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DIMENSION AND LOCAL BASES OF HOMOGENEOUS SPLINE SPACES*

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Abstract. Recently, we have introduced spaces of splines defined on triangulations lying on the sphere or on sphere-like surfaces. These spaces arose out of a new kind of Bernstein-Bézier theory on such surfaces. The purpose of this paper is to contribute to the development of a constructive theory for such spline spaces analogous to the well-known theory of polynomial splines on planar triangulations. Rather than working with splines on sphere-like surfaces directly, we instead investigate more general spaces of homogeneous splines in $\mathbb{R}^3$. In particular, we present formulae for the dimensions of such spline spaces, and construct locally supported bases for them.

Key words. multivariate splines, piecewise polynomial functions, homogeneous spline spaces, dimensions, sphere-like surfaces, sphere, interpolation, approximation, data fitting

AMS subject classifications. 41A63, 41A15, 65D07

1. Introduction. Let $\Delta := \{T^{[i]}\}_1^N$ be a planar triangulation of a set Ω, and let $0 \le r \le d$ be integers. The classical *space of splines of degree d and smoothness r* is defined by

$$\mathcal{S}_d^r(\Delta) := \{s \in C^r(\Omega) \; : \; s|_{T^{[i]}} \in \mathcal{P}_d, \quad i = 1, \ldots, N\}, \tag{1}$$

where $\mathcal{P}_d$ is the space of bivariate polynomials of degree at most d. These spaces of spline functions have found numerous applications in interpolation, data fitting, finite element solutions of boundary value problems, computer aided geometric design, image processing, and elsewhere.

There is a well-developed (albeit incomplete) constructive theory for the polynomial spline spaces $\mathcal{S}_d^r(\Delta)$ which includes

1) dimension formulae
2) construction of local bases
3) estimates on the approximation power
4) algorithms for manipulating the splines
5) algorithms for interpolation, data fitting, etc.

Recently [4], we introduced analogous spaces of splines defined on a triangulation on the sphere or on a sphere-like surface. As suggested by our companion paper [6], we believe that such spaces have important applications, and hence it is important to develop the analogous constructive theory.

Following [4], we will analyze spherical splines by investigating a more general class of splines associated with a *trihedral decomposition* $\mathcal{T} := \{T^{[i]}\}_1^N$ of a set $\Omega \subseteq \mathbb{R}^3$ (see Sect. 2 below). Given such a decomposition, the associated spaces of *homogeneous splines* are defined by

$$\mathcal{H}_d^r(\mathcal{T}) := \{s \in C^r(\Omega) : s|_{T^{[i]}} \in \mathcal{H}_d, \quad i = 1, \ldots, N\}, \tag{2}$$

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where $\mathcal{H}_d$ denotes the space of trivariate polynomials of degree d which are homogeneous of degree d (recall that a function f defined on $\mathbb{R}^3$ is *homogeneous of degree* d provided $f(\alpha v) = \alpha^d f(v)$ for all real numbers α and all $v \in \mathbb{R}^3$). Splines on the sphere or on a sphere-like surface S are then obtained by restricting $\mathcal{H}_d^r(\mathcal{T})$ to S.

The main purpose of this paper is to establish dimension formulae for spaces of homogeneous splines, and to show how to construct bases of locally supported splines. Homogeneous splines can be stored and evaluated using the algorithms presented in [4] for homogeneous polynomials. The question of the approximation power of homogeneous and spherical splines will be dealt with elsewhere. Applications to the interpolation and fitting of scattered data on the sphere or on a sphere-like surface are discussed in [6]. Even though we are working in $\mathbb{R}^3$, because of the nature of homogeneous polynomials—which are essentially bivariate functions—the entire development is closely modelled after the analysis of the bivariate spaces of splines $\mathcal{S}_d^r(\Delta)$ carried out in [8, 15, 17].

2. Homogeneous spline spaces. We begin by introducing some notation, closely following [4].

DEFINITION 1. *Let $\{v_1, v_2, v_3\}$ be a set of linearly independent unit vectors in $\mathbb{R}^3$. We call*

$$T = \{v \in \mathbb{R}^3 : v = b_1 v_1 + b_2 v_2 + b_3 v_3 \quad \textit{with} \quad b_i \geq 0\} \tag{3}$$

the trihedron generated by $\{v_1, v_2, v_3\}$. *As in [4], we call the real numbers b_1, b_2, b_3 the* trihedral coordinates *of v with respect to T. They are homogeneous linear functions in the coordinates of v.*

We call the set $\{v \in T : b_i = 0\}$ the (i-th) *face* of T, and the set $\{\alpha v_i : \alpha \geq 0\}$ the (i-th) *ray* of T (or the ray generated by v_i). To avoid awkward repetitions, we abuse our notation slightly: in addition to writing v for a unit vector, we also use v to denote the associated point in $\mathbb{R}^3$ and the associated ray generated by v.

DEFINITION 2. *Let $\mathcal{T} = \{T^{[i]}\}_{i=1}^N$ be a non-empty set of trihedra, and let $\Omega := \cup T^{[i]}$. Then we call $\mathcal{T}$ a* trihedral decomposition of Ω *provided*

1) the interiors of the trihedra in $\mathcal{T}$ are pairwise disjoint,

2) the set $\Omega \cap S$ is homeomorphic to a two-dimensional disk or equals S, where S is the unit sphere,

3) each face of a trihedron in $\mathcal{T}$ is either on the boundary of Ω or it is a common face of precisely two trihedra in $\mathcal{T}$.

Each of the $T^{[i]} \cap S$ is a spherical triangle, and $\Delta = \{T^{[i]} \cap S\}_{i=1}^N$ is a *spherical triangulation*, cf. [19]. We say a trihedral decomposition $\mathcal{T}$ is *total* if $\Omega = \mathbb{R}^3$. Otherwise, we say that it is *partial*.

It will be convenient to denote the set of unit vectors defining the rays of the trihedra in $\mathcal{T}$ by $\mathcal{V}$. If $\mathcal{T}$ is a partial trihedral decomposition, it is natural to define rays to be *boundary rays* of $\mathcal{T}$ provided they are associated with vectors $v \in \mathcal{V}$ which lie on the boundary of Ω. All other rays will be called *interior rays*. We denote the sets of boundary and interior rays in $\mathcal{T}$ by $\mathcal{V}_B$ and $\mathcal{V}_I$, respectively. Clearly, all rays of a total trihedral decomposition are interior rays. Following the notation used for planar triangulations, we denote the number of boundary and interior rays of $\mathcal{T}$ by V_B and V_I, respectively. Similarly, we denote the number of boundary and interior faces of $\mathcal{T}$ by E_B and E_I. For a partial decomposition, the number of rays is given by $V := V_B + V_I$, and the number of faces is given by $E := E_B + E_I$. For a total decomposition, $V = V_I$ and $E = E_I$.

