Smooth Macro-Elements on Powell-Sabin-12 Splits

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Abstract. Macro-elements of smoothness C^r are constructed on Powell-Sabin-12 splits of a triangle for all $r \geq 0$. These new elements complement those recently constructed on Powell-Sabin-6 splits [5,12], and can be used to construct convenient superspline spaces with stable local bases and full approximation power that can be applied to the solution of boundary-value problems and for interpolation of Hermite data.

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Short title: Macro-elements on Powell-Sabin-12 splits
Classification: 41A15, 65M60, 65N30

§1. Introduction

A bivariate macro-element defined on a triangle T consists of a finite dimensional linear space $\mathcal{S}$ defined on T, and a set Λ of linear functionals forming a basis for the dual of $\mathcal{S}$. Usually the space $\mathcal{S}$ is chosen to be a space of polynomials or a space of piecewise polynomials defined on some subtriangulation of T. The members of Λ, called the degrees of freedom, are usually taken to be point evaluations of derivatives, although here we will also work with sets of linear functionals which pick off certain spline coefficients.

A macro-element defines a local interpolation scheme. In particular, if f is a sufficiently smooth function, then we can define the corresponding interpolant as the unique function $s \in \mathcal{S}$ such that $\lambda s = \lambda f$ for all $\lambda \in \Lambda$. We say that a macro-element has smoothness C^r provided that if the element is used to construct an interpolating function locally on each triangle of a triangulation $\triangle$, then the resulting piecewise function is C^r continuous globally. Macro-elements are useful tools for building spaces of smooth splines with stable local bases and full approximation power.

Several families of C^r macro-elements have been developed using polynomials [17,19], and piecewise polynomials on appropriate splits, see [4,5,11,12,13,15], and references therein. The purpose of this paper is to describe a family of C^r

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macro-elements based on the Powell-Sabin-12 split, see Definition 3.1. These new macro-elements complement the existing families of C^r macro-elements based on the Powell-Sabin-6 split [5,12], and for compatibility make use of splines of the same degrees, see however Remark 7.2. A major advantage of our new elements is that certain geometric constraints required in the Powell-Sabin-6 case can be removed, see Remark 7.4.

The paper is organized as follows. In Sect. 2 we review some well-known Bernstein-Bézier notation. Our C^r family of macro-elements is introduced and studied in Sect. 3, while Sect. 4 contains several supporting lemmas. We discuss the approximation power of our new macro-elements in Sect. 5. In Sect. 6 we translate our degrees of freedom into nodal functionals, and discuss a related Hermite interpolation method and associated error bound. We conclude with remarks in Sect. 7.

§2. Preliminaries

We use Bernstein–Bézier techniques as in [1–13,16,17]. In particular, we represent polynomials p of degree d on a triangle $T := \langle v_1, v_2, v_3 \rangle$ in their B-form

$$p = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d,$$

where B_{ijk}^d are the Bernstein basis polynomials of degree d associated with T. As usual, we associate the coefficients c_{ijk}^T with the domain points $\xi_{ijk}^T := \frac{(iv_1+jv_2+kv_3)}{d}$. We write $\mathcal{D}_{d,T} := \{\xi_{ijk}^T\}_{i+j+k=d}$.

Given a triangulation $\triangle$, let $\mathcal{D}_{d,\triangle} := \bigcup_{T\in\triangle} \mathcal{D}_{d,T}$, and let $\mathcal{S}_d^0(\triangle)$ be the space of continuous splines of degree d on $\triangle$. Then it is well known that each spline in $\mathcal{S}_d^0(\triangle)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$, where the coefficients of the polynomial $s|_T$ are precisely $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}\cap T}$. We recall that if $T := \langle v_1, v_2, v_3\rangle$, then the ring of radius m around v_1 is $R_m^T(v_1) := \{\xi_{d-m,j,k}^T : j+k=m\}$ and the disk of radius m around v_1 is $D_m^T(v_1) := \{\xi_{ijk}^T : i \ge d-m\}$. If v is a vertex of $\triangle$, we define the ring $R_m(v)$ of radius m around v to be the set of all domain points on rings $R_m^T(v)$ where T is a triangle with vertex at v. The disk $D_m(v)$ of radius m around v is defined similarly.

In this paper we are interested in subspaces $\mathcal{S}$ of $\mathcal{S}_d^0(\triangle)$ which satisfy additional smoothness conditions. Following [6], to describe smoothness we shall make use of smoothness functionals defined as follows. Let $T := \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2\rangle$ be two adjoining triangles which share the edge $e := \langle v_2, v_3\rangle$, and let c_{ijk} and $\tilde{c}_{ijk}$ be the coefficients of the B-representations of s_T and $s_{\widetilde{T}}$, respectively. Then for any $n \le m \le d$, let $\tau_{e,m}^n$ be the linear functional defined on $\mathcal{S}_d^0(\triangle)$ by

$$\tau_{e,m}^n s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B_{ijk}^n(v_4), \tag{2.1}$$

where B^n_{ijk} are the Bernstein polynomials of degree n on the triangle T. In terms of these linear functionals, the condition that s be C^r smooth across the edge e is equivalent to

$$\tau^n_{e,m} s = 0, \qquad n \le m \le d, \quad 0 \le n \le r.$$

Smoothness conditions can be used to directly compute coefficients of one piece of a spline from another. They can also be used in situations where some of the coefficients of two different pieces of s are known. The following well-known lemma [4] (see also Lemma 3.3 of [7]) shows how this works for computing coefficients on the ring $R^T_m(v_2) \cup R^{\widetilde{T}}_m(v_2)$.

Lemma 2.1. *Suppose $T := \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} := \langle v_4, v_3, v_2 \rangle$ are two triangles sharing an edge $e := \langle v_2, v_3 \rangle$, and suppose the points v_1, v_2, v_4 are not collinear. Let $s \in \mathcal{S}^0_d(\triangle)$, where $\triangle := T \cup \widetilde{T}$, and suppose that $\tau^n_{e,m} s = 0$, $n = \ell+1, \ldots, q+\tilde{q}-\ell$ for some $\ell, m, q, \tilde{q}$ with $0 \le q, \tilde{q}$, $-1 \le \ell \le q, \tilde{q}$, and $q + \tilde{q} - \ell \le m \le d$. Suppose that all coefficients c_{ijk} and $\tilde{c}_{ijk}$ of the polynomials $s|_T$ and $s|_{\widetilde{T}}$ corresponding to domain points in $D_m(v_2)$ are known except for*

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, \qquad && \nu = \ell+1, \ldots, q, \\ \tilde{c}_\nu &:= \tilde{c}_{\nu,m-\nu,d-m}, \qquad && \nu = \ell+1, \ldots, \tilde{q}, \end{aligned} \tag{2.2}$$

Then these coefficients are uniquely determined by the smoothness conditions.

If s is a spline in $\mathcal{S}^0_d(\triangle)$ which satisfies additional smoothness conditions beyond C^0 continuity, then clearly we cannot independently choose all of its coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. We recall that a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points $\mathcal{D}_{d,\triangle}$ such that if we set $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $s \equiv 0$. The set $\mathcal{M}$ is called a minimal determining set (MDS) for $\mathcal{S}$ if there is no smaller determining set. It is known that $\mathcal{M}$ is a MDS for $\mathcal{S}$ if and only if every spline $s \in \mathcal{S}$ is *uniquely* determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$.

