. Using the fact that \( A_T/A \) is bounded by a constant \( K_9 \) depending only on the smallest angle in \( \Delta \) (cf. (3.8) in [12] in the bivariate case), we get (5.4) with \( K_8 := K_3 K_7 K_9^{1/p} \# I_T \). \( \Box \)

We are now ready to prove the main result of this paper. Instead of Theorem 1.1, we prove a slightly stronger result.

**Theorem 5.3.** Let \( 1 \leq p \leq \infty, d \geq 3r + 2, \) and \( 0 \leq k \leq d \). Then there exists a constant \( C_T \), depending only on \( d, p \), and the smallest angle in \( \Delta \), such that

\[
|f - Qf|_{k,p,T} \leq C \text{diam}(\Omega_T)^{d+1-k}|f|_{d+1,p,\Omega_T},
\]

for all \( f \in W^{d+1,p}(S) \) and all \( T \in \Delta \). Moreover, there exists a constant \( C' \), depending only on \( d, p \), and the smallest angle in \( \Delta \), such that

\[
|f - Qf|_{k,p,S} \leq C' |\Delta|^{d+1-k}|f|_{d+1,p,S},
\]

for all \( f \in W^{d+1,p}(S) \) and all \( 0 \leq k \leq d \) such that \( Qf \in W^{k,p}(S) \).

**Proof:** For \( T \in \Delta \), let \( s \in \Pi_d(\Omega) \) be such that

\[
|f - s|_{k,p,T} \leq |f - s|_{k,p,\Omega_T} \leq C'_T \text{diam}(\Omega_T)^{d+1-k}|f|_{d+1,p,\Omega_T}
\]

and

\[
\|f - s\|_{L_p(\Omega_T)} \leq C''_T \text{diam}(\Omega_T)^{d+1}|f|_{d+1,p,\Omega_T}.
\]

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where $C'_T$ and $C''_T$ depend only on $d$, $p$, the smallest angle in the triangulation of $\Omega_T$, and the Lipschitz constant of $\partial \Omega_T$, which in turn depends only on the smallest angle in the triangulation $\Delta$. The existence of such a polynomial $s$ is guaranteed by Theorem 4.2. By the linearity of $Q$ and the fact that $Q$ reproduces polynomials of degree $d$, we can write

$$|f - Qf|_{k,p,T} \leq |f - s|_{k,p,T} + |Q(s)|_{k,p,T}. \quad (5.9)$$

We now estimate the second term on the right-hand side of (5.9). It follows from 1) of Proposition 5.1 that there exists a constant $D \geq 1$, depending only on the minimum angle in the triangulation of $\Omega_T$, such that $\text{diam}(\Omega_T) \leq D \rho_T$, or equivalently, $\rho_T^{-1} \leq D \text{diam}(\Omega_T)^{-1}$. By (5.4) and (5.8), we thus have

$$|Q(s)|_{k,p,T} \leq K_s \rho_T^{-k} \|f - s\|_{L_2(\Omega_T)} \leq K_s C''_T \rho_T^{-k} \text{diam}(\Omega_T)^{d+1} |f|_{d+1,p,\Omega_T} \leq K_s C''_T D^k \text{diam}(\Omega_T)^{d+1-k} |f|_{d+1,p,\Omega_T}. \quad (5.10)$$

Combining (5.9), (5.7), and (5.10), we get the first assertion of the theorem with $C := C'_T + K_s C''_T D^d$.