Let T be a trihedron generated by $\{v_1, v_2, v_3\}$, and let b_1, b_2, b_3 denote the corresponding trihedral coordinates as functions of $v \in \mathbb{R}^3$. The *homogeneous Bernstein basis polynomials of degree d associated with T* are the polynomials

$$B^d_{ijk}(v) := \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \qquad i+j+k=d. \tag{4}$$

The space $\mathcal{H}_d$ of trivariate homogeneous polynomials is a $\binom{d+2}{2}$-dimensional linear space, and as observed in [4], it is spanned by the set of $\binom{d+2}{2}$ Bernstein basis polynomials defined in (4). Thus each $p \in \mathcal{H}_d$ can be written uniquely in the form

$$p(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v). \tag{5}$$

In [4], p is referred to as a *homogeneous Bernstein-Bézier* (HBB-) *polynomial of degree d*.

It will be convenient to define the *domain points* associated with T to be the points

$$P_{ijk} := \frac{iv_1 + jv_2 + kv_3}{d}, \qquad i+j+k=d. \tag{6}$$

In contrast to the case of polynomial splines on planar triangles, this definition of P_{ijk} is not the only natural one (see Remark 24).

If we look at all of the domain points for all of the trihedra in a trihedral decomposition, it is clear that the domain points associated with a common face of two trihedra coincide. If we eliminate such repetitions, we see that for a given trihedral decomposition $\mathcal{T}$, there is one point associated with each ray, $d-1$ points associated with each face, and $\binom{d-1}{2}$ associated with the interior of each trihedron. Thus the set $\mathcal{G}$ of distinct domain points has cardinality

$$\#(\mathcal{G}) = V + (d-1)E + \binom{d-1}{2} N. \tag{7}$$

The importance of the HBB-form of homogeneous polynomials is that it provides a simple way to describe when two such polynomials defined on adjoining trihedra join together smoothly. Indeed, suppose $T^{[1]}$ and $T^{[2]}$ are two trihedra generated by the sets $\{v_1, v_2, v_3\}$ and $\{v_1, v_3, v_4\}$, respectively. Then as shown in [4], the two associated homogeneous polynomials $p^{[1]}$ and $p^{[2]}$ of degree d agree on the face shared by $T^{[1]}$ and $T^{[2]}$ in value and all derivatives up to order r if and only if

$$c^{[2]}_{ijk} = \sum_{\mu+\nu+\kappa=k} c^{[1]}_{i+\mu,\nu,j+\kappa} B^k_{\mu\nu\kappa}(v_4) \quad \text{for all} \quad k \le r, \quad i+j+k=d, \tag{8}$$

where $B^k_{\mu\nu\kappa}$ are the Bernstein basis polynomials of degree k associated with $T^{[1]}$.

By (8), $p^{[1]}$ and $p^{[2]}$ join *continuously* across their common face if and only if

$$c^{[2]}_{ij0} = c^{[1]}_{i0j}, \qquad i+j=d. \tag{9}$$

We conclude that a spline $s \in \mathcal{H}^0_d(\mathcal{T})$ is uniquely defined by a set of $\#(\mathcal{G})$ coefficients, one associated with each point $P \in \mathcal{G}$. This implies that the space $\mathcal{H}^0_d(\mathcal{T})$ has dimension $\#(\mathcal{G})$.

For later use, for each $P \in \mathcal{G}$, it will be convenient to define a linear functional λ_P defined on $\mathcal{H}_d^0(\mathcal{T})$ with the property that for any $s \in \mathcal{H}_d^0(\mathcal{T})$,

$$\lambda_P s = c_P, \tag{10}$$

where c_P is the coefficient associated with the point P. We denote the set of all such linear functionals by Λ. Clearly $\#(\Lambda) = \#(\mathcal{G})$.

For each $\lambda \in \Lambda$, there is a unique spline $s_\lambda \in \mathcal{H}_d^0(\mathcal{T})$ such that

$$\gamma s_\lambda = \delta_{\gamma,\lambda}, \quad \text{all } \gamma \in \Lambda. \tag{11}$$

The spline s_λ has all coefficients equal to 0 except for the coefficient λs_λ which has value 1. By construction, s_λ has one of the following supports:

1) a single trihedron T if the coefficient λs_λ is associated with a domain point in the interior of T,
2) a pair of adjoining trihedra if the coefficient λs_λ is associated with a domain point in the interior of a face separating two trihedra,
3) the union of all trihedra which share the ray v if the coefficient λs_λ is associated with the domain point v.

In view of these properties, we say that such splines have *local support.* The duality property (11) assures that the splines s_λ for $\lambda \in \Lambda$ are linearly independent, and since there are precisely $\#(\mathcal{G})$ of them, they form a basis for $\mathcal{H}_d^0(\mathcal{T})$.

To obtain analogous results for $\mathcal{H}_d^r(\mathcal{T})$, we follow [8, 15, 17]. To get an upper bound on dimension, we construct a *determining set* $\Gamma \subset \Lambda$ such that if $s \in \mathcal{H}_d^r(\mathcal{T})$,

$$\gamma s = 0 \quad \text{for all} \quad \gamma \in \Gamma \qquad \text{implies} \qquad s \equiv 0. \tag{12}$$

Then as shown in [8], $\dim \mathcal{H}_d^r(\mathcal{T})$ is bounded above by the cardinality of Γ. We can get a lower bound for the dimension (and construct a basis at the same time) if Γ is chosen so that for each $\lambda \in \Gamma$, there exists a spline $s_\lambda \in \mathcal{H}_d^r(\mathcal{T})$ satisfying

$$\gamma s_\lambda = \delta_{\gamma,\lambda}, \quad \text{all } \gamma \in \Gamma. \tag{13}$$

This duality implies that the splines $\{s_\lambda\}$ are linearly independent, and it follows that the dimension of $\mathcal{H}_d^r(\mathcal{T})$ is equal to the cardinality of Γ and that these splines form a basis. Such a set Γ is called a *minimal determining set.*

We close this section by presenting the main result of the paper. Its proof will be developed in the following sections.

THEOREM 3. *Let $r \geq 0$ and $d \geq 3r+2$. Suppose $\mathcal{T}$ is a trihedral decomposition of a set $\Omega \subseteq \mathbb{R}^3$. Let*

$$\sigma := \sum_{v \in \mathcal{V}_I} \sigma_v, \quad \textit{where} \quad \sigma_v := \sum_{m=1}^{d-r} (r+m+1-me_v)_+ \tag{14}$$

and e_v is the number of distinct planes containing the faces that meet at the ray v. Then

$$\dim \mathcal{H}_d^r(\mathcal{T}) = (d-r)(d-2r)V - 2d^2 + 6dr - 3r^2 + 3r + 2 + \sigma, \tag{15}$$

if $\mathcal{T}$ is a total decomposition, and

$$\dim \mathcal{H}_d^r(\mathcal{T}) = \frac{(d-r+1)(d-r)}{2} V_B + (d-r)(d-2r)V_I - \frac{2d^2 - 6dr + 3r^2 - 3r - 2}{2} + \sigma, \tag{16}$$

if $\mathcal{T}$ is a partial decomposition. In either case there exists a basis for $\mathcal{H}_d^r(\mathcal{T})$ consisting of splines such that the support of each spline is either a single trihedron, an adjoining pair, or the set of trihedra containing a single ray.