A MDS $\mathcal{M}$ is called local provided that there is an integer n such that for every $\xi \in \mathcal{D}_{d,\triangle} \cap T$ and every triangle T in $\triangle$, c_ξ is a linear combination of $\{c_\eta\}_{\eta \in \Gamma_\xi}$ where Γ_ξ is a subset of $\mathcal{M}$ with $\Gamma_\xi \subset \operatorname{star}^n(T)$. Here $\operatorname{star}^n(T) := \operatorname{star}(\operatorname{star}^{n-1}(T))$ for $n \ge 2$, where if U is a cluster of triangles, $\operatorname{star}(U)$ is the set of all triangles which have a nonempty intersection with U. Moreover, $\mathcal{M}$ is called stable, provided that there is a constant K depending on the smallest angle in $\triangle$ such that

$$|c_\xi| \le K \max_{\eta \in \Gamma_\xi} |c_\eta|, \qquad \text{for all } \xi \in \mathcal{D}_{d,\triangle}. \tag{2.3}$$

A linear functional λ defined on $\mathcal{S}^0_d(\triangle)$ is called a nodal functional provided that λs is a combination of values and/or derivatives of s at some point η. A collection $\{\lambda\}_{\lambda \in \mathcal{N}}$ is called a nodal determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ if $\lambda s = 0$ for all $\lambda \in \mathcal{N}$ implies $s \equiv 0$. $\mathcal{N}$ is called a nodal minimal determining set (NMDS) for $\mathcal{S}$ if there is no smaller nodal determining set.

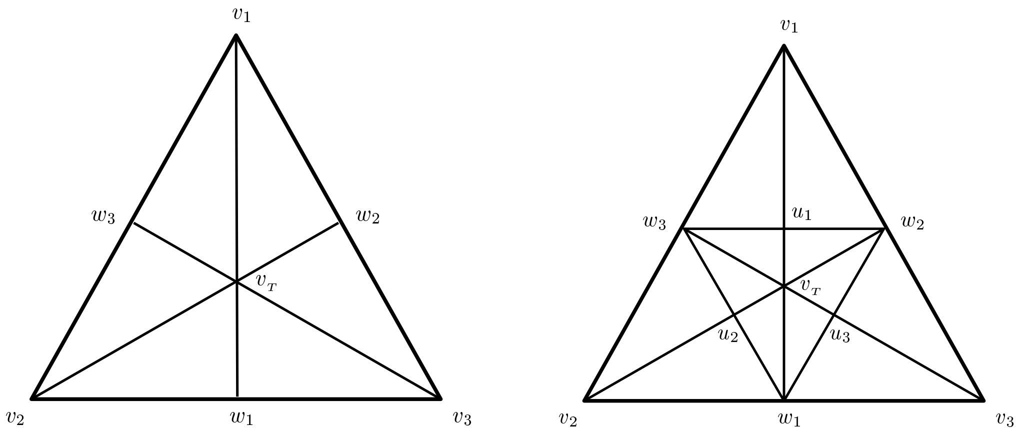

Fig. 1. The Powell-Sabin-6 and Powell-Sabin-12 splits.

§3. A Family of C^r Powell-Sabin-12 Macro-elements

We now define the Powell-Sabin split of interest in this paper.

Definition 3.1. *Given a triangle $T = \{v_1, v_2, v_3\}$, for each $1 \le i \le 3$, let w_i be the midpoint of the edge $e_i := \langle v_{i+1}, v_{i+2}\rangle$ opposite to v_i, where we set $v_4 := v_1$. Draw in the line segments $\langle v_i, w_i\rangle$, $i = 1, 2, 3$. Then it is easy to see that these three line segments intersect at the barycenter $v_T := (v_1 + v_2 + v_3)/3$ of T. The resulting partition T_{PS6} of T into six triangles is called the* Powell-Sabin-6 split *of T, see Fig. 1 (left). If we now draw in the line segments $\langle w_i, w_{i+1}\rangle$, $i = 1, 2, 3$, where $w_4 := w_1$, then the resulting partition T_{PS12} of T into twelve triangles is called the* Powell-Sabin-12 split *of T, see Fig. 1 (right).*

We need some additional notation and terminology connected with Powell-Sabin-12 splits. For each $i = 1, 2, 3$, we write u_i for the intersection of $\langle w_{i+1}, w_{i+2}\rangle$ with $\langle v_i, v_T\rangle$. Note that the u_i are midpoints of the edges $\langle w_{i+1}, w_{i+2}\rangle$, and are singular vertices of T_{PS12}, i.e., vertices which are formed by two crossing lines. We refer to the edges of the form $\langle v_i, u_i\rangle$ as type-1 edges, to edges of the form $\langle w_i, v_T\rangle$ as type-2 edges, and to edges of the form $\langle u_i, v_T\rangle$ as type-3 edges.

Given a triangulation $\triangle$ of a domain Ω, we write $\mathcal{V}$ and $\mathcal{E}$ for the sets of vertices and edges of $\triangle$. To define our macro-element spaces, we shall work with the refinement $\triangle_{PS12}$ of $\triangle$ which is obtained by applying the Powell-Sabin-12 split to each triangle of $\triangle$. We write $\mathcal{W}$ for the set of midpoints of edges of $\triangle$. For $i = 1, 2, 3$, we write $\mathcal{E}_i$ for the set of edges of $\triangle_{PS12}$ of type i. Let $\tilde{\mathcal{E}}_2$ be a subset of $\mathcal{E}_2$ obtained by selecting exactly one edge of $\mathcal{E}_2$ for each macro-triangle in $\triangle$. As usual in spline theory, m_+ is defined to be m if $m > 0$, and is zero otherwise.

We now introduce the spline spaces of interest in this paper. The definition depends on the value of r mod 4. Given $r > 0$, we define the C^r Powell-Sabin-12 macro-element space to be

$$
\begin{aligned}
\mathcal{S}_r(\triangle_{PS12}) := \{s \in \mathcal{S}^r_d(\triangle_{PS12}) : & \; s \in C^\rho(v) \text{ all } v \in \mathcal{V}, \\
& s \in C^\mu(w) \text{ all } w \in \mathcal{W}, \\
& \tau s = 0 \text{ for all } \tau \in \mathcal{T}_1 \cup \mathcal{T}_2\},
\end{aligned}
\tag{3.1}
$$

where for all $\ell \geq 0$,

r	ρ	μ	d
$4\ell+1$	$6\ell+1$	$6\ell+1$	$9\ell+2$
$4\ell+2$	$6\ell+3$	$6\ell+3$	$9\ell+5$
$4\ell+3$	$6\ell+4$	$6\ell+5$	$9\ell+7$
$4\ell+4$	$6\ell+6$	$6\ell+7$	$9\ell+10$

$$
\mathcal{T}_1 := \begin{cases} \bigcup_{e \in \mathcal{E}_1} \{\tau^{r+i}_{e,\rho+j}\}_{i=1,j=1}^{2\ell-2(\ell-j+1)_+, d-\rho}, & \text{if } r \text{ is odd}, \\ \bigcup_{e \in \mathcal{E}_1} \{\tau^{r+i}_{e,\rho+j}\}_{i=1,j=1}^{2\ell+1-2(\ell-j+1)_+, d-\rho}, & \text{otherwise}, \end{cases}
$$

$$
\mathcal{T}_2 := \begin{cases} \bigcup_{e \in \mathcal{E}_2} \{\tau^{r+i}_{e,\mu+j}\}_{i=1,j=2}^{2j-2,\ell}, & r = 4\ell+1, \\ \bigcup_{e \in \mathcal{E}_2} \{\tau^{r+i}_{e,\mu+j}\}_{i=1,j=1}^{2j-1,\ell} \cup \bigcup_{e \in \widetilde{\mathcal{E}}_2} \{\tau^{r+1}_{e,\mu+\ell+1}\} \cup \bigcup_{e \in \mathcal{E}_3} \{\tau^{r+1}_{e,r+j}\}_{j=1}^{\ell}, & r = 4\ell+2, \\ \bigcup_{e \in \mathcal{E}_2} \{\tau^{r+i}_{e,\mu+j}\}_{i=1,j=1}^{2j,\ell}, & r = 4\ell+3, \\ \bigcup_{e \in \mathcal{E}_2} \{\tau^{r+i}_{e,\mu+j}\}_{i=1,j=1}^{2j+1,\ell} \cup \bigcup_{e \in \widetilde{\mathcal{E}}_2} \{\tau^{r+1}_{e,\mu+\ell+1}\} \cup \bigcup_{e \in \mathcal{E}_3} \{\tau^{r+1}_{e,r+j}\}_{j=1}^{\ell+1}, & r = 4\ell+4. \end{cases}
$$

