Designing NURBS Cam Profiles using Trigonometric Splines

by

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Abstract. We show how to design cam profiles using NURBS curves whose support functions are appropriately scaled trigonometric splines. In particular, we discuss the design of cams with various side conditions of practical interest, such as interpolation conditions, constant diameter, minimal acceleration or jerk, and constant dwells. In contrast to general polynomial curves, these NURBS curves have the useful property that their offsets are of the same type, and hence also have an exact NURBS representation.

1. Introduction

There are a number of schemes based on ordinary polynomial splines for the design of displacement functions describing cam profiles, see e.g. [20,21]. However, the existing schemes have the drawback that the parametric representation of the resulting cam profiles is not suitable for immediate practical application. In particular, these profile curves are not industry standard NURBS curves. Thus, to use polynomial splines in the manufacturing process, their representations must be converted (approximately) to a form accepted by CNC milling machines.

The purpose of this paper is to show that trigonometric splines are an attractive alternative to classical polynomial splines for cam design. Indeed, as observed in

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our paper [13], trigonometric splines can be used to define rational curves, and therefore do not need conversion as the polynomial splines do. Moreover, curves constructed with trigonometric splines have the useful property that their offsets are also rational, and hence have the same convenient representation.

In this paper we show how trigonometric splines can be used to to design profiles for cam mechanisms with a flat face follower. A typical objective is to determine a cam shape by means of a displacement (or support) function satisfying a given set of side conditions, including for example prescribed values of the displacement function and its first and second derivatives (velocity and acceleration of the cam). In addition, we may seek to enforce certain global characteristics on the cam, such as constant diameter, or small values for the maximum acceleration or jerk.

2. Curves Defined by Support Functions

In this section we review properties of curves defined by support functions. This is the natural setting for the design of profiles of cam mechanisms with a flat face follower.

Let $h(\theta)$ be a positive continuous function defined on an interval $I = [a, b]$. One natural way to associate a curve in $\mathbb{R}^2$ with h is via its *polar graph*

$$G_h := \{P(\theta) := (x(\theta), y(\theta)) \ : \ a \le \theta \le b\}, \tag{2.1}$$

where $x(\theta) = h(\theta)\cos(\theta)$ and $y(\theta) = h(\theta)\sin(\theta)$. If $[a, b] = [0, 2\pi]$ and if h is 2π-periodic, then G_h is a *closed curve*. The polar representation is useful for the description of profiles for cam mechanisms with a roller follower.

An alternative way to associate a curve with a univariate function $h(\theta)$ can be used for the design of profiles of cam mechanisms with a *flat face follower*, see Fig. 1. For each θ, let $L(\theta)$ be the line passing through the point $P(\theta)$ in (2.1) which is perpendicular to the ray $\overline{OP(\theta)}$ from the origin to $P(\theta)$, and let

$$C_h := \{\text{the envelope of the lines } L(\theta) \ : \ a \le \theta \le b\}. \tag{2.2}$$

The function h is called the *support function* of the curve C_h. In the present application, it describes the *displacement* of the follower as a function of the rotation angle of the cam.

The curve C_h can also be expressed as a *parametric curve*. In fact [19],

$$C_h = \{(x(\theta), y(\theta)) \ : \ a \le \theta \le b\},$$

where now

$$x(\theta) = h(\theta)\cos(\theta) - h'(\theta)\sin(\theta),$$
$$y(\theta) = h(\theta)\sin(\theta) + h'(\theta)\cos(\theta).$$

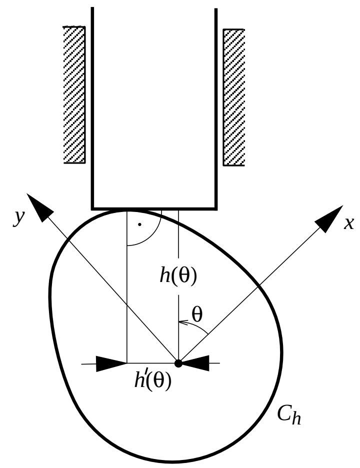

Fig. 1. Cam mechanism with a flat face follower.

This representation also reflects the fact that the curve normals have distance $h'(\theta)$ from the origin, see Fig. 1.