3. Minimal determining sets for splines on oranges. In [8, 15] the key to analyzing the dimension of bivariate spline spaces was first to examine the special case of a *cell* consisting of a set of triangles sharing one vertex. In this section we construct minimal determining sets for spline spaces on the trihedral analog of cells. In the context of tetrahedral decompositions these were called oranges in [10, 20]. Throughout this section we assume only that $0 \le r < d$.

DEFINITION 4. *A trihedral decomposition $\mathcal{O}$ consisting of a set of trihedra sharing one ray v is called an* orange. *We call v the axis of the orange, see Fig. 1.*

— insert file `orange.fig` here —

FIG. 1. *An orange with axis v.*

Suppose the trihedra in $\mathcal{O}$ are labeled in counterclockwise order as $T^{[1]}, T^{[2]}, \ldots, T^{[N]}$ as we move around the axis v, where the rays of $T^{[\ell]}$ are v, v_ℓ, and $v_{\ell+1}$. If v is an interior ray, we have $v_{N+1} = v_1$. We can label the domain points in these trihedra as

$$P_{ijk}^{[\ell]} := \frac{iv + jv_\ell + kv_{\ell+1}}{d}, \qquad i+j+k = d. \tag{17}$$

THEOREM 5. *If $\mathcal{O}$ is an orange associated with a boundary ray v, then*

$$\dim H_d^r(\mathcal{O}) = \binom{d+2}{2} + (N-1)\binom{d-r+1}{2}. \tag{18}$$

If $\mathcal{O}$ is an orange associated with an interior ray v, then

$$\dim \mathcal{H}_d^r(\mathcal{O}) = \binom{r+2}{2} + N\binom{d-r+1}{2} + \sum_{m=1}^{d-r} (r+m+1-me)_+, \tag{19}$$

where e denotes the number of distinct planes shared by trihedra in $\mathcal{O}$.

Proof. Let Π be a plane which intersects the axis of $\mathcal{O}$ at a point w which is not the origin, and so that Π is perpendicular to the axis. The intersections with Π of those faces of $\mathcal{O}$ which contain the axis are rays in Π emanating from w. If we replace them with unit line segments with one end at w, and then connect their endpoints in order, we get a planar triangulation Δ consisting of a set of triangles sharing the vertex w. Clearly, the restriction of a spline in $\mathcal{H}_d^r(\mathcal{O})$ to Δ is a spline in $\mathcal{S}_d^r(\Delta)$. Conversely, by the homogeneity of the splines in $\mathcal{H}_d^r(\mathcal{O})$, a spline in $\mathcal{S}_d^r(\Delta)$ extends uniquely to a spline in $\mathcal{H}_d^r(\mathcal{O})$. The two spaces $\mathcal{S}_d^r(\Delta)$ and $\mathcal{H}_d^r(\mathcal{O})$ are therefore isomorphic, and the dimension assertion follows from Theorem 2.2 in [17]. □

Following the proofs of Lemma 3.1 in [15] and Lemma 3.1 in [8], we now construct minimal determining sets for $\mathcal{H}_d^r(\mathcal{O})$ when $\mathcal{O}$ is an orange. We need the concept of a ring of domain points around a ray v.

DEFINITION 6. *Let $\mathcal{O}$ be an orange as above. Then given an integer d, the m-*th ring *of $\mathcal{O}$ is the set of domain points*

$$\left\{P_{d-m,j,k}^{[\ell]} : j+k=m,\ \ell=1,2,\ldots,N\right\}. \tag{20}$$

*The m-*disk *in $\mathcal{O}$ is the union of the 0-th through m-th rings.*

The concepts of ring and disk are illustrated in Fig. 1. In particular, the domain points in the 5-ring around the vertex v in the figure are marked with + signs. The domain points in the 5-disk include all points marked with * or with +. To avoid cluttering the picture, the domain points in the far face (with vertices v, v_1, and v_2) have been omitted.

THEOREM 7. *Suppose $\mathcal{O}$ is an orange associated with a boundary ray v. Then the set*

$$\left\{P_{ijk}^{[1]} : i+j+k=d\right\} \cup \bigcup_{\ell=2}^{N} \left\{P_{ijk}^{[\ell]} : k \ge r+1\right\} \tag{21}$$

is a minimal determining set for $\mathcal{H}_d^r(\mathcal{T})$. Suppose $\mathcal{O}$ is an orange surrounding an interior ray v, and let e be as in Theorem 5. Let

$$\mu_{N-e+1} < \mu_{N-e+2} < \cdots < \mu_{N-1} = N, \quad \mu_N = N+1, \tag{22}$$

be such that the associated edges are pairwise noncollinear, and let

$$\mu_1 < \mu_2 < \cdots < \mu_{N-e} \tag{23}$$

be the complementary set so that

$$\{\mu_1, \mu_2, \ldots, \mu_N\} = \{2, 3, \ldots, N+1\}. \tag{24}$$

Let $\Gamma_0 \subseteq \Lambda$ be the set of functionals corresponding to domain points in the trihedron $T^{[1]}$. In addition, for each $m = 1, \ldots, d-r$, let Γ_m be the set of functionals corresponding to the first $Nm - (r+m+1) + (r+m+1-me)_+$ points in the ordered set

$$\{P_{d-m-r,m-1,r+1}^{[\mu_1]}, \ldots, P_{d-m-r,0,m+r}^{[\mu_1]}, \ldots, P_{d-m-r,m-1,r+1}^{[\mu_N]}, \ldots, P_{d-m-r,0,m+r}^{[\mu_N]}\}. \tag{25}$$

Then

$$\Gamma = \bigcup_{m=0}^{d-r} \Gamma_m \tag{26}$$

is a minimal determining set for $H_d^r(\mathcal{O})$.