Let n_V and n_E be the numbers of vertices and edges of $\triangle$, respectively. For each $v \in \mathcal{V}$, let T_v be some triangle in $\triangle_{PS12}$ with vertex at v. For each $e := \langle v_1, v_2 \rangle$ of $\triangle$, let v_{T_e} be the barycenter of a triangle T_e in $\triangle$ that contains e, and let $T^1_e := \langle u_1, v_1, w_e \rangle$ and $T^2_e := \langle u_2, w_e, v_2 \rangle$ be the two subtriangles of T_e sharing the edge e, where w_e is the midpoint of e. In addition, let $T^3_e := \langle v_{T_e}, w_e, u_2 \rangle$ be one of the triangles in $\triangle_{PS12}$ containing the edge $\langle w_e, v_{T_e} \rangle$, see Fig. 1 (right).

Theorem 3.2. *For all $r \geq 1$,*

$$
\dim \mathcal{S}_r(\triangle_{PS12}) = \binom{\rho+2}{2} n_V + \Big[\frac{(\mu+1)^2}{4} + \ell(\ell+1)\Big] n_E. \tag{3.2}
$$

Moreover, the set

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} (\mathcal{M}_e^1 \cup \mathcal{M}_e^2 \cup \mathcal{M}_e^3) \tag{3.3}$$

is a stable local minimal determining set for $\mathcal{S}_r(\triangle_{PS12})$, *where*

1) $\mathcal{M}_v := D_\rho(v) \cap T_v$,

2) $\mathcal{M}_e^1 := \bigcup_{i=1}^{\ell} \{\xi^{T_e^1}_{\rho+\mu-d+i+1,d-\rho-j,d-\mu-i+j-1}\}_{j=1}^{i}$,

3) $\mathcal{M}_e^2 := \bigcup_{i=1}^{\ell} \{\xi^{T_e^2}_{\rho+\mu-d+i+1,d-\mu-i+j-1,d-\rho-j}\}_{j=1}^{i}$,

4) $\mathcal{M}_e^3 := \bigcup_{j=0}^{(\mu-1)/2} \{\xi^{T_e^3}_{i+j,d-i-2j,j}\}_{i=1}^{\mu-2j}$.

Proof: To show that $\mathcal{M}$ is a stable local minimal determining set, we show that we can set the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of a spline in $\mathcal{S}_r(\triangle_{PS12})$ to arbitrary values, and that all other coefficients of s are then uniquely, locally, and stably determined. First, for each $v \in \mathcal{V}$, we set the coefficients corresponding to $\mathcal{M}_v$. Then using the C^ρ smoothness at v, we can uniquely compute the coefficients of s corresponding to all other domain points in $D_\rho(v)$. This is a stable local process.

At this point it is not obvious that the coefficients which we have determined so far are compatible with each other since they may be connected by smoothness conditions. Indeed, for any two vertices u and v which are connected by an edge of $\triangle$, there exist chains of smoothness conditions which involve coefficients in both of the disks $D_\rho(u)$ and $D_\rho(v)$ along with other yet undetermined coefficients. As we progress we have to be sure that as we compute these undetermined coefficients, all of these smoothness conditions are verified.

For each $e := \langle u, v\rangle \in \mathcal{E}$, we now apply Lemma 4.1 to determine the coefficients of s corresponding to domain points in the disk $D_\mu(w_e)$, where w_e is the midpoint of e. Due to the C^μ smoothness at w_e, we can regard the coefficients of s in this disk as coefficients of a polynomial g of degree μ. The lemma insures that we can set the coefficients of s corresponding to the domain points in $\mathcal{M}_e^3$ to arbitrary values, and that all coefficients corresponding to the remaining domain points in $D_\mu(w_e)$ are uniquely and stably determined. Since the lemma allows arbitrary values for the coefficients corresponding to domain points in the sets $D_\rho(u) \cap D_\mu(w_e)$ and $D_\rho(v) \cap D_\mu(w_e)$, it follows that all smoothness conditions connecting coefficients associated with domain points in $[D_\rho(u) \cup D_\rho(v)] \cap D_\mu(w_e)$ are satisfied, i.e., there are no incompatibilities due to these smoothness conditions. We still have to watch for possible incompatibilities due to other smoothness conditions involving domain points outside of the disks $\{D_\rho(v)\}_{v\in\mathcal{V}}$ and $\{D_\mu(w)\}_{w\in\mathcal{W}}$.

Our next step is to set the coefficients corresponding to the sets $\mathcal{M}_e^1$ and $\mathcal{M}_e^2$ for each edge e of $\triangle$. If $e := \langle v_1, v_2 \rangle$ is an interior edge of $\triangle$ with midpoint w_e, then using the C^r smoothness conditions across the edge e, we can uniquely determine the coefficients corresponding to the domain points in the sets

$$\widetilde{\mathcal{M}}_e^1 := \bigcup_{i=1}^{\ell} \{\xi^{\widetilde{T}_e^1}_{\rho+\mu-d+i+1, d-\rho-j, d-\mu-i+j-1}\}_{j=1}^{i},$$

$$\widetilde{\mathcal{M}}_e^2 := \bigcup_{i=1}^{\ell} \{\xi^{\widetilde{T}_e^2}_{\rho+\mu-d+i+1, d-\mu-i+j-1, d-\rho-j}\}_{j=1}^{i},$$

where $\widetilde{T}_e^1 := \langle \tilde{u}_1, v_1, w_e \rangle$ and $\widetilde{T}_e^2 := \langle \tilde{u}_2, w_e, v_2 \rangle$ are the triangles in $\triangle_{PS12}$ which share edges with T_e^1 and T_e^2, respectively. At this point we have made sure that all smoothness conditions up to order r across e are satisfied.

For each type-1 edge $e := \langle v, u \rangle$, we now show how to use Lemma 2.1 to compute coefficients on the rings $R_{\rho+j}(v)$ for $j = 1, \ldots, d-\rho$. Fix $1 \le j \le d-\rho$. Then it is easy to see that there are exactly $n := 2(d-\mu) - 1 - 2(\ell - j + 1)_+$ unset coefficients on $R_{\rho+j}(v)$. Now combining the C^r smoothness conditions across e with the special conditions in $\mathcal{T}_1$ associated with this edge, gives us a set of exactly n (univariate) smoothness conditions which uniquely determine these coefficients, see Lemma 2.1. By the geometry, the matrix of this nonsingular $n \times n$ linear system is the same for all edges $e \in \mathcal{E}_1$, and thus the computation is stable in the sense that (2.3) holds.

