Two Lagrange Interpolation Methods Based on C^1 Splines on Tetrahedral Partitions

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Abstract. We describe two methods which can be used to interpolate function values at a set of given points in a volumetric domain using quadratic and cubic trivariate splines which are C^1 smooth. The methods start with a tetrahedral partition of the data points, and use well-known refinement methods to create finer partitions on which the spline spaces are built. The construction of the interpolating splines requires some additional function values at selected points in the domain, but no derivatives are needed at any points in the three-dimensional space. Both interpolation methods are local and stable, and provide approximation order three for smooth functions.

§1. Introduction

Given a set of points $\mathcal{V} := \{\eta_i\}_{i=1}^n$ in $\mathbb{R}^3$, our aim in this paper is to provide a constructive method for solving the following problem.

Problem 1. *Find a tetrahedral partition whose set of vertices includes $\mathcal{V}$, an N-dimensional space $\mathcal{S}$ of C^1 splines defined on this partition, and a set of additional points $\{\eta_i\}_{i=n+1}^N$ such that for every choice of the data $\{z_i\}_{i=1}^N$, there is a unique spline $s \in \mathcal{S}$ satisfying*

$$s(\eta_i) = z_i, \qquad i = 1, \ldots, N. \tag{1}$$

We call $P := \{\eta_i\}_{i=1}^N$ and $\mathcal{S}$ a Lagrange interpolation pair, *and refer to s as a* Lagrange interpolating spline.

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We emphasize that the spline s solving Problem 1 must be uniquely determined from function values only, in contrast to most known trivariate spline interpolation methods which make use of derivative information. Constructing Lagrange interpolation pairs is a complicated problem, especially since we want a method which

1) is local in the sense that the value of s at a given point η is only influenced by the data values z_i at points η_i which are near η,
2) is stable in the sense that small changes in the data z_i will result in a small change in s,
3) has linear complexity in the sense that given n data points and an initial tetrahedral partition of them, the number of operations required to complete the solution of the problem should be $\mathcal{O}(n)$,
4) yields optimal approximation order in the sense that if $z_i = f(\eta_i)$ for some smooth function f, then the interpolant approximates f to the same order as the best approximation of f from $\mathcal{S}$.

To reach these goals, both $\mathcal{S}$ and P must be carefully chosen. The only result we have found in the literature [21] deals with the special case where the points of $\mathcal{V}$ lie at the corners of a collection of boxes in $\mathbb{R}^3$. The method makes use of C^1 quintic splines. Our aim here is to solve the problem for a general set $\mathcal{V}$. We give two solutions, one based on C^1 cubic splines, and the other on C^1 quadratic splines. Starting with a given tetrahedral partition $\triangle$ of the points $\mathcal{V}$, both methods approximate smooth functions to order three, which is optimal for quadratic splines, but one order less than optimal for cubic splines. Both methods also make use of a certain priority list of the tetrahedra in $\triangle$ for organizing the additional points in P. Our construction is based on the following steps:

1) choose an initial tetrahedral partition $\triangle$ with vertices at the points of $\mathcal{V}$,
2) refine $\triangle$ by splitting each of the tetrahedra of $\triangle$ into subtetrahedra,
3) define an appropriate C^1 spline space $\mathcal{S}$ over the refined tetrahedral partition,
4) use a priority list of the tetrahedra in $\triangle$ to insert additional interpolation points on the edges of the tetrahedra in $\triangle$ to create the point set P.

The paper is organized as follows. In Section 2 we introduce some notation and describe the Bernstein-Bézier representation of splines. In Section 3 we discuss a key algorithm for classifying tetrahedra which also establishes a priority list for later processing the tetrahedra to create a local interpolant. The main results of the paper are contained in Sections 4 and 5, where we introduce the Lagrange interpolating pairs, and show that the corresponding interpolation processes are stable and local. Error bounds for the methods are given in Section 6, and we conclude the paper with several remarks in Section 7.