Proof. We prove the result only for the case where the axis of the orange is an interior ray; the other case is similar. It is easy to check that the cardinality of Γ is given by the formula (19), and so we only need to show that Γ is a minimal determining set. To that end, consider the plane Π that is perpendicular to the vector v and passes through the point v. Explicitly,

$$\Pi = \left\{ u \in \mathbb{R}^3 : (u - v) \cdot v = 0 \right\}, \tag{27}$$

where $\cdot$ denotes the ordinary dot product. Let w_ℓ denote the orthogonal projection of v_ℓ onto Π, i.e.,

$$w_\ell = v_\ell + \left(1 - \frac{v_\ell \cdot v}{v \cdot v}\right) v, \quad \ell = 1, 2, \ldots, N. \tag{28}$$

The intersection of $\mathcal{O}$ with Π forms a two-dimensional cell Δ in the sense of [17]. Let Γ_Δ denote the functionals defined on $S_d^r(\Delta)$ corresponding to the projections of the points defining Γ. In view of the correspondence between bivariate polynomials and trivariate homogeneous polynomials, by Theorem 3.3 of [17], Γ_Δ is a (minimal) determining set of $S_d^r(\Delta)$. The fact that Γ is a determining set for $\mathcal{H}_d^r(\mathcal{O})$ now follows from a careful comparison of the smoothness conditions for $S_d^r(\Delta)$ and $\mathcal{H}_d^r(\mathcal{O})$. Any spline $s \in \mathcal{H}_d^0(\mathcal{O})$ can be expressed on the trihedron $T^{[\ell]}$ in the form (5) with $c_{ijk} = c_{ijk}^{[\ell]}$. To obtain the smoothness conditions for $\mathcal{H}_d^r(\mathcal{O})$ we write

$$v_{\ell+1} = r_\ell v + s_\ell v_{\ell-1} + t_\ell v_\ell, \tag{29}$$

where for convenience we treat all rays, domain points, and coefficients cyclically as we move around v (so that $v_0 = v_N$ for example). By (8), a spline in $H_d^0(\mathcal{O})$ belongs to $H_d^r(\mathcal{O})$ if and only if for all $\ell = 1, 2, \ldots, N$,

$$c_{ijk}^{[\ell+1]} = \sum_{\mu+\nu+\kappa=k} c_{i+\mu,\nu,j+\kappa}^{[\ell]} \frac{k!}{\mu!\nu!\kappa!} r_\ell^\mu s_\ell^\nu t_\ell^\kappa, \tag{30}$$

for $k \le r$ and $i + j + k = d$. Consider now the corresponding smoothness conditions for $S_d^r(\Delta)$. It can be checked that the projections of the v_ℓ satisfy

$$w_{\ell+1} = \tilde{r}_\ell v + s_\ell w_{\ell-1} + t_\ell w_\ell, \tag{31}$$

where

$$\tilde{r}_\ell := 1 - s_\ell - t_\ell. \tag{32}$$

Using a tilde to denote the coefficients of a spline in $S_d^r(\Delta)$ we obtain the conditions

$$\tilde{c}_{ijk}^{[\ell+1]} = \sum_{\mu+\nu+\kappa=k} \tilde{c}_{i+\mu,\nu,j+\kappa}^{[\ell]} \frac{k!}{\mu!\nu!\kappa!} \tilde{r}_\ell^\mu s_\ell^\nu t_\ell^\kappa, \qquad \ell = 1, \ldots, N. \tag{33}$$

We now show that Γ is a determining set for $\mathcal{H}_d^r(\mathcal{O})$. Consider a spline $s \in \mathcal{H}_d^r(\mathcal{O})$. We work our way through the rings of the orange. The 0-th ring is v itself. It is in Γ and therefore the coefficient corresponding to it must be zero. Suppose now that the coefficients corresponding to the first m rings are all zero and consider the $(m+1)$-th ring and the smoothness conditions (30) and (33) for $k = m$. In spite of r_ℓ and $\tilde{r}_\ell$ being different, these equations are equivalent since the terms where $r_\ell \neq 0$ and $\tilde{r}_\ell \neq 0$ contain coefficients which are zero by the induction hypothesis. Thus, the coefficients of s must vanish on the $(m+1)$-th ring and it follows that Γ is a determining set. Since it has cardinality equal to the dimension of $H_d^r(\mathcal{O})$, it follows that Γ is minimal. □

Remark 8. The argument used in the proof of Theorem 7 applies to all minimal determining sets which have $\#\Gamma_m$ points on the $(r+m)$-th ring for $m = 1, \ldots, d-r$. However, it is not true in general that the analog of a minimal determining set for a two-dimensional cell is also a minimal determining set for a corresponding orange as is shown in the following example.

— insert file **nondet.fig** here —

FIG. 2. *A non-determining set.*

Example 9. Let $\mathcal{O}$ be an orange with $N = 4$, $v_3 = -v_1$ and $v_4 = -v_2$.

Discussion: Figure 2 shows a minimal determining set for $S_2^1(\Delta)$ that is not determining for $H_2^1(\mathcal{O})$, where points corresponding to functionals not in the set are marked with a dot, and the functionals corresponding to all other points are in the set. Note that in particular the functional corresponding to the center point (which is at v) is *not* in the set. Clearly, in the two-dimensional cell the coefficients at the points marked with a crosshair ($\oplus$) or a triangle ($\triangle$) determine the coefficient at v. In fact we have

$$\bar{c}_v = \frac{\bar{c}_\oplus + c_\triangle}{2}. \tag{34}$$

On the other hand, the relevant smoothness condition for $H_2^1(\mathcal{O})$ is

$$c_\oplus = -c_\triangle, \tag{35}$$

and so the two points marked with $\oplus$ or $\triangle$ cannot both be in the minimal determining set. It is of course easy to construct sets that are minimal determining for both $S_2^1(\Delta)$ and $H_2^1(\mathcal{O})$. An example (conforming to Theorem 7) can be obtained from Figure 2 by replacing the point marked with $\oplus$ with v. □

4. A minimal determining set for $\mathcal{H}_d^r(\mathcal{T})$ when $d \geq 3r+2$. In this section we construct a minimal determining set Γ for $\mathcal{H}_d^r(\mathcal{T})$ in the case where $d \geq 3r+2$. As in the bivariate case [8, 15], the key to the construction is to partition the Bézier coefficients into suitable subsets. Consider a trihedron T generated by the vectors v_1, v_2, v_3, and let $\mathcal{P} := \{P_{ijk}\}_{i+j+k=d}$ be the associated set of Bézier coefficients. To make the description of Γ easier, we recall the correspondence between coefficients, domain points, and the associated linear functionals

$$c_{ijk} \sim P_{ijk} \sim \lambda_{P_{ijk}}, \tag{36}$$

and work only with domain points P_{ijk} here. We define the *distance of P_{ijk} from the ray v* to be

$$\operatorname{dist}(P_{ijk}, v) := \begin{cases} d-i, & \text{if } v = v_1, \\ d-j, & \text{if } v = v_2, \\ d-k, & \text{if } v = v_3. \end{cases} \tag{37}$$