We now show that the coefficients corresponding to the remaining domain points are also uniquely determined while maintaining all smoothness conditions. These remaining domain points lie inside triangles of the form $T := \langle w_1, w_2, w_3 \rangle$, where the $w_i \in \mathcal{W}$. Let $T_{\rm PS6}$ be the Powell-Sabin-6 split of T, see Fig. 4. We have already determined all coefficients corresponding to domain points in the disks $D_\mu(w_i)$ for $i = 1, 2, 3$. In addition, by the C^r smoothness across the edges $e_i := \langle w_i, w_{i+1} \rangle$ for $i = 1, 2, 3$, the coefficients corresponding to domain points on the rings $R_{d-j}(v_T)$ for $j = 0, \ldots, r$ are also determined. For each $i = 1, 2, 3$, the fact that the midpoint u_{i-1} of e_i is a singular vertex insures that all C^r smoothness conditions across the edge $\langle u_{i-1}, v_T \rangle$ are automatically satisfied, and there are no incompatibilities. Now we can apply Lemma 4.2 to uniquely and stably determine all coefficients of s corresponding to the remaining domain points in T. We have shown that $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}_r(\triangle_{PS12})$.

To complete the proof, we note that the dimension of $\mathcal{S}_r(\triangle_{PS12})$ is equal to the cardinality of $\mathcal{M}$, which is easily seen to be the number in (3.2). $\square$

For the Powell-Sabin-12 split $T_{\rm PS12}$ of a single triangle, Tab. 1 shows the values of r, ρ, μ, d and $\dim \mathcal{S}_r(T_{\rm PS12})$ for $1 \le r \le 12$. Fig. 2 shows the corresponding minimal determining sets for $r = 1, 2, 3, 4$, where the points in $\mathcal{M}$ are marked with black dots.

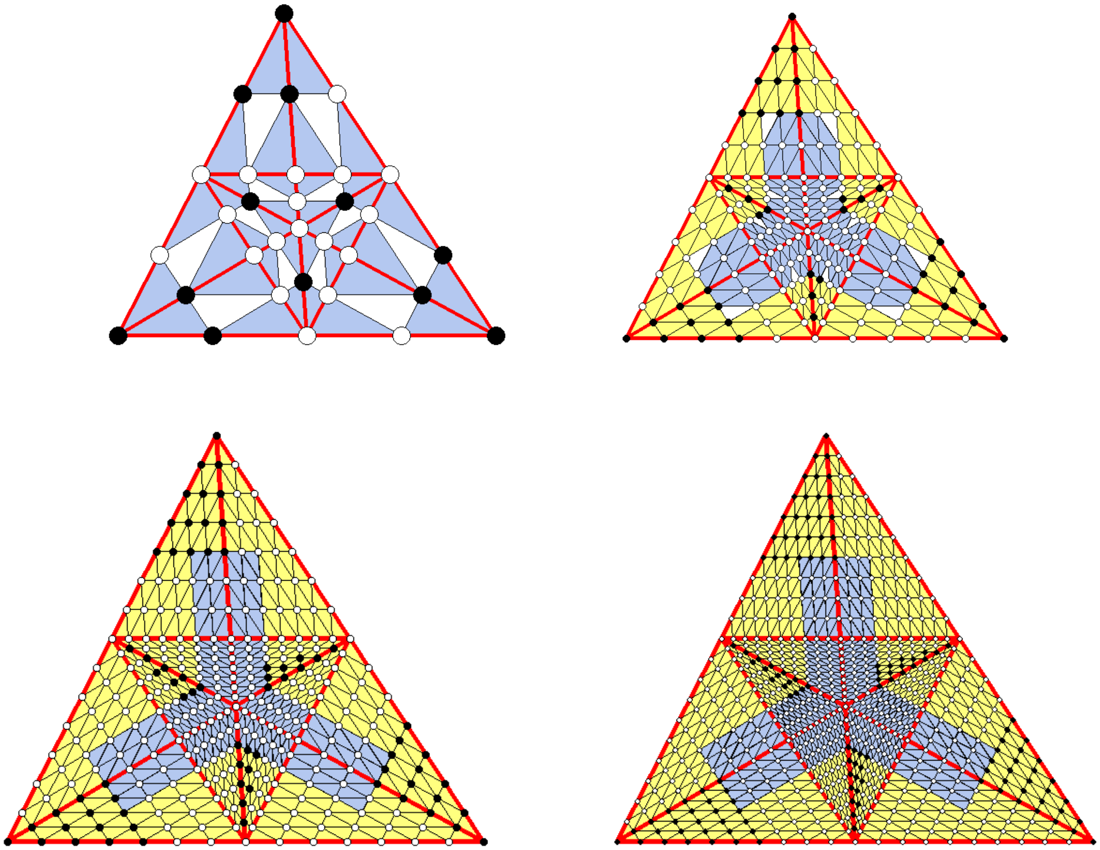

Fig. 2. Minimal determining sets for $\mathcal{S}_r(T_{\mathrm{PS12}})$ for $r = 1, 2, 3, 4$.

r	ρ	μ	d	dim
1	1	1	2	12
2	3	3	5	42
3	4	5	7	72
4	6	7	10	132
5	7	7	11	162
6	9	9	14	246
7	10	11	16	312
8	12	13	19	426
9	13	13	20	480
10	15	15	23	618
11	16	17	25	720
12	18	19	28	888

Tab. 1. The dimension of $\mathcal{S}_r(T_{\mathrm{PS12}})$.

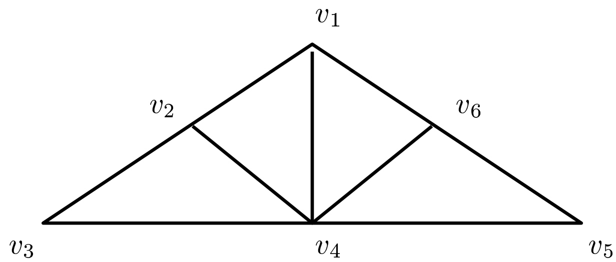

Fig. 3. The triangulation of Lemma 4.1.

§4. Two lemmas

In this section we establish two lemmas which are needed for the proof of Theorem 3.2. Our first lemma concerns a special MDS for the space of polynomials $\mathcal{P}_\mu$ in the case where μ is odd.

Lemma 4.1. *Let $\triangle$ be the triangulation shown in Fig. 3 with six vertices $v_1, \ldots, v_6$, where we suppose that v_4 is the midpoint of the edge $e := \langle v_3, v_5 \rangle$. Let $T := \langle v_1, v_4, v_6 \rangle$. Suppose μ is odd, and let $m := \frac{\mu-1}{2}$. Let $\mathcal{M} := D_m(v_3) \cup D_m(v_5) \cup \mathcal{M}_e \subset \mathcal{D}_{\triangle,\mu}$, where*

$$\mathcal{M}_e := \bigcup_{j=0}^{m} \{\xi^T_{i+j,\mu-i-2j,j}\}_{i=1}^{\mu-2j}.$$

Then $\mathcal{M}$ is a stable mimimal determining set for $\mathcal{P}_\mu$.

Proof: It is easy to check that $\#\mathcal{M} = \binom{\mu+2}{2} = \dim \mathcal{P}_\mu$, and thus it suffices to prove that if we set the coefficients of $s \in \mathcal{P}_\mu$ corresponding to $\xi \in \mathcal{M}$, then all other coefficients are stably determined. To this end, we consider the B-representation of $\tilde{s} \equiv s$ relative to the triangulation $\widetilde{\triangle}$ consisting of the two triangles $\widetilde{T}_1 := \langle v_1, v_3, v_4 \rangle$ and $\widetilde{T}_2 := \langle v_1, v_4, v_5 \rangle$. We denote the corresponding coefficients of $\tilde{s}$ by $\tilde{c}_\eta$ for $\eta \in \mathcal{D}_{\mu,\widetilde{\triangle}}$. The values of c_ξ for $\xi \in D_m(v_3) \cap \mathcal{D}_{\mu,\triangle}$ stably determine all derivatives of s up to order m at v_3, which in turn stably determine the coefficients $\tilde{c}_\eta$ for all $\eta \in D_m(v_3) \cap \mathcal{D}_{\mu,\widetilde{\triangle}}$. A similar argument shows that $\tilde{c}_\eta$ are stably determined for all $\eta \in D_m(v_5) \cap \mathcal{D}_{\mu,\widetilde{\triangle}}$.