The *signed radius of curvature* of the curve C_h is given by

$$\rho(\theta) := h(\theta) + h''(\theta). \tag{2.3}$$

Thus, a closed curve C_h is *convex* if and only if $\rho(\theta) \geq 0$ for all $0 \leq \theta \leq 2\pi$.

Using support functions one can easily describe offset curves of a given curve: the *offset curve at signed distance d from C_h* is C_{h_d} with $h_d(\theta) := d + h(\theta)$. The closed curve C_h agrees with its offset at distance $-d$ and thus is of *constant diameter* $d > 0$ provided

$$h(\theta) + h(\theta + \pi) = d, \quad \theta \in [0, \pi). \tag{2.4}$$

For more details on the use of support functions for describing (constant diameter) curves, see [19].

3. Trigonometric Splines

In this section we briefly recall some facts about trigonometric splines. For more details, see [18]. Given integers $N \geq 0, k \geq 1$, a positive constant α, and an interval $I = [a, b]$, let

$$t_1 \leq t_2 \leq \cdots \leq t_{N+k} \tag{3.1}$$

be a *knot sequence* such that

$$0 < t_{i+k} - t_i < \pi/\alpha, \quad i = 1, \ldots, N. \tag{3.2}$$

Associated with the knot sequence, let

$$T_i^1(\theta) := \begin{cases} 1, & \text{if } t_i \leq \theta < t_{i+1} \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

and

$$T_i^k(\theta) := \frac{s(\theta - t_i)}{s(t_{i+k-1} - t_i)} T_i^{k-1}(\theta) + \frac{s(t_{i+k} - \theta)}{s(t_{i+k} - t_{i+1})} T_{i+1}^{k-1}(\theta), \qquad k > 1, \tag{3.4}$$

where from now on $s(\theta) := \sin(\alpha\theta)$ and $c(\theta) := \cos(\alpha\theta)$. Here T_i^k is defined to be identically zero if $t_{i+k} = t_i$, and terms in (3.4) with zero denominator are treated as zero. The T_i^k are the well-known *normalized trigonometric B-splines of order* k. Each $T_i^k(\theta)$ is positive for $\theta \in (t_i, t_{i+k})$, and is zero for all $\theta \notin [t_i, t_{i+k}]$. Moreover, on each subinterval (t_i, t_{i+1}), the B-spline T_i^k belongs to the space of *trigonometric polynomials of order* k, defined by

$$\mathcal{T}_k := \begin{cases} \operatorname{span}\{1, s(2\theta), c(2\theta), s(4\theta), c(4\theta), \ldots, s((k-1)\theta), c((k-1)\theta)\}, & k \text{ odd} \\ \operatorname{span}\{s(\theta), c(\theta), s(3\theta), c(3\theta), \ldots, s((k-1)\theta), c((k-1)\theta)\}, & k \text{ even.} \end{cases}$$

The space $\mathcal{S}$ of *trigonometric splines of order* k is defined as the linear span of the B-splines, namely

$$\mathcal{S} := \left\{ \sum_{i=1}^{N} a_i T_i^k, \quad a_i \in \mathbb{R}, \quad i = 1, \ldots, N \right\}.$$

When k is odd there exist coefficients ξ_i^k such that

$$1 \equiv \sum_{i=1}^{N} \xi_i^k \, T_i^k(\theta), \qquad t_k \le \theta \le t_{N+1}. \tag{3.5}$$

The coefficients ξ_i^k in the partition of unity (3.5) were computed in [11] using results on trigonometric polar forms. In particular, it follows from Lemma 5.3 of [11] that

$$\xi_i^k = \frac{1}{(k-1)!} \sum_{\mu} \prod_{j=1}^{(k-1)/2} c(t_{i+\mu(2j)} - t_{i+\mu(2j-1)}), \tag{3.6}$$

where the sum is taken over all permutations $\mu : \{1, \ldots, k-1\} \to \{1, \ldots, k-1\}$. The cases of $k = 3$ and $k = 5$ are of particular practical interest:

$$\xi_i^3 = c(t_{i+2} - t_{i+1}), \tag{3.7}$$

and

$$\xi_i^5 = \frac{1}{3}\big[c(t_{i+2} - t_{i+1})c(t_{i+4} - t_{i+3}) + c(t_{i+3} - t_{i+1})c(t_{i+4} - t_{i+2}) + c(t_{i+4} - t_{i+1})c(t_{i+3} - t_{i+2})\big], \tag{3.8}$$

for $i = 1, \ldots, N$.