For $i = 1, 2, 3$, let

$$\begin{aligned} \mathcal{D}_\mu(v_i) &:= \{P \in \mathcal{P} : \operatorname{dist}(P, v_i) \leq \mu\} \\ \mathcal{A}(v_i) &:= \{P \in \mathcal{P} : \operatorname{dist}(P, v_i) > \mu,\ \operatorname{dist}(P, v_{i+1}) \geq d-r, \\ &\qquad \operatorname{dist}(P, v_{i+2}) \geq d-r\} \\ \mathcal{F}(v_i) &:= \{P \in \mathcal{P} : \operatorname{dist}(P, v_i) \geq d-r\} \\ \mathcal{E}(v_i) &:= \{P \in \mathcal{F}(v_i) : |\operatorname{dist}(P, v_{i+1}) - \operatorname{dist}(P, v_{i+2})| \leq d-3r-2\} \\ \mathcal{B}_L(v_i) &:= [\mathcal{F}(v_i) \cap \mathcal{D}_{2r}(v_{i+1})] \setminus [\mathcal{D}_\mu(v_{i+1}) \cup \mathcal{A}(v_{i+1}) \cup \mathcal{E}(v_i)] \\ \mathcal{B}_R(v_i) &:= [\mathcal{F}(v_i) \cap \mathcal{D}_{2r}(v_{i+2})] \setminus [\mathcal{D}_\mu(v_{i+2}) \cup \mathcal{A}(v_{i+2}) \cup \mathcal{E}(v_i)] \\ \mathcal{C} &:= \{P \in \mathcal{P} : \operatorname{dist}(P, v_j) < d-r, \quad j = i, i+1, i+2\}, \end{aligned} \tag{38}$$

where

$$\mu := r + \lfloor \frac{r+1}{2} \rfloor, \tag{39}$$

and we identify $v_4 = v_1$ and $v_5 = v_2$.

— insert file **div.fig** here —

FIG. 3. *Division of domain points by Algorithm 12, $d = 23$, $r = 6$, $\mu = 9$.*

The set $\mathcal{D}_\mu(v_i)$ contains the points in a disk around v_i of radius μ. $\mathcal{A}(v_i)$ (called a *cap* in [15]) is the set of points not in $\mathcal{D}_\mu(v_i)$ but whose corresponding coefficients are involved in smoothness conditions of order up to r across the two faces sharing v_i. The sets $\mathcal{E}(v_i)$, $\mathcal{B}_L(v_i)$ and $\mathcal{B}_R(v_i)$ include only domain points whose corresponding coefficients are involved in smoothness conditions across the face opposite the ray v_i. Finally, $\mathcal{C}$ corresponds to coefficients which do not enter any smoothness conditions.

In Figure 3 we have marked the domain points associated with one trihedron for the case $d = 23$ and $r = 6$ to show which of the above sets they belong to. Dots correspond to points in the sets $\mathcal{D}_\mu(v_i)$, circles to points in the sets $\mathcal{E}(v_i)$, asterisks to points in the caps $\mathcal{A}(v_i)$, plus signs to points in the sets $\mathcal{B}_L(v_i)$ and $\mathcal{B}_R(v_i)$, and $\times$'s to points in the set $\mathcal{C}$.

As in the bivariate case, in order to describe a minimal determining set for $\mathcal{H}_d^r(\mathcal{T})$, we have to take account of certain degenerate faces. In [15] an edge F of a planar triangulation is defined to be degenerate at one of its endpoints v if the edges preceding and succeeding F and connected to v are collinear. We require a similar concept for trihedral decompositions:

DEFINITION 10. *Let F be an interior face of a trihedral decomposition $\mathcal{T}$, and let v be one of the two rays generating it. We say that F is* degenerate *at v if the faces other than F of the two trihedra sharing F and meeting in v are coplanar.*

We also need to adapt the familiar concept of a singular vertex.

DEFINITION 11. *An interior ray v of a trihedral decomposition $\mathcal{T}$ is said to be* singular *if it has precisely four faces meeting at v which lie in two distinct planes.*

In contrast to the planar case where an edge can be degenerate at only one endpoint, for trihedral decompositions, it is possible for a face to be degenerate at both of the rays defining it, see Example 19 below. We are now ready to describe a minimal determining set Γ for $\mathcal{H}_d^r(\mathcal{T})$ in the case $d \geq 3r + 2$.

Algorithm 12. If $d \geq 3r + 2$, choose the set Γ as follows:

1) For each *interior ray* v of $\mathcal{T}$, choose a minimal determining set as described in Theorem 7 for the space $\mathcal{H}_d^r(\mathcal{T})$ restricted to the μ-disk of $\mathcal{O}_v$, where $\mathcal{O}_v$ is the orange surrounding v.
2) For each *boundary ray* v of $\mathcal{T}$, choose a minimal determining set as described in Theorem 7 for the space $\mathcal{H}_d^r(\mathcal{T})$ restricted to the μ-disk of $\mathcal{O}_v$, where $\mathcal{O}_v$ is the orange containing v.
3) For each *trihedron* T in $\mathcal{T}$, choose the functionals corresponding to $\mathcal{C}$ and all three of the sets $\mathcal{A}(v_i)$ associated with T.
4) For each *face* F in T, include the functionals corresponding to the set $\mathcal{E}(v)$ associated with a ray v in an adjoining trihedron and opposite to F. If F is a boundary face, there is only one such trihedron, while if it is an interior face, we can work with either of the two trihedra sharing it. If F is a boundary face, also include the functionals associated with the two sets $\mathcal{B}_L(v)$ and $\mathcal{B}_R(v)$.
5) Suppose v is an interior vertex, and that m of the faces attached to v are degenerate at v. Then for each such face F, remove the functionals corresponding to the cap nearest to v in the triangle preceding F (in counterclockwise order), and replace them by the functionals in the set $\mathcal{B}_L$ associated with F and lying in the same triangle. If F is degenerate at both of its ends, carry out this step at each end. It is easy to see that m can only be 1,2, or 4. For an illustration of this step in the case $m = 1$, see Figures 1 and 2 in [15]).
6) If v is singular, add the functionals corresponding to one cap $\mathcal{A}(v)$ in one of the trihedra containing v.

THEOREM 13. *Let $\mathcal{T}$ be a trihedral decomposition and let $d \geq 3r+2$ and $r \geq 0$. Then the set Γ constructed in Algorithm 12 is a minimal determining set for $\mathcal{H}_d^r(\mathcal{T})$, and its cardinality is given by (15) if $\mathcal{T}$ is total, and by (16) if $\mathcal{T}$ is partial. For each $\lambda \in \Gamma$, there exists a unique spline $s_\lambda \in \mathcal{H}_d^r(\mathcal{T})$ such that (13) holds. Then $\{s_\lambda\}_{\lambda\in\Gamma}$ forms a basis for $\mathcal{H}_d^r(\mathcal{T})$ such that the support of each spline is either a single trihedron, an adjoining pair, or an orange.*

Proof. We give the proof only in the case where $\mathcal{T}$ is total as the case where it is partial is very similar. First we observe that the cardinalities of the sets defined in (38) are as follows:

$$\begin{aligned} \#\mathcal{D}_\mu(v_i) &= \binom{\mu+2}{2} \\ \#\mathcal{A}(v_i) = \#\mathcal{B}_L(v_i) = \#\mathcal{B}_R(v_i) &= \binom{2r-\mu+1}{2} \\ \#\mathcal{E}(v_i) &= dr + d - 12\mu r - 3\mu - 1 + 6r^2 + 4\mu^2 \\ \#\mathcal{C} &= \binom{d-3r-1}{2}. \end{aligned} \tag{40}$$

Moreover, the sets are pairwise disjoint and their union is the set of all domain points in the trihedron $\mathcal{T}$.