We now claim that all coefficients of $\tilde{s}$ corresponding to domain points in the set $\widetilde{\mathcal{M}}_e := \bigcup_{j=0}^{m} \{\eta^{\widetilde{T}_2}_{i+j,\mu-i-2j,j}\}_{i=1}^{\mu-2j}$ are stably determined from the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_e}$. To see this, note that since v_6 lies on the edge $\langle v_1, v_5 \rangle$, the barycentric coordinates of v_5 relative to T have the form $(b_1, 0, b_2)$ with $b_1 + b_2 = 1$. Then using the de Casteljau algorithm to convert the B-coefficients of s relative to T into B-coefficients of $\tilde{s}$ relative to $\widetilde{T}_2$, we find that for each $1 \le i \le \mu - 2j$ and $0 \le j \le m$, the coefficient of $\tilde{s}$ corresponding to $\eta^{\widetilde{T}_2}_{i+j,\mu-i-2j,j}$ is a stable linear combination of the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_e}$.

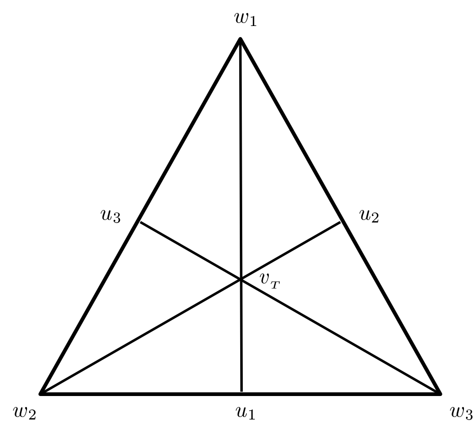

Fig. 4. The labelling of the Powell-Sabin-6 split for Lemma 4.2.

It is easy to check that $D_m^{\widetilde{T}_2}(v_1) \cap \mathcal{D}_{\mu,\widetilde{\triangle}} \subset \widetilde{\mathcal{M}}_e$. Then using the smoothness across the edge $\langle v_1, v_4 \rangle$ of $\widetilde{\triangle}$, we can stably compute the coefficients of $\tilde{s}$ corresponding to the remaining domain points in $D_m(v_1) \cap \mathcal{D}_{\mu,\widetilde{\triangle}}$. We have now determined all coefficients of $\tilde{s}$ except for those corresponding to $\mu - j$ domain points on $R_{\mu-j}(v_1) \cap \mathcal{D}_{\mu,\widetilde{\triangle}}$ for each $j = 0, \ldots, m$. Since the coefficients associated with $R_{\mu-j}(v_1)$ are subjected to precisely $\mu - j$ (univariate) smoothness conditions across the edge $\langle v_4, v_1 \rangle$, we can use Lemma 2.1 to stably compute them. Finally, to complete the proof, we note that the coefficients c_ξ of s can now be stably computed from those of $\tilde{s}$ by subdivision. $\square$

Our second lemma deals with splines on the Powell-Sabin-6 split T_{PS6} of a single triangle. Since we want to apply this lemma to the triangle $T := \langle w_1, w_2, w_3 \rangle$ which is inside the Powell-Sabin-12 split shown in Fig. 1 (right), we label its vertices as in Fig. 4, where we assume u_i is the midpoint of the edge opposite w_i for $i = 1, 2, 3$, and $v_T := (w_1 + w_2 + w_3)/3$ is the barycenter of T. As in Sect. 3, we write $\mathcal{E}_2$ for the set of edges of T_{PS6} of the form $\langle w_i, v_T \rangle$, and $\mathcal{E}_3$ for the set of edges of T_{PS6} of the form $\langle u_i, v_T \rangle$. Given r, μ, and d as in (3.1), let $\mathcal{T}_2$ be the corresponding set of special smoothness conditions defined there. Then we consider the following superspline space

$$\begin{aligned} \mathcal{S}_r(T_{\text{PS6}}) := \{ s \in \mathcal{S}_d^r(T_{\text{PS6}}) : s \in C^\mu(w_i) \text{ all } i = 1, 2, 3, \\ \tau s = 0 \text{ for all } \tau \in \mathcal{T}_2 \}. \end{aligned} \tag{4.1}$$

For each $n = 1, 2, 3$, let $T_n := \langle v_T, w_n, u_{n+2} \rangle$.

Lemma 4.2. *For all* $r \geq 1$,

$$\dim \mathcal{S}_r(T_{\text{PS6}}) = 3\binom{\mu + 2}{2} + 3(2d - 2\mu - r - 1)(r + 1). \tag{4.2}$$

Moreover, the set

$$\mathcal{M} := \bigcup_{n=1}^{3} \big(\mathcal{M}_{w_n} \cup \mathcal{M}_{u_n}\big), \tag{4.3}$$

is a stable minimal determining set, where $\mathcal{M}_{w_n} := D_\mu(w_n) \cap T_n$ *and*

$$\mathcal{M}_{u_n} := \begin{cases} \bigcup_{j=1}^{2\ell} \{\xi^{T_n}_{i,d-\mu-j,\mu-i+j}\}_{i=0}^{r}, & \text{if } r \text{ is odd,} \\ \bigcup_{j=1}^{2\ell+1} \{\xi^{T_n}_{i,d-\mu-j,\mu-i+j}\}_{i=0}^{r}, & \text{otherwise.} \end{cases} \tag{4.4}$$

Proof: We show below that $\mathcal{M}$ is a determining set, and thus $\dim \mathcal{S}_r(T_{\rm PS6}) \le \#\mathcal{M}$. It is easily seen that

$$\#\mathcal{M} = \begin{cases} 78\ell^2 + 57\ell + 9, & r = 4\ell+1, \\ 78\ell^2 + 111\ell + 39, & r = 4\ell+2, \\ 78\ell^2 + 141\ell + 63, & r = 4\ell+3, \\ 78\ell^2 + 195\ell + 123, & r = 4\ell+4, \end{cases} \tag{4.5}$$

which is equal to the expression in (4.2). We now derive a lower bound for $\dim \mathcal{S}_r(T_{\rm PS6})$. By Theorem 2.2 of [16],

$$\dim \mathcal{S}^r_d(T_{\rm PS6}) = \binom{r+2}{2} + 6\binom{d-r+1}{2} + \sigma,$$

where

$$\sigma = \begin{cases} (r^2-1)/4, & \text{if } r \text{ is odd,} \\ r^2/4, & \text{if } r \text{ is even.} \end{cases}$$

Enforcing the C^μ continuity at the vertices w_1, w_2, w_3 of T requires $3\binom{\mu-r+1}{2}$ conditions. Since

$$\#\mathcal{T}_2 = \begin{cases} 3\ell^2 - 3\ell, & r = 4\ell+1, \\ 3\ell^2 + 3\ell + 1, & r = 4\ell+2, \\ 3\ell^2 + 3\ell, & r = 4\ell+3, \\ 3\ell^2 + 9\ell + 4, & r = 4\ell+4, \end{cases}$$

the dimension of $\mathcal{S}_r(T_{\rm PS6})$ in each of the four cases is bounded below by the same quantities appearing in (4.5), which implies that $\dim \mathcal{S}_r(T_{\rm PS6})$ is given by the formula in (4.2).