The first derivative of a trigonometric spline is

$$DT_i^k(\theta) := \alpha(k-1)\left[\frac{c(\theta - t_i)T_i^{k-1}(\theta)}{s(t_{i+k-1} - t_i)} - \frac{c(t_{i+k} - \theta)T_{i+1}^{k-1}(\theta)}{s(t_{i+k} - t_{i+1})}\right], \qquad (3.9)$$

see [12]. Higher derivatives can be calculated by using this formula recursively. Thus, for example, the second derivative is

$$D^2T_i^k(\theta) := \alpha(k-1)\left[\frac{c(\theta - t_i)DT_i^{k-1}(\theta) - \alpha s(\theta - t_i)T_i^{k-1}(\theta)}{s(t_{i+k-1} - t_i)} - \frac{c(t_{i+k} - \theta)DT_{i+1}^{k-1}(\theta) + s(t_{i+k} - \theta)T_{i+1}^{k-1}(\theta)}{s(t_{i+k} - t_{i+1})}\right]. \qquad (3.10)$$

4. Connection with Rational Curves

Curves G_h of the form (2.1) with $h(\theta) = 1/f(\theta)$, where f is a trigonometric spline, have been studied in several papers of Sánchez-Reyes [16,17], although without reference to the trigonometric spline literature. They were also studied independently by de Casteljau [2], who calls them *focal splines*, and more recently by Casciola and Morigi [1] who call them *p-splines*. These curves can be viewed as *rational geometric spline curves*. Their segments are rational curves of degree $k-1$ with a special arrangement of the control points.

These types of curves are useful in several engineering applications, including the design of cams with roller followers. However, as mentioned above, for flat face follower cams it is more natural to consider curves C_h corresponding to a support function $h \in \mathcal{S}$. It turns out that with $\alpha = (k-1)^{-1}$, C_h is a *piecewise rational curve* (NURBS curve) of degree $2k-4$, see [13]. Explicit formulae for the Bézier representations of the segments of the curve can be found there.

A striking property of the curves C_h is that their *offsets* are also rational curves. In fact, for k odd, this is easy to see, since the space $\mathcal{T}_k$ contains constants and hence the functions $h(\theta) + d, d \in \mathbb{R}$, are in this space whenever h is. A more general argument which also applies to even values of k can be found in [13].

5. Cam Design

In this section we discuss the practical aspects of using trigonometric spline support functions to design cams. In particular, we discuss making the support function take on prescribed values at given angles, constructing cams with prescribed dwells, creating constant diameter cams, and designing cams with minimal acceleration or jerk.

5.1. Closed Curves

The curve C_h will be closed if and only the corresponding support function h is 2π-periodic. There is a standard way to create periodic splines [18]. Let $\{t_i\}_{i=1}^{n+2k-1}$ be an extended partition with $t_k = 0$, $t_{n+k} = 2\pi$, and

$$t_{i+n} = t_i + 2\pi, \quad i = 1, \dots, 2k-1, \tag{5.1}$$

and let $\{T_i^k\}_{i=1}^{n+k-1}$ be the corresponding trigonometric B-splines of order k. Suppose $a_1, \dots, a_n$ are given real numbers, and that

$$a_{i+n} = a_i, \qquad i = 1, \dots, k-1. \tag{5.2}$$

Then the trigonometric spline

$$h(\theta) = \sum_{i=1}^{n+k-1} a_i T_i^k(\theta) \tag{5.3}$$

is 2π-periodic, i.e., $h^{(j)}(0) = h^{(j)}(2\pi)$, for $j = 0, \dots, k-2$.