Next, we show that the cardinality of the set Γ is given by (15) when $\mathcal{T}$ is total. It can be shown that in this case

$$N = 2(V-2) \quad \text{and} \quad E = 3(V-2), \tag{41}$$

where E is the number of faces of $\mathcal{T}$. Note that step 5 of Algorithm 12 does not change the cardinality of Γ, and that step 2 does not contribute since there are no boundary rays. With these observations it follows from Algorithm 12 and Theorem 7 that

$$\begin{aligned} \#\Gamma = \quad & \sum_{v\in\mathcal{V}} \left[\binom{r+2}{2} + E_v\binom{\mu-r+1}{2} + \tilde{\sigma}_v\right] && \text{(step 1)} \\ + \quad & N\left[\binom{d-3r-1}{2} + 3\binom{2r-\mu+1}{2}\right] && \text{(step 3)} \\ + \quad & E\left[dr + d - 12\mu r - 3\mu - 1 + 6r^2 + 4\mu^2\right] && \text{(step 4)} \\ + \quad & K\binom{2r-\mu+1}{2}, && \text{(step 6)} \end{aligned} \tag{42}$$

where E_v is the number of interior faces meeting at the ray v, K is the number of singular rays, and

$$\tilde{\sigma}_v := \sum_{m=1}^{\mu-r} (r+m+1-me_v)_+ . \tag{43}$$

Using

$$\sum_{v\in\mathcal{T}} E_v = 2E \tag{44}$$

and (41), the equality of the right hand sides of (42) and (15) follows after a straightforward manipulation. (Note that for singular rays, the $\tilde{\sigma}_v$ and the factor multiplying K combine to produce σ_v).

We now show that Γ is a determining set for $\mathcal{H}_d^r(\mathcal{T})$. In the absence of degenerate faces this follows as in [15]. For a degenerate face, note that the coefficients corresponding to points in the cap moved in step 5 of Algorithm 12 are implied to be zero by the smoothness conditions (8) across the degenerate face, independent of the possible relocation of other caps.

To complete the proof, we now construct a basis for $\mathcal{H}_d^r(\mathcal{T})$ satisfying (13). Clearly, for a given $\lambda \in \Gamma$, we can set the coefficient $\lambda s = 1$ and all other coefficients corresponding to $\gamma \in \Gamma$ with $\gamma \neq \lambda$ to zero, we can solve for the remaining coefficients using the smoothness conditions. If the domain point P corresponding to λ is contained in a set $\mathcal{C}$, then the resulting spline s_λ has support on the trihedron T containing P. If P is in a set of the form $\mathcal{E}(v_i)$, then s_λ has support on the union of the two trihedra containing the face opposite v_i. In all other cases, s_λ has support on an orange. □

Remark 14. Instead of constructing an explicit basis, it is also possible to prove the dimension statement in Theorem 13 by showing that the expressions in (15) provides a *lower bound* on $\dim \mathcal{H}_d^r(\mathcal{T})$ as was done in [1] in the planar case. This is done by thinking of $\mathcal{H}_d^r(\mathcal{T})$ as a subspace of $\mathcal{H}_d^0(\mathcal{T})$, enforcing the smoothness conditions in the μ-disks via Theorem 7, and then subtracting the number of appropriate smoothness conditions (8) needed to enforce smoothness across the interior faces of $\mathcal{T}$.

5. A minimal determining set for $\mathcal{H}_d^r(\mathcal{T})$ when $d \geq 4r+1$. As in the case of splines defined on a planar triangulation [8], the construction of a minimal determining set can be greatly simplified if $d \geq 4r+1$. In this case the disks of radius $2r$ around rays of T do not overlap, and the remaining smoothness conditions across faces of $\mathcal{T}$ decouple. In that case the following much simpler algorithm can be used:

Algorithm 15. If $d \geq 4r+1$, choose the set Γ as follows:

1) For each *interior ray* v of $\mathcal{T}$, choose a minimal determining set for $\mathcal{H}_{2r}^r(\mathcal{O}_v)$ as described in Theorem 7, where $\mathcal{O}_v$ is the orange surrounding v.
2) For each *boundary ray* v of $\mathcal{T}$, choose a minimal determining set for $\mathcal{H}_{2r}^r(\mathcal{O}_v)$ as described in Theorem 7. where $\mathcal{O}_v$ is the orange containing v.
3) For each *trihedron* T in $\mathcal{T}$, choose $\mathcal{C}$.
4) For each *face* in $\mathcal{T}$, choose

$$\tilde{\mathcal{E}}(v_1) = \mathcal{F}(v_1) \setminus [\mathcal{D}_{2r}(v_2) \cup \mathcal{D}_{2r}(v_3)], \tag{45}$$

where v_1, v_2, v_3 define a trihedron such that v_2 and v_3 span the face.

For the case $d = 23$ and $r = 5$, Figure 4 shows the choice of the domain points for a single trihedron T using Algorithm 15. As in Figure 3, dots correspond to points in sets of the form $\mathcal{D}_{2r}(v_i)$ and circles correspond to points in $\tilde{\mathcal{E}}(u_i)$, while $\times$'s mark the points in $\mathcal{C}$.

6. The case $d \leq 3r+1$. As in the planar case, it is also possible to treat spline spaces for $d \leq 3r+1$ provided we restrict the class of trihedral decompositions somewhat.

THEOREM 16. *Let $d = 3r+1$, and suppose that the trihedral decomposition $\mathcal{T}$ does not possess any degenerate faces. Then the dimension of $H_{3r+1}^r(\mathcal{T})$ is given by (15) or (16), depending on whether $\mathcal{T}$ is total or partial. Moreover, there exists a basis with local supports as in Theorem 13.*

Proof. A minimal determining set can be constructed by an obvious adaptation of the prescription given in [9] for the planar case. □

— insert file **div2.fig** here —

FIG. 4. *Division of domain points by Algorithm 15,* $d = 23$, $r = 5$.

It is also of interest to consider certain generic decompositions, see [11] for the planar case.

DEFINITION 17. *A trihedral decomposition* $\mathcal{T}$ *is said to be* generic with respect to r and d *provided that for all sufficiently small perturbations of the rays of* $\mathcal{T}$, *the resulting trihedral decomposition* $\tilde{\mathcal{T}}$ *satisfies*

$$\dim H_d^r(\tilde{\mathcal{T}}) = \dim H_d^r(\mathcal{T}). \tag{46}$$

THEOREM 18. *Fix* $d \in \{2,3,4\}$, *and suppose* $\mathcal{T}$ *is a generic trihedral decomposition with respect to* $r = 1$ *and* d. *Then the dimension of* $H_d^1(\mathcal{T})$ *is given by (15) or (16), depending on whether* $\mathcal{T}$ *is total or not.*

Proof. The space $H_d^1(\mathcal{T})$ is isomorphic to the space $\mathcal{S}_d^1(\Delta)$, where Δ is the generalized triangulation (see [11]) obtained by projecting the points in $\mathcal{V}$ through the origin onto a plane that does not contain the origin and is not parallel to any of the rays in $\mathcal{T}$. The result then follows from Theorems 27 and 33 in [11]. □

The proof of Theorem 18 does not involve finding a minimal determining set. For $d = 4$, it may be possible to construct one using the techniques in [7]. However, in the case $d \in \{2,3\}$, no general procedure for finding a minimal determining set is known even in the (generic) planar case.