To complete the proof, we need to show that $\mathcal{M}$ is a determining set and that it is stable. Suppose we set $\{c_\xi\}_{\xi\in\mathcal{M}}$. We now show how to use the smoothness conditions to stably compute all other coefficients. For each $i = 1, 2, 3$, we use the

C^μ smoothness at w_i to stably compute the coefficients corresponding to all other domain points in $D_\mu(w_i)$. Next, consider the sets of domain points

$$L^i_j := R_{d-j}(v_T) \cap \langle v_T, w_{i+1}, w_{i+2}\rangle, \qquad j = 0, \ldots, d,$$

for $i = 1, 2, 3$. For each $0 \le j \le r$, L^i_j contains exactly r domain points for which the corresponding coefficients have not yet been determined. The C^r smoothness across the edge $\langle u_i, v_T\rangle$ gives r univariate smoothness conditions involving these coefficients, and they can thus be determined from Lemma 2.1. At this point the proof divides into four cases.

Case 1. $r = 4\ell+1$. For each $j = 1, \ldots, \ell$ and $i = 1, 2, 3$, there are $r+2j-2$ domain points on $R_{\mu+j}(w_i)$ whose corresponding coefficients have not yet been determined. These coefficients are subject to $r+2j-2$ smoothness conditions which correspond to the C^r smoothness conditions combined with the $2j-2$ functionals in $\mathcal{T}_2$. We can again get these coefficients from Lemma 2.1. Now for each $r+1 \le j \le r+\ell$ and $i = 1, 2, 3$, the set L^i_j contains r domain points whose corresponding coefficients have not yet been computed. But then using the C^r smoothness across the edge $\langle u_i, v_T\rangle$, Lemma 2.1 gives the values of these coefficients. To complete the proof in this case, we use Lemma 2.1 to perform the following cycle of computations: for each $j = 1, \ldots, d-\mu-\ell$:

a) compute the r unset coefficients on the ring $R_{\mu+\ell+j}(w_i)$ for $i = 1, 2, 3$,

b) compute the $r-2j$ unset coefficients on $L^i_{r+\ell+j}$ for $i = 1, 2, 3$.

This cycle of computations gives all the remaining coefficients.

Case 2. $r = 4\ell+2$. For each $j = 1, \ldots, \ell$ and $i = 1, 2, 3$ there are $r+2j-1$ domain points on $R_{\mu+j}(w_i)$ whose corresponding coefficients have not yet been determined. These coefficients are subject to $r+2j-1$ smoothness conditions obtained by combining the C^r smoothness with the $2j-1$ functionals in $\mathcal{T}_2$ corresponding to the set $\mathcal{E}_2$. We can thus compute these coefficients from Lemma 2.1. Now for each $r+1 \le j \le r+\ell$ and $i = 1, 2, 3$, the set L^i_j contains $r+1$ domain points whose corresponding coefficients have not yet been computed. But then using the C^r smoothness across the edge $e := \langle u_i, v_T\rangle$ together with the smoothness condition corresponding to $\tau^{r+1}_{e,r+j}$ in $\mathcal{T}_2$ with $e \in \mathcal{E}_3$, Lemma 2.1 gives these coefficients. Now for each $i = 1, 2, 3$, we examine the ring $R_{\mu+\ell+1}(w_1)$, where we assume the edge $e := \langle w_1, v_T\rangle$ is the edge chosen for $\widetilde{\mathcal{E}}_2$. There are $r+1$ domain points on this ring whose corresponding coefficients are not yet determined. Using the smoothess condition described by the functional $\tau^{r+1}_{e,\mu+\ell+1}$ in $\mathcal{T}_2$, we can use Lemma 2.1 to compute all of these coefficients. The lemma then gives the coefficients corresponding to domain points on the layers $L^i_{r+\ell+1}$ for $i = 2, 3$. We can now do the two rings $R_{\mu+\ell+1}(w_2)$ and $R_{\mu+\ell+1}(w_3)$, followed by the layer $L^1_{r+\ell+1}$. To complete the proof in this case, we use Lemma 2.1 to perform the following cycle of computations: for each $j = 2, \ldots, d-\mu-\ell$:

a) compute the $r-2j+3$ unset coefficients on the layer $L^i_{\mu+\ell+j}$ for $i = 1, 2, 3$,

b) compute the $r-1$ unset coefficients on the ring $R_{\mu+\ell+j}(w_i)$ for $i=1,2,3$.

This cycle of computations shows that all the remaining coefficients are determined.

The cases $r=4\ell+3$ and $r=4\ell+4$ can be handled in a similar way, and the proof is complete. □

§5. Approximation Power

Let $\triangle$ be a triangulation of a polygonal domain Ω, and let $\mathcal{S}_r(\triangle_{PS12})$ be the macro-element space defined in (4.1). Let $|\triangle|$ be the mesh size of $\triangle$, i.e., the diameter of the largest triangle in $\triangle$. In this section we use the fact that $\mathcal{S}_r(\triangle_{PS12})$ has a stable local minimal determining set $\mathcal{M}$ to show that the space has full approximation power. More precisely, we give bounds on how well functions f in Sobolev spaces $W_q^{m+1}(\Omega)$ can be approximated in terms of $|\triangle|$ and the smoothness of f as measured by the usual Sobolev semi-norm $|f|_{m+1,q,\Omega}$. Let

$$\|g\|_{q,\Omega} := \begin{cases} \left(\sum_{T\in\triangle} \|g\|_{q,T}^q\right)^{1/q}, & 1\le q<\infty, \\ \max_{T\in\triangle} \|g\|_{\infty,T}, & q=\infty. \end{cases}$$

Unless otherwise stated, all constants appearing in this section depend only on the smallest angle θ in the triangulation $\triangle_{PS12}$, or equivalently on the smallest angle in $\triangle$, see Remark 7.8. It is easy to see that $|\triangle_{PS12}| \le |\triangle|/2$.

Theorem 5.1. *For all $f\in W_q^{m+1}(\Omega)$ with $1\le q\le\infty$ and $m\le d$, there exists a spline $s_f\in\mathcal{S}_r(\triangle_{PS12})$ such that*

$$\left\|D_x^\alpha D_y^\beta (f-s_f)\right\|_{q,\Omega} \le C\ |\triangle|^{m+1-\alpha-\beta}\ |f|_{m+1,q,\Omega}, \tag{5.1}$$

for all $0\le\alpha+\beta\le m$. Here the constant C depends only on the smallest angle in $\triangle$, and if $q<\infty$ also on the Lipschitz constant associated with the boundary of Ω.

Proof: We give the proof only for $1\le q<\infty$. The case $q=\infty$ is similar and simpler. We begin by constructing a quasi-interpolant Q mapping $L_1(\Omega)$ into the spline space $\mathcal{S}_r(\triangle_{PS12})$. Fix $f\in L_1(\Omega)$. Then for each triangle $T\in\triangle_{PS12}$, we choose the largest disk contained in T, and let F_T be the corresponding averaged Taylor polynomial of degree d approximating f, see e.g. [10]. Then for each $\xi\in\mathcal{M}\cap T$, let $c_\xi := \gamma_\xi(F_T f)$, where γ_ξ is the linear functional which picks off the B-coefficient associated with domain point ξ. We now define Qf be the spline in $\mathcal{S}_r(\triangle_{PS12})$ whose other coefficients are determined from $\{c_\xi\}_{\xi\in\mathcal{M}}$ by using smoothness conditions as in the proof of Theorem 3.2. Q is a linear projector mapping $L_1(\Omega)$ onto $\mathcal{S}_r(\triangle_{PS12})$.