5.2. Interpolation

It is clear from the previous section that in designing a cam using support functions h of the form (5.3) with coefficients satisfying (5.2), we have n free parameters at our disposal, namely the coefficients $a_1, \dots, a_n$. One way to use (some of) these degrees of freedom is to force h to take on prescribed values at given angles. In particular, given $0 \le \theta_1 < \cdots < \theta_m < 2\pi$ and positive numbers $h_1, \dots, h_m$, the function h satisfies

$$h(\theta_j) = h_j, \qquad j = 1, \dots, m \tag{5.4}$$

if and only the following set of m *linear conditions* on the coefficients a_i hold:

$$\sum_{i=1}^{n+k-1} a_i T_i^k(\theta_j) = h_j, \qquad j = 1, \dots, m. \tag{5.5}$$

It should be pointed out that it is not always possible to choose coefficients so that (5.5) holds. For example, if more than k of the θ_j fall in one knot interval $[t_i, t_{i+1}]$, we would be forcing more than k conditions on a single trigonometric polynomial, which is impossible. The following result shows that in general, interpolation is possible if the θ_j are sufficiently spread out.

Theorem 5.1. *Given $\{\theta_j\}_{j=1}^m$, suppose there exists a sequence $1 \le i_1 < \cdots < i_m \le n$ such that*

$$t_{i_j} < \theta_j < t_{i_j+k}, \qquad i = 1, \dots, m. \tag{5.6}$$

Then the interpolation problem (5.4) is solvable for any given $h_1, \ldots, h_m$.

Proof: A general result concerning the nonsingularity of interpolation matrices for trigonometric splines can be found in [10]. The result stated here is the periodic version, and can be derived from the nonperiodic case using the same arguments as for the case of periodic polynomial splines, cf. [18]. The conditions (5.6) are called the *Schoenberg-Whitney conditions*, and it can be shown that they are also necessary. ■

Using results in [10], it is possible to show that prescribed derivatives can also be interpolated under appropriate conditions.

5.3. Dwells

A cam is said to have a *dwell of displacement* η *between the angles* ϕ_1 *and* ϕ_2 provided that the support function h satisfies

$$h(\theta) \equiv \eta, \qquad \phi_1 \le \theta \le \phi_2. \tag{5.7}$$

This means that the follower remains at a fixed distance η above the center of the cam as the cam turns through the angles ϕ_1 to ϕ_2 [8].

Assuming that we represent the support function h as a trigonometric spline of odd order, we now show how to create a dwell of specified displacement η between two angles chosen from the knot set $\{t_i\}_{i=k}^{n+k}$.

Theorem 5.2. *Suppose* $\phi_1 = t_l$ *and* $\phi_2 = t_r$ *with* $k \le l < r \le n+k$. *Then given* η, *the support function* h *defined in (5.3) satisfies (5.7) if and only if*

$$c_i = \eta \xi_i^k, \qquad i = l+1-k, \ldots, r-1. \tag{5.8}$$

Proof: By the support properties of the trigonometric B-splines, (5.7) holds if and only if

$$\sum_{i=l+1-k}^{r-1} c_i T_i^k(\theta) \equiv \eta, \qquad \phi_1 \le \theta \le \phi_2. \tag{5.9}$$

Now in view of the partition of unity equation (3.5), this is equivalent to (5.8). ■

The requirement that a dwell begin and end at a knot is no restriction in practice as we are free to choose the number and location of the knots of h. It is possible to create cams with more than one dwell, but of course if two successive dwell displacements differ, then the corresponding dwell intervals must be separated by at least one knot interval.

5.4. Constant Diameter Cams

To design constant diameter cams, we must make the support function h satisfy the condition (2.4). As noted above, this is only possible if we work with odd order trigonometric splines. In addition, we have to choose the knots and coefficients of h carefully. Let $n = 2m$, and suppose

$$t_{i+k+m} = t_{i+k} + \pi, \qquad i = 1, \ldots, m.$$

Theorem 5.3. *Suppose the support function (5.3) has coefficients* $\{a_i\}_{i=1}^{n+k-1}$ *satisfying*

$$a_{i+m} = d\,\xi_i^k - a_i, \qquad i = 1, \ldots, m, \tag{5.10}$$

and (5.2), where $n = 2m$ *and* ξ_i^k *are the coefficients appearing in (3.6). Then the curve* C_h *has constant diameter* d.