We now consider the second assertion. To prove (5.6), we sum (5.5) over all triangles in $\Delta$. To do this, we first note that by the local support property of the basis functions $B_i$, there exists a constant $L \geq 1$ which depends only on the smallest angle in $\Delta$ such that $\text{diam}(\Omega_T) \leq L \text{diam}(T)$, for every $T \in \Delta$. By (5.5), we obtain

$$|f - Qf|_{k,p,S} = \sum_{T \in \Delta} |f - Qf|_{k,p,T} \leq C \sum_{T \in \Delta} \text{diam}(\Omega_T)^{d+1-k} |f|_{d+1,p,\Omega_T} \leq CL^{d+1-k} \sum_{T \in \Delta} \text{diam}(T)^{d+1-k} |f|_{d+1,p,\Omega_T} \leq CL^{d+1-k} \text{diam}(\Delta)^{d+1-k} \sum_{T \in \Delta} \sum_{T' \subset \Omega_T} |f|_{d+1,p,T'} \leq CL^{d+1} M' \text{diam}(\Delta)^{d+1-k} \sum_{T' \in \Delta} |f|_{d+1,p,T'} = CL^{d+1} M' \text{diam}(\Delta)^{d+1-k} |f|_{d+1,p,S},$$

where $M' := \max\{\#\{T : T' \subset \Omega_T\}, \ T' \in \Delta\}$, which is a constant depending only on the minimum angle in $\Delta$. \qed
\section{Remarks}

\textbf{Remark 6.1.} The seminorms defined in (3.12) can be used to define a norm on the space $W^{k,p}(\Omega)$ as

$$
\|f\|_{W^{k,p}(\Omega)} := \sum_{t=0}^{k} \|f_t\|_{L^p(\Omega)}.
$$

Using the results of Sect. 3 and a partition of unity $\{\alpha_j\}$ associated with an atlas $\{(\Gamma_j, \phi_j)\}$ for $\Omega$, it can be shown that this norm is equivalent to the norm $\|f\|_{W^{k,p}(\Omega)}$ defined in (3.3). Another alternative is to define Sobolev norms using covariant derivatives, see \textit{e.g.} [5,11].

\textbf{Remark 6.2.} In Section 3, we mentioned that the operators defined by (3.4) were not well suited for defining Sobolev seminorms for domains $\Omega$ that are proper subsets of $S$. The following example illustrates this fact. Consider the function

$$
f(x, y, z) := \log \left( \frac{1+z}{1-z} \right), \quad (x, y, z) \in S.
$$

Clearly, $f$ is a spherical function that is infinitely differentiable except at the south and north poles ($z = \pm 1$). Let $\Omega$ be any open subset of $S$ not containing the poles. Then one can easily check that

$$
\Delta_0 f = \Delta^* f = 0 \quad \text{on} \quad \Omega,
$$

\textit{i.e.}, in addition to constants, which are spherical harmonics of degree $d = 0$, the operator $\Delta_0$ also annihilates other smooth functions defined on $\Omega$.

\textbf{Remark 6.3.} Let $s \in \Pi_d(T)$ be a spherical polynomial defined on a spherical triangle $T$ with $\text{diam}(T) \leq 1$, and let $c$ be the vector of Bézier coefficients of $s$ (see [2]). Then the Bernstein-Bézier representation of $s$ is \textit{stable} in the sense that there exist positive constants $D_1$ and $D_2$, depending only on $d$, such that

$$
D_1 \|c\|_{L^\infty} \leq \|s\|_{L^\infty(T)} \leq D_2 \|c\|_{L^\infty}.
$$

This is a standard result for bivariate polynomials, see \textit{e.g.}, [12], Lemma 4.1. It is easy to see that a similar proof works in the spherical case. The inequalities can also be proved using radial projection.

\textbf{Remark 6.4.} Since $Qf \in C^r(S)$, the largest integer $k$ in (5.6) for which $Qf \in W^{k,p}(S)$ is always at least $r + 1$.

\textbf{Remark 6.5.} Theorem 5.3 can easily be generalized to provide bounds on $\|f - Qf\|_{k,p,\Omega}$, where $\Omega$ is a set formed from a union of triangles of $\Delta$. The only change in this case is that the constant $C'$ on the right-hand side of (5.6) then depends on the Lipschitz constant of the boundary of $\Omega$, and the semi-norm on the right-hand side of (5.6) has to be taken over the set $\hat{\Omega} := \cup \{\Omega_i : \Omega_i \cap \Omega \neq \emptyset\}$. 
References