Using the L_q stability of the B-form and properties of F_T, see [10], we have

$$|c_\xi| = |\gamma_\xi(F_T f)| \le \frac{K_1}{A_T^{1/q}}\|F_T f\|_{q,T} \le \frac{K_1K_2}{A_T^{1/q}}\|f\|_{q,T}, \qquad \xi\in\mathcal{M}\cap T,$$

where A_T is the area of T. By the locality and stability of $\mathcal{M}$, it follows that if η is a domain point lying in T, then

$$|c_\eta| \le \frac{K_1 K_2 K_3}{A_{min}^{1/q}} \|f\|_{q,\Omega_T},$$

where A_{min} is the area of the smallest triangle in $\Omega_T := \mathrm{star}^3(T)$. It is shown in [10] that the area of the largest triangle in Ω_T is bounded by a constant (depending only on θ) times the area of the smallest triangle. Using the fact that the Bernstein basis polynomials form a partition of unity, we get $\|Qf\|_{q,T} \le K_4 \|f\|_{q,\Omega_T}$.

Now suppose $f \in W_q^{m+1}(\Omega)$ with $m \le d$. Fix $0 \le \alpha+\beta \le m$ and $T \in \triangle_{PS12}$. Then using the Markov inequality [18], it follows that for any $p \in \mathcal{P}_d$,

$$\begin{aligned} \|D_x^\alpha D_y^\beta (f - Qf)\|_{q,T} &\le \|D_x^\alpha D_y^\beta (f-p)\|_{q,T} + \|D_x^\alpha D_y^\beta Q(f-p)\|_{q,T} \\ &\le \|D_x^\alpha D_y^\beta (f-p)\|_{q,T} + \frac{K_5}{\rho_T^{\alpha+\beta}} \|Q(f-p)\|_{q,T} \\ &\le \|D_x^\alpha D_y^\beta (f-p)\|_{q,T} + \frac{K_4 K_5}{\rho_T^{\alpha+\beta}} \|f-p\|_{q,\Omega_T}, \end{aligned} \tag{5.2}$$

where ρ_T is the diameter of the largest disk contained in T. It is shown in [10] that $|\Omega_T| \le K_6 \rho_T$. Now (cf. Lemma 4.6 of [10]), there exists a polynomial $p \in \mathcal{P}_m$ depending on f with

$$\left\|D_x^i D_y^j (f-p)\right\|_{q,\Omega_T} \le K_7 |\Omega_T|^{m+1-i-j} |f|_{m+1,q,\Omega_T}, \tag{5.3}$$

for all $0 \le i+j \le m$, where K_7 is a constant depending on θ and the Lipschitz constant of the boundary of Ω. Inserting this in (5.2) leads to

$$\left\|D_x^\alpha D_y^\beta (f - Qf)\right\|_{q,T} \le K_8 \, |\triangle|^{m+1-\alpha-\beta} \, |f|_{m+1,q,\Omega_T}, \qquad \text{all } 0 \le \alpha+\beta \le m. \tag{5.4}$$

Summing over all triangles $T \in \triangle_{PS12}$ and using the fact that the number of triangles in Ω_T is bounded by a constant depending only on θ, we get (5.1). □

§6. A nodal determining set for $\mathcal{S}_r(\triangle_{PS12})$

In this section we describe a nodal minimal determining set for $\mathcal{S}_r(\triangle_{PS12})$ and a corresponding Hermite interpolation projector. For each triangle T in $\triangle$, let v_T be its barycenter. For each edge $e := \langle u, v\rangle$ of $\triangle$, let w_e be its midpoint, and let $w_e^1 := \frac{\mu u + (d-\mu) w_e}{d}$ and $w_e^2 := \frac{\mu v + (d-\mu) w_e}{d}$. Let D_e be the directional derivative associated with a unit vector perpendicular to e. For each $i > 0$, let

$$\begin{aligned} \eta_{e,1,j}^i &:= \frac{(i-j+1)u + j w_e^1}{i+1}, \\ \eta_{e,2,j}^i &:= \frac{(i-j+1)v + j w_e^1}{i+1}, \\ \eta_{e,3,j}^i &:= \frac{(i-j+1)w_e^1 + j w_e^2}{i+1}, \end{aligned}$$

for $j = 1, \dots, i$. Finally, for any point $t \in \mathbb{R}^2$, let ε_t be the point evaluation functional at t.

Theorem 6.1. *The set*

$$\mathcal{N} := \bigcup_{v \in \mathcal{V}} \mathcal{N}_v \cup \bigcup_{e \in \mathcal{E}} \left(\mathcal{N}_e^1 \cup \mathcal{N}_e^2 \cup \mathcal{N}_e^3\right) \tag{6.1}$$

is a nodal determining set for $\mathcal{S}_r(\triangle_{PS12})$, *where*

1) $\mathcal{N}_v := \{\varepsilon_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le \rho}$,
2) $\mathcal{N}_e^1 := \bigcup_{i=1}^{\ell} \{\varepsilon_{\eta_{e,1,j}^i} D_e^{\rho+\mu-d+1+i}\}_{j=1}^i$,
3) $\mathcal{N}_e^2 := \bigcup_{i=1}^{\ell} \{\varepsilon_{\eta_{e,2,j}^i} D_e^{\rho+\mu-d+1+i}\}_{j=1}^i$,
4) $\mathcal{N}_e^3 := \bigcup_{i=1}^{m} \{\varepsilon_{\eta_{e,3,j}^i} D_e^{i}\}_{j=1}^i \cup \bigcup_{i=1}^{m+1} \{\varepsilon_{\eta_{e,3,j}^i} D_e^{\mu-i+1}\}_{j=1}^i$,

with $m = (\mu-1)/2$.

Proof: It is easy to check that the cardinality of $\mathcal{N}$ is equal to the dimension of $\mathcal{S}_r(\triangle_{PS12})$ as given in (3.2). Thus, it suffices to show that $\mathcal{N}$ is a nodal determining set, i.e., setting $\{\lambda s\}_{\lambda \in \mathcal{N}}$ determines all coefficients of s. For every vertex v of $\triangle$, we can compute all coefficients corresponding to domain points in the disk $D_\rho(v)$ directly from the data $\{\lambda s\}_{\lambda \in \mathcal{N}_v}$.

Given an edge e of $\triangle$, let w_e be its midpoint. We now compute all coefficients of s corresponding to domain points in $D_\mu(w_e)$. By the C^μ smoothness at w_e, these coefficients can be regarded as the coefficients of a polynomial g of degree μ. Suppose we represent this polynomial in B-form relative to the triangle $\widetilde{T} := \langle v_e, w_e^1, w_e^2 \rangle$, where $v_e := (\mu v_{T_e} + (d-\mu) w_e)/d$ and v_{T_e} is the center of some triangle T_e containing the edge e. As in Lemma 4.1, we can immediately compute the coefficients of g in the disks $D_m(w_e^1) \cap \mathcal{D}_{\mu,\widetilde{T}}$ and $D_m(w_e^2) \cap \mathcal{D}_{\mu,\widetilde{T}}$. For each $i = 1, \dots, \mu$, we now compute the coefficients of g corresponding to the remaining domain points on $R_{\mu-i}(v_e) \cap \mathcal{D}_{\mu,\widetilde{T}}$ from the derivative information given in 4). We can now get the coefficients of s corresponding to domain points in $D_\mu(w_e) \cap T_e$ by applying subdivision to $\widetilde{T}$. If e is an interior edge, the coefficients of s corresponding to the remaining domain points in $D_\mu(w_e)$ can be computed from the C^μ smoothness at w_e.