Proof: By construction, h is periodic on $[0, 2\pi]$. By the way in which the knots were defined, it is clear that for $0 \le \theta < \pi$,

$$h(\theta) = \sum_{i=1}^{k+m-1} a_i\, T_i^k(\theta)$$

$$h(\theta + \pi) = \sum_{i=1}^{k+m-1} a_{i+m}\, T_i^k(\theta + \pi).$$

Inserting (5.10) and taking note of (3.5), we see that

$$h(\theta) + h(\theta + \pi) = d \sum_{i=1}^{k+m-1} \xi_i^k\, T_i^k(\theta) = d, \qquad 0 \le \theta < \pi,$$

i.e., (2.4) holds and thus C_h has constant diameter d. ■

5.5. Minimizing Acceleration and Jerk

One of the goals in designing cams is to create profiles for which the acceleration and jerk of the follower arm are small. In terms of our geometry, the acceleration is given by the second derivative of h, while the jerk is given by the third derivative of h.

Let h be the desired displacement function of the form (5.3) satisfying (5.2). We wish to design a cam minimizing the expression

$$\|Mh\|_\infty := \max_{\theta \in [0, 2\pi]} |Mh(x)|, \tag{5.11}$$

where M is a given differential operator. Setting $M = D^2$ corresponds to minimizing the acceleration, whereas $M = D^3$ leads to the minimization of the jerk.

Another possibility is to set $M = D^2 + 1$ in which case by (2.3) we would be minimizing the curvature of the cam.

Typically, various side constraints are imposed on the cam such as interpolation and dwelling conditions, or the requirement of constant diameter, described in earlier sections. These conditions are linear in the coefficients $a = (a_1, \ldots, a_n)$, and assuming there are m of them, they can be expressed in matrix form as

$$Ca = b, \tag{5.12}$$

where C is an m by n matrix, and b is an m-vector, both depending on the type of constraints. Unfortunately, the problem of minimizing (5.11) subject to (5.12) is nonlinear, and hence a numerical method must be employed. This is in contrast to the least squares problem, which leads to a linear system of equations. However, least squares solutions may not be optimal, since they may have large values of acceleration or jerk at certain angles of the cam.

To solve the minimization problem numerically, we discretize the interval $[0, 2\pi]$ by replacing it with a sufficiently dense set of points $\{u_l\}_{l=1}^{L}$. We now approximate the minimization problem by the following linear programming problem:

$$\begin{aligned} &\text{minimize} \quad \varepsilon \\ &\text{subject to} \\ &\qquad \sum_{i=1}^{n+k-1} a_i M T_i^k(u_l) - \varepsilon \le 0, \quad l = 1, \ldots, L, \\ &\qquad \sum_{i=1}^{n+k-1} a_i M T_i^k(u_l) + \varepsilon \le 0, \quad l = 1, \ldots, L, \\ &\qquad Ca = b. \end{aligned} \tag{5.13}$$

Here ε is an auxiliary variable whose minimization clearly forces the expression

$$\max_{l=1,\ldots,L} |Mh(u_l)|,$$

approximating $\|Mh\|_\infty$, to be as small as possible. The standard linear programming problem (5.13) can be be solved using well-established methods.

5.6. Convex Cams

Assuming a cam is designed by the optimization process described in the previous section, it is a relatively easy matter to add conditions which will help insure that the resulting cam is *convex*. We simply require that the signed radius of curvature be nonnegative at some (relatively dense) set of points $\{u_l\}_{l=1}^{L}$ in $[0, 2\pi]$:

$$h(u_l) + h''(u_l) \ge 0, \qquad l = 1, \ldots, L.$$

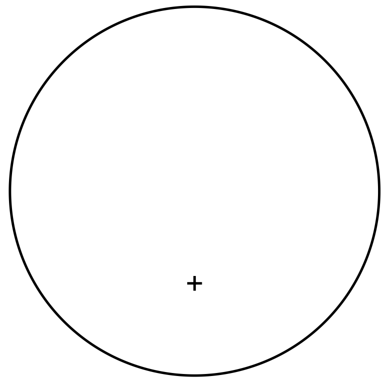

Fig. 2. The cam in Example 6.1.

This is just a set of L additional side constraints on the optimization problem (5.13).