7. Doubly degenerate faces. While the structure of bivariate splines on planar triangulations and homogeneous splines on trihedral decompositions in $\mathbb{R}^3$ are very similar, there is a situation which can occur in the homogeneous case but cannot occur in the planar case: it is possible for a face to be degenerate at both rays. We illustrate this in the following example.

Example 19. Let

$$v_i = -v_{i+3} = e_i, \quad i = 1,2,3, \tag{47}$$

where e_i denote the standard unit vectors, and let $\mathcal{T}^*$ be the set of trihedra generated by the sets

$$\begin{array}{llll} \{v_1,v_2,v_3\}, & \{v_1,v_2,v_6\}, & \{v_1,v_3,v_5\}, & \{v_1,v_5,v_6\}, \\ \{v_2,v_3,v_4\}, & \{v_2,v_4,v_6\}, & \{v_3,v_4,v_5\}, & \{v_4,v_5,v_6\}. \end{array} \tag{48}$$

The convex hull of these points forms a regular octahedron, see Fig. 5. In the resulting trihedral decomposition each face is degenerate at each of its two rays, and at each ray each face sharing the ray is contained in one of only two planes. Thus, all rays of $\mathcal{T}^*$ are singular.

As a check on our formulae and to provide actual numbers for comparison purposes, we have computed the dimensions of $H_d^r(\mathcal{T}^*)$ in Example 19 for $1 \le r \le 5$ and $1 \le d \le 15$ by setting up the smoothness conditions and numerically computing the rank of the matrix describing the smoothness conditions using the Goliath package [2,3] and other special purpose software. For the trihedral decomposition $\mathcal{T}^*$ there are 6 singular rays. Thus for $d \ge 2r$ the expression (15) becomes

$$\begin{aligned} \phi_d^r &:= 4d^2 - 12dr + 9r^2 + 3r + 2 + 6\sum_{m=1}^{d-r}(r+1-m)_+ \\ &= 2\,(2d^2 - 6dr + 6r^2 + 3r + 1)\,. \end{aligned} \tag{49}$$

This gives

$$\phi_d^r = \begin{cases} 4d^2 - 12d + 20, & \text{if } d \ge 2 \text{ and } r = 1 \\ 4d^2 - 24d + 62, & \text{if } d \ge 4 \text{ and } r = 2 \\ 4d^2 - 36d + 128, & \text{if } d \ge 6 \text{ and } r = 3 \\ 4d^2 - 48d + 218, & \text{if } d \ge 8 \text{ and } r = 4 \\ 4d^2 - 60d + 252, & \text{if } d \ge 10 \text{ and } r = 5. \end{cases} \tag{50}$$

— insert file **octa.fig** here —

FIG. 5. *The regular octahedron.*

In Table 1 we have used an asterisk to mark those cases where the computed dimensions of $H_d^r(\mathcal{T}^*)$ differ from the values of ϕ_d^r. As a curiosity, we note that for the trihedral decomposition $\mathcal{T}^*$, the formulae are in fact correct for $d = 3r + 1$ (and of course all larger values) but not for $d \le 3r$, even though $\mathcal{T}^*$ is not generic and all faces are degenerate.

8. Super splines. As in the planar case [15], the above methods can also be used to compute the dimension and to construct locally supported bases for spaces of homogeneous *super splines*:

$$\mathcal{H}_d^{r,\theta}(\mathcal{T}) := \{s \in \mathcal{H}_d^r(\mathcal{T}) : s \in C^{\rho_v}(v), \quad v \in \mathcal{V}\}, \tag{51}$$

d:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$r=1$:	3*	9*	19*	36	60	92	132	180	236	300	372	452	540	636	740
$r=2$:	3*	6*	13*	24*	39*	61*	90	126	170	222	282	350	426	510	602
$r=3$:	3*	6*	10*	18*	30*	46*	66*	93*	127*	168	216	272	336	408	488
$r=4$:	3*	6*	10*	15*	24*	37*	54*	75*	100*	132*	171*	217*	270	330	398
$r=5$:	3*	6*	10*	15*	21*	31*	45*	63*	85*	111*	141*	178*	222*	273*	331*

Table 1. Dimensions of $H_d^r(\mathcal{T}^*)$ on the regular octahedron.

where $\theta := \{\rho_v\}_{v\in\mathcal{V}}$ and $r \le \rho_v < d$ for all v. Here $s \in C^{\rho_v}(v)$ means that all of the derivatives up to order ρ_v of the pieces of s which join at v have a common value at v. We assume throughout that the μ and ρ_v disks around neighboring rays do not overlap, i.e.,

$$\max\{\mu,\rho_u\} + \max\{\mu,\rho_v\} < d \tag{52}$$

for all pairs of vertices u and v which generate a face of $\mathcal{T}$, where μ is defined in (39).

THEOREM 20. *Let $\mathcal{T}$ be a partial trihedral decomposition and suppose that $d \ge 3r+2$ and that (52) holds. Then*

$$\begin{aligned}\dim \mathcal{H}_d^{r,\theta}(\mathcal{T}) = {} & \left[(d-r)(d-2r) - \binom{r+2}{2}\right] V + \sum_{v\in\mathcal{V}} \binom{\rho_v+2}{2} \\ & - \sum_{v\in\mathcal{V}} \left[E_v \binom{\rho_v - r+1}{2}\right] - 2d^2 + 6dr - 3r^2 + 3r + 2 \\ & + \sum_{v\in\mathcal{V}} \sum_{m=\rho_v-r+1}^{d-r} (r+m+1-me_v)_+ \end{aligned} \tag{53}$$

if $\mathcal{T}$ is a total trihedral partition, and

$$\begin{aligned}\dim \mathcal{H}_d^{r,\theta}(\mathcal{T}) = {} & \frac{(d-r+1)(d-r)}{2} V_B + \left[(d-r)(d-2r) - \binom{r+2}{2}\right] V_I \\ & - \frac{2d^2 - 6dr + 3r^2 - 3r - 2}{2} \\ & - \sum_{v\in\mathcal{V}} \left[E_v \binom{\rho_v - r+1}{2}\right] + \sum_{v\in\mathcal{V}_I} \binom{\rho_v+2}{2} \\ & + \sum_{v\in\mathcal{V}_I} \sum_{m=\rho_v-r+1}^{d-r} (r+m+1-me_v)_+ \end{aligned} \tag{54}$$

if $\mathcal{T}$ is a partial trihedral decomposition. Here E_v is the number of interior faces attached to the vertex v for each $v \in \mathcal{V}$. Moreover, there exists a basis of splines for $\mathcal{H}_d^{r,\theta}(\mathcal{T})$ such that the support of each spline is either a single trihedron, an adjoining pair, or an orange.