Next for each edge e, we use the values $\{\lambda s\}_{\lambda \in \mathcal{N}_e^1}$ to compute all coefficients of s corresponding to domain points in $\mathcal{M}_e^1$. First, we consider $i = 1$ in the definition of $\mathcal{M}_e^1$, i.e., the domain point $\xi^{T_e^1}_{\rho+\mu-d+2,d-\rho-1,d-\mu-1}$. This coefficient is determined by $D_{e,1}^{\rho+\mu-d+2} s(\eta_{e,1,1}^1)$, since all other coefficients involved in this derivative have already been computed. Then assuming we have dealt with the points in $\mathcal{M}_e^1$ up to $i-1$, we can use the values $\{D_{e,1}^{\rho+\mu-d+1+i} s(\eta_{e,1,j}^i)\}_{j=1}^i$ to find the coefficients corresponding to $\{\xi^{T_e^1}_{\rho+\mu-d+i+1,d-\rho-j,d-\mu-i+j-1}\}_{j=1}^i$. This involves solving an $i \times i$ linear system. A similar argument leads to the coefficients of s corresponding to domain points in $\mathcal{M}_e^2$.

At this point we have determined all coefficients corresponding to domain points in the minimal determining set $\mathcal{M}$ of Theorem 3.2, and it follows from that theorem that all other coefficients are also determined. □

Theorem 6.1 shows that for any function $f \in C^\mu(\Omega)$, there is a unique spline $s \in \mathcal{S}_r(\triangle_{PS12})$ solving the Hermite interpolation problem $\lambda s = \lambda f$ for all $\lambda \in \mathcal{N}$. The mapping which takes functions $f \in C^\mu(\Omega)$ to this Hermite interpolating spline defines a linear projector $\mathcal{I}$ mapping $C^\mu(\Omega)$ onto $\mathcal{S}_r(\triangle_{PS12})$. We now give an error bound for how well $\mathcal{I}f$ approximates smooth functions f in the maximum norm. We write $|\triangle|$ for the mesh size of the initial trianglulation $\triangle$ before applying the Powell-Sabin-12 splits.

Given a triangle $T \in \triangle$ and a domain point $\xi \in T$ of $\mathcal{S}_r(\triangle_{PS12})$, it is easy to see that if the coefficient c_ξ of $\mathcal{I}f$ is computed from derivatives as in the proof of Theorem 6.1, then

$$|c_\xi| \le K_1 \sum_{\nu=0}^{\mu} |T|^\nu |f|_{\nu,T}, \tag{6.2}$$

where K_1 is a constant depending only on the smallest angle in $\triangle$. Since the computation of all other coefficients from smoothness conditions (cf. the proofs of Theorems 3.2 and 6.1) is a stable process, it follows that (6.2) holds for all domain points ξ lying in T. Since the Bernstein basis polynomials form a partition of unity, (6.2) implies

$$\|\mathcal{I}f\|_T \le K_1 \sum_{\nu=0}^{\mu} |T|^\nu |f|_{\nu,T}. \tag{6.3}$$

Theorem 6.2. *There exists a constant K depending only on the smallest angle in $\triangle$ such that for every $f \in C^{m+1}(\Omega)$ with $\mu - 1 \le m \le d$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_\Omega \le K |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,\Omega}, \tag{6.4}$$

for all $0 \le \alpha + \beta \le m$.

Proof: Fix $f \in C^{m+1}(\Omega)$ and a triangle $T \in \triangle$. Then Lemma 4.6 in [10] implies that there exists a polynomial $p \in \mathcal{P}_m$ such that

$$\|D_x^i D_y^j (f - p)\|_T \le K_2 |T|^{m+1-i-j} |f|_{m+1,T}, \tag{6.5}$$

for all $0 \le i+j \le m$. Now fix $0 \le \alpha+\beta \le m$. Then since $\mathcal{I}$ reproduces polynomials, we have

$$\|D_x^\alpha D_y^\beta (f - \mathcal{I}f)\|_T \le \|D_x^\alpha D_y^\beta (f - p)\|_T + \|D_x^\alpha D_y^\beta \mathcal{I}(f - p)\|_T,$$

and to complete the proof it suffices to estimate the second term. Let $T_1,\ldots,T_{12}$ be the subtriangles in the Powell-Sabin-12 split of T. Then using the Markov inequality, cf. [10,18], it follows that

$$\|D_x^\alpha D_y^\beta \mathcal{I}(f-p)\|_{T_j} \le \frac{K_3}{\rho_{T_j}^{\alpha+\beta}} \|\mathcal{I}(f-p)\|_{T_j} \le \frac{K_1 K_3}{\rho_{T_j}^{\alpha+\beta}} \sum_{\nu=0}^{\mu} |T|^\nu |f-p|_{\nu,T}, \tag{6.6}$$

for all $j = 1,\ldots,12$, where ρ_{T_j} is the diameter of the largest disk contained in T_j. By the geometry of the Powell-Sabin-12 split, $|T| \le K_4 \rho_{T_j}$, and taking the maximum over all $T \in \triangle$, we immediately get (6.4). □

§7. Remarks

Remark 7.1. We were first motivated to construct a family of smooth macro-elements on the Powell-Sabin-12 split after hearing a lecture by Rong-Qing Jia in which he used a mixture of C^1 Powell-Sabin-6 and Powell-Sabin-12 elements in order to construct continuously differentiable wavelets on triangulations, see [8].

Remark 7.2. It was shown in [5,12] that it is not possible to construct C^r macro-elements on the Powell-Sabin-6 split using splines of lower degree than those considered here. Here we have constructed our macro-elements on Powell-Sabin-12 splits with the same degrees for the purposes of compatibility, cf. Remark 7.1. However, due to the special geometry of the Powell-Sabin-6 split of the triangle $\langle w_1, w_2, w_3\rangle$ inside the Powell-Sabin-12 split (see Definition 3.1 and Fig. 4), we have found that it is possible to construct macro-elements in the Powell-Sabin-12 case with lower degrees. We plan to report on this elsewhere.

Remark 7.3. The Powell-Sabin-12 split was introduced in [14], where it was used to define a C^1 macro-element based on quadratic splines. This corresponds to our element for $r = 1$. In this case the macro-element space has dimension 12, and the nodal degrees of freedom consist of the values and gradients at the three vertices of T along with one cross-boundary derivative at the midpoint of each edge, see Fig. 2.

Remark 7.4. The C^r macro-elements constructed in [5,12] provide global C^r smoothness for a triangulation $\triangle$ which has been refined with Powell-Sabin-6 splits only if for each interior edge e of $\triangle$, the split point w_e on the edge e lies on the line joining the interior points v_T and $\tilde{v}_T$ of the two triangles T and $\widetilde{T}$ which share e, and thus in general, w_e will not be at the midpoint of e. This geometric constraint is not required for our Powell-Sabin-12 macro-elements.

Remark 7.5. In developing the macro-element spaces of this paper, we have made extensive use of the java code of Alfeld for examining determining sets for superspline spaces. The code is described in [1], and can be used or downloaded from `http://www.math.utah.edu/~alfeld`. The code not only checks whether a given set of domain points is a MDS, but also produces the equations needed to compute all unset coefficients from those that have been set.

Remark 7.6. The construction described here is not unique in the sense that there are other choices of the extra smoothness conditions which also lead to macro-elements based on the degrees of freedom used here.

Remark 7.7. Frequently in practice one has to interpolate given values at scattered data points where no derivative information is provided. In this case, macro-element methods can still be applied, but the needed derivatives (or the equivalent set of B-coefficients) have to be estimated from the data.

Remark 7.8. Simple trigonometry shows that if $T_{\rm PS12}$ is the Powell-Sabin split of a triangle T, then $\sin(\theta_{PS}) \geq \sin(\theta)/3$, where θ_{PS} is the smallest angle in $T_{\rm PS12}$ and θ is the smallest angle in T.

Remark 7.9. In [3] it was noted that the classical C^1 Clough-Tocher and Powell-Sabin macro-elements have natural analogs in terms of spherical splines. Since the algebra of spherical splines is essentially the same as for bivariate splines [2], it is clear that the entire family of macro-elements constructed here can also be immediately carried over to the sphere.

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