6. Examples of Cam Design

To illustrate the above theory, we now present several examples of cam designs with constant diameters using trigonometric splines of order $k = 3$ and $k = 5$. To get cams with NURBS curve profiles, throughout this section we take $\alpha = 1/(k-1)$. All of our examples are based on equally spaced knots with a spacing $2\pi/n$.

6.1. Cams Based on Trigonometric Splines of Order 3

For $k = 3$ the support function for each curve segment is of the form $h_i(\theta) = r_i + m_i \cos(\theta) + n_i \sin(\theta)$. Thus each segment of the support function is a circle with center (m_i, n_i) and radius r_i. Since $h \in C^1[0, 2\pi]$, it follows that the cam profile consists of a collection of *circular arcs* joined together with first order geometric continuity (continuous tangent lines). With $k = 3$, the acceleration of the cam follower is a discontinuous function of θ with jumps at the angles corresponding to knots of h.

Example 6.1. *Construct a cam with the following properties:*

1) *a displacement of 1.0 at angle* $\theta = 0$,
2) *a displacement of 1.5 at angle* $\theta = \pi/2$.
3) *a constant diameter of 2.*

Discussion: We seek h in the form (5.3) with $n = 4$. The conditions (5.10) for constant diameter 2 are

$$a_3 + a_1 = 2\cos(\pi/4) = 1.4142136$$
$$a_4 + a_2 = 2\cos(\pi/4) = 1.4142136.$$

Combining these with the two interpolation conditions, we have a system of four linear equations for the four coefficients. The solution is $a_1 = 0.3524855$, $a_2 = 1.0448228$, $a_3 = 1.0761710$, and $a_4 = 0.3695643$. The corresponding cam is shown in Fig. 2. ■

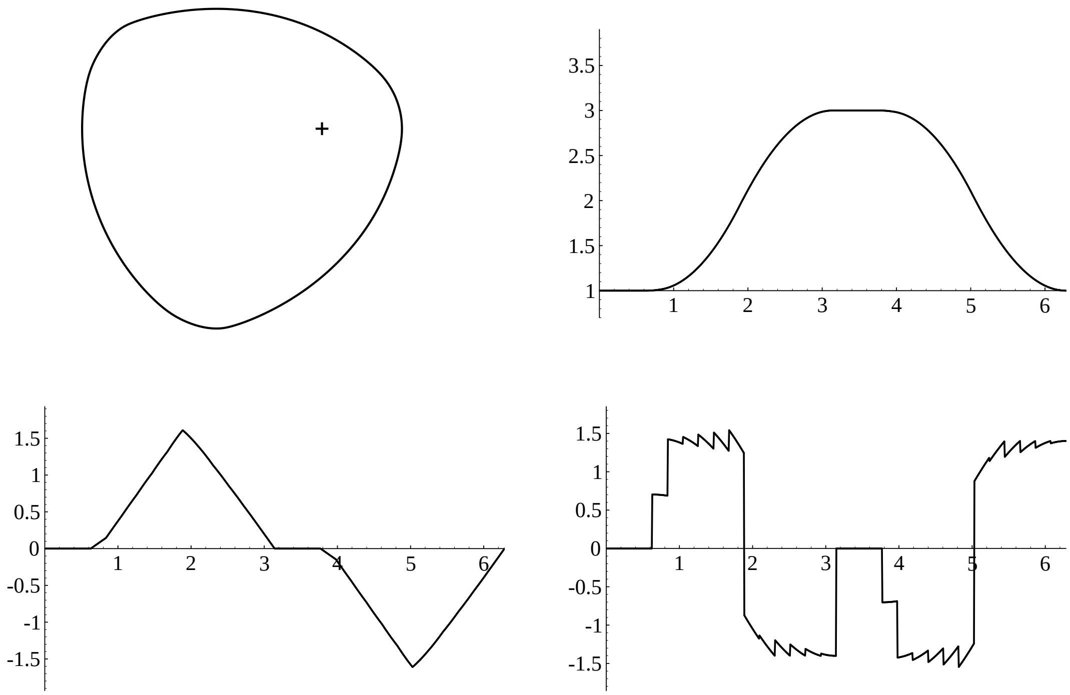

Fig. 3. The cam in Example 6.3.