Proof. We give the proof for the case of a partial trihedral decomposition. The proof when the decomposition is total is similar (and simpler). The key observation is that the set of points chosen by Algorithm 12 and lying inside the disk $\mathcal{D}_{\rho_v}(v)$ is a minimal determining set for $\mathcal{H}_{\rho_v}^r(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the orange with axis v. Thus, if we now impose C^{ρ_v} continuity at v, then we can replace those $\dim \mathcal{H}^r_{\rho_v}(\mathcal{O}_v)$ points by $\binom{\rho_v+2}{2}$ points lying in one trihedron in $\mathcal{O}_v$. This shows that for each ray v, the change in the number of points in the minimal determining set Γ constructed by Algorithm 12 is given by

$$\dim H^r_{\rho_v}(\mathcal{O}_v) - \dim H^{\rho_v}_{\rho_v}(\mathcal{O}_v) = \dim H^r_{\rho_v}(\mathcal{O}_v) - \binom{\rho_v+2}{2}. \tag{55}$$

Thus,

$$\begin{aligned}
\dim \mathcal{H}^{r,\theta}_d(\mathcal{T}) &= \dim \mathcal{H}^r_d(\mathcal{T}) - \sum_{v\in\mathcal{V}} \left[\dim H^r_{\rho_v}(\mathcal{O}_v) - \binom{\rho_v+2}{2}\right] \\
&= \frac{(d-r+1)(d-r)}{2} V_B + (d-r)(d-2r)V_I \\
&\quad - \frac{2d^2-6dr+3r^2-3r-2}{2} + \sum_{v\in\mathcal{V}} \sum_{m=1}^{d-r} (r+m+1-me_v)_+ \\
&\quad - \sum_{v\in\mathcal{V}_I} \left[\binom{r+2}{2} + E_v \binom{\rho_v-r+1}{2} + \sum_{m=1}^{\rho_v-r} (r+m+1-me_v)_+\right] \\
&\quad - \sum_{v\in\mathcal{V}_B} \left[\binom{\rho_v+2}{2} + E_v\binom{\rho_v-r+1}{2}\right] + \sum_{v\in\mathcal{V}} \binom{\rho_v+2}{2}.
\end{aligned} \tag{56}$$

Now combining terms, we get (54).

Our new minimal determining set can now be used to construct a basis of locally supported splines as was done in the proof of Theorem 3. ☐

The case where all ρ_v are equal is of particular interest.

Corollary 21. *Suppose*

$$\rho_v = \rho \geq r, \quad v \in \mathcal{V}, \tag{57}$$

and that $2\rho < d$. *Then*

$$\begin{aligned}
\dim \mathcal{H}^{r,\theta}_d(\mathcal{T}) &= \frac{(2d^2-6dr-3r^2+12r\rho+3r-5\rho^2-3\rho)}{2} V \\
&\quad + (-2d^2+6rd+3r^2-3r+6\rho^2-12r\rho+6\rho+2) \\
&\quad + \sum_{v\in\mathcal{V}} \sum_{m=\rho-r+1}^{d-r} (r+m+1-me_v)_+
\end{aligned} \tag{58}$$

if $\mathcal{T}$ *is a total trihedral partition, and*

$$\begin{aligned}
\dim \mathcal{H}^{r,\theta}_d(\mathcal{T}) &= \frac{(d^2-2rd-r^2+d+r-2\rho^2+4\rho r-2\rho)}{2} V_B \\
&\quad + \frac{(2d^2-6rd-3r^2+12\rho r+3r-5\rho^2-3\rho)}{2} V_I \\
&\quad + \frac{(-2d^2+6rd+3r^2-3r+6\rho^2-12r\rho+6\rho+2)}{2} \\
&\quad + \sum_{v\in\mathcal{V}_I} \sum_{m=\rho-r+1}^{d-r} (r+m+1-me_v)_+
\end{aligned} \tag{59}$$

if $\mathcal{T}$ is a partial trihedral decomposition. Moreover, there exists a basis of splines for $\mathcal{H}_d^{r,\theta}(\mathcal{T})$ such that the support of each spline is either a single trihedron, an adjoining pair, or an orange.

Proof. Substituting (57) in (53) and using (41) and (44) leads to (58). For a partial decomposition, the classical Euler relations for a triangulation imply

$$\sum_{v \in \mathcal{V}} E_v = 2E_I = 2(V_B + 3V_I - 3). \tag{60}$$

Now substituting (57) in (54) and using (60) leads to (59). □

In both Theorem 20 and Corollary 21, the formula for a total trihedral decomposition can be obtained from the formula for a partial one by dropping the term with V_B and doubling the constant term. Moreover, if we set $\rho = r$ in the corollary, of course we recover the formulae in Theorem 3.

9. Remarks. *Remark 22.* The proof of Theorem 7 is based on the proof of Theorem 3.3 of [17] for polynomial splines on planar triangulations. The description of the minimal determining set for a cell given there is not quite correct in that it allows $\mu_{N-1} < N$ which could lead to the same point being included in Γ twice. This is easily fixed by requiring that $\mu_{N-1} = N$ as we have done here.

Remark 23. As in our paper [5], it is possible to develop a theory of homogeneous splines defined on a (total or partial) decomposition of $\mathbb{R}^2$ by wedges (the two-dimensional analogs of trihedra). Such splines can be restricted to a circle or a similar curve to obtain univariate functions along the curve. The corresponding dimensions and minimal determining sets can be obtained in a straightforward manner by considering a single ring in Theorem 7.

Remark 24. In the bivariate polynomial spline case, there is no question that the right way to define domain points P_{ijk} associated with the Bernstein-Bézier coefficients of a polynomial is by the formula in (6). In that case the set of pairs $\{P, c_P\}_{P \in \mathcal{G}}$ is called the *Bézier net* of s, and has an important geometric interpretation. However, in the trihedral setting, it is not so clear what is the best way to define the analogous points. As discussed in [4], there are reasonable alternatives, although it appears that there is no definition which carries the full geometric significance of the domain points in the planar case. Our choice here is a useful way to label control coefficients.

Remark 25. For polynomial spline spaces on planar triangulations, there are well-known lower and upper bounds on the dimension of $\mathcal{S}_d^r(\Delta)$ which are of interest for $d < 3r + 2$, see e.g. [18] and references therein. Similar bounds can be derived for our homogeneous spline spaces, and will be treated elsewhere.

Remark 26. The formula (54) given in Theorem 20 for a partial trihedral decomposition is much simpler than the corresponding formula in Theorem 2.4 of [15]. Since our proof of Theorem 20 can also be used in the bivariate case, the simpler formula (54) is also valid there.

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