Example 6.2. *Construct a cam with the same properties as in Example 6.1 except with displacement 1.3 at angle* $\theta = \pi/2$.

Discussion: In this case the four equations turn out to be *incompatible*, and so these design requirements cannot be met with $n = 4$. ■

We now consider a more ambitious example which includes most of the design elements discussed above.

Example 6.3. *Construct a cam with the following properties:*

1) *constant diameter 4,*
2) *a dwell of displacement 1.0 in the interval* $[0, \pi/5]$,
3) *a displacement of 1.5 at angle* $\theta = \pi/2$.

Discussion: We choose $n = 30$ and solve (5.13). The resulting cam is shown in Fig. 3 along with plots of the displacement, velocity, and acceleration. Because the cam is of constant diameter, it also exhibits a dwell of displacement 2.5 for $\pi \le \theta \le 6\pi/5$. For later comparison, we note that $\|h'\| = 1.627$ and $\|h''\| = 1.535$. ■

6.2. Cams Based on Trigonometric Splines of Order 5

In this subsection we choose $k = 5$ and $\alpha = .25$. In this case, $s \in C^3[0, 2\pi]$, and thus both the acceleration and jerk of the cam follower are continuous functions of θ.

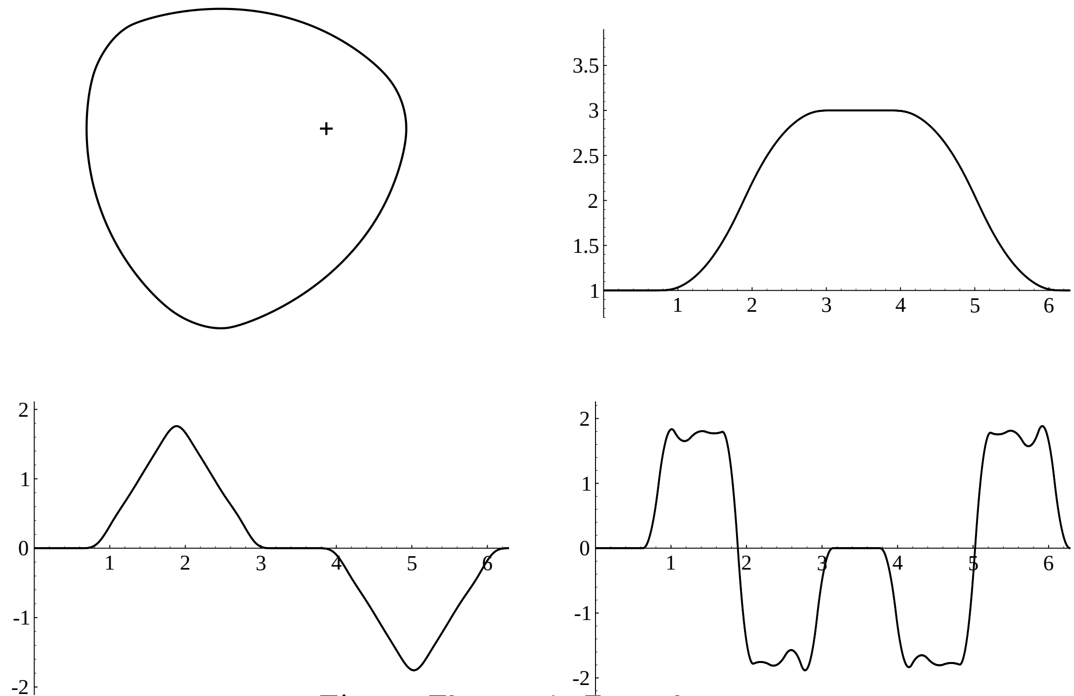

Fig. 4. The cam in Example 6.4.

Example 6.4. *Construct a C^3 cam with the properties listed in Example 6.3.*

Discussion: We choose $n = 30$ and solve (5.13). The resulting cam is shown in Fig. 4 along with plots of the displacement, velocity, and acceleration. For this cam, For this cam, $\|h'\| = 1.761$ and $\|h''\| = 1.878$. Thus, the only cost for the smooth acceleration is a slight increase in the maximum norms of velocity and acceleration. ■

Example 6.5. *Construct a cam with the following properties:*

1) $h(0) = 1$ *and* $h'(0) = h''(0) = 0$.

2) $h(\pi) = \pi + 1$ *and* $h'(\pi) = h''(\pi) = 0$.

Discussion: We choose $n = 30$ and solve (5.13). The resulting cam is shown in Fig. 5 along with plots of the displacement, velocity, and acceleration. This example should be compared with Example 1 in [21], and in particular with their cubic spline design shown in Fig. 5 of that paper. The norm of velocity for our design is 2.17 as compared to 1.89 for theirs, and (after normalization), the norm of acceleration for our design is 4.2 as compared to 5.4 for their design. ■

7. Remarks

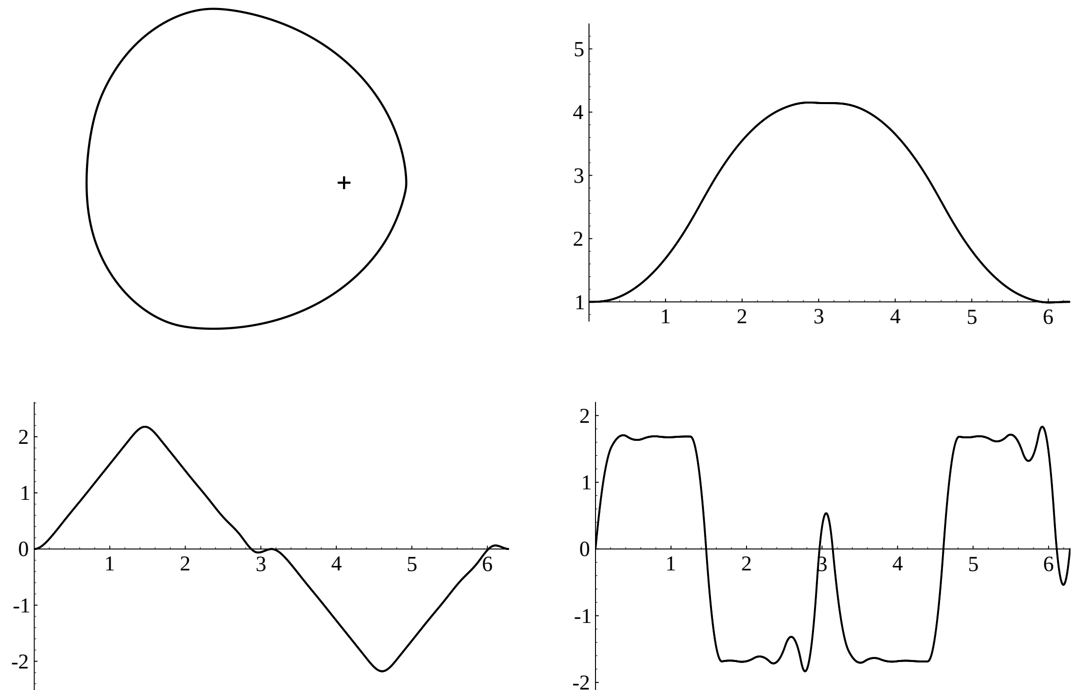

Fig. 5. The cam in Example 6.5.

Remark 7.1. In designing cams with dwells, it is important that the displacement be constant throughout the dwell interval (i.e., the cam curve is a circular arc in this interval). While this is difficult to achieve with many curve schemes, as we have seen above it is easy to achieve with our scheme (for odd order splines).

Remark 7.2. Rational curves with rational offsets have been an active area of research in computer-aided geometric design in the past few years, and have important applications in NC milling and layered manufacturing (see [4,5,6,14] and references therein). However, the simple generation of our family of rational geometric splines with rational offsets does not seem to have been exploited previously.

Remark 7.3. Using the presented approach might be advantageous for solving various design problems where the length of the curve is specified as an additional constraint. While most existing approaches give rise to a nonlinear problem [3,7,15], using the curve representation in terms of support functions gives rise to a system of linear equations for the spline coefficients. The linearity follows from the fact that the length is given by $\int_a^b [h(\theta) + h''(\theta)] d\theta$.

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