§2. Preliminaries

Suppose $\triangle$ is a finite set of closed tetrahedra in $\mathbb{R}^3$ whose union is a connected set Ω. Then we call $\triangle$ a regular tetrahedral partition $\triangle$ of Ω provided any pair of tetrahedra intersect only at a common vertex, along a common edge, or along a common triangular face, Given $\triangle$ and an integer $1 < d$, we write

$$\mathcal{S}_d^1(\triangle) := \{s \in C^1(\Omega): \ s|_T \in \mathcal{P}_d, \text{ all } T \in \triangle\}$$

for the space of C^1 polynomial splines of degree d, where $\mathcal{P}_d$ is the $\binom{d+3}{3}$ dimensional space of trivariate polynomials of degree d.

Given a (non-degenerate) tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$ in $\triangle$ with vertices v_1, v_2, v_3, v_4, let

$$\mathcal{D}_{d,T} := \{\xi_{ijk\ell}^T := (iv_1 + jv_2 + kv_3 + \ell v_4)/d\}_{i+j+k+\ell=d}$$

be the associated set of domain points. We write $\mathcal{D}_{d,\triangle}$ for the union of the sets $\mathcal{D}_{d,T}$ over all $T \in \triangle$. Given an integer $0 \le m < d$, let $R_m^T(v_1) := \{\xi_{ijk\ell}^T : \ i = d - m\}$, $D_m^T(v_1) := \{\xi_{ijk\ell}^T : \ i \ge d - m\}$, and associated with the edge $e := \langle v_3, v_4 \rangle$, let $E_m^T(e) := \{\xi_{ijk\ell}^T : \ i, j \le m\}$. We call $R_m(v_1) := \bigcup\{R_m^T(v_1) : \ T$ has a vertex at $v_1\}$ the shell of radius m around v_1, and $D_m(v_1) := \bigcup\{D_m^T(v_1) : \ T$ has a vertex at $v_1\}$ the ball of radius m around v_1. The analogous sets associated with other vertices and edges are defined similarly.

Throughout the paper we make use of the well-known Bernstein-Bézier representation of trivariate splines: for every spline s in $\mathcal{S}_d^1(\triangle)$,

$$s|_T = \sum_{i+j+k+\ell=d} c_{ijk\ell} \ B_{ijk\ell}^d, \tag{2}$$

where $B_{ijk\ell}^d = \frac{d!}{i!j!k!\ell!} \ \lambda_1^i \lambda_2^j \lambda_3^k \lambda_4^\ell, \ i + j + k + \ell = d$, are the Bernstein basis polynomials of degree d associated with T. Here, $\lambda_\nu \in \mathcal{P}_1, \ \nu = 1, \ldots, 4$, are the barycentric coordinates associated with T. It is well known that each continuous spline on $\triangle$ of degree d is uniquely determined by its corresponding set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. To describe smoothness conditions for splines, suppose that $T := \langle v_1, v_2, v_3, v_4 \rangle$ and $\tilde{T} := \langle v_1, v_2, v_3, v_5 \rangle$ are two adjoining tetrahedra from $\triangle$ which share the oriented triangular face $F := \langle v_1, v_2, v_3 \rangle$ (i.e. T and $\tilde{T}$ are neighboring tetrahedra), and let $s|_T$ be given as in (2) and

$$s|_{\tilde{T}} := \sum_{i+j+k+\ell=d} \tilde{c}_{ijk\ell} \ \widetilde{B}_{ijk\ell}^d, \tag{3}$$

where $\widetilde{B}^d_{ijk}$ are the Bernstein basis polynomials of degree d associated with the tetrahedron $\tilde{T}$. Then, the C^1 smoothness of s across F is equivalent to $c_{ijk0} = \tilde{c}_{ijk0},\ i+j+k=d$, and

$$c_{ijk1} = \lambda_1(v_5)\ \tilde{c}_{i+1,j,k,0} + \lambda_2(v_5)\ \tilde{c}_{i,j+1,k,0} + \lambda_3(v_5)\ \tilde{c}_{i,j,k+1,0} + \lambda_4(v_5)\ \tilde{c}_{i,j,k,1}, \tag{4}$$

where $i+j+k=d-1$. The smoothness condition (4) can be used to calculate the B-coefficient on the left provided the B-coefficients on the right are given, and clearly this calculation is stable since the weights $\{\lambda_\nu(v_5)\}_{\nu=1}^4$ depend only on the smallest angle in $\{T,\tilde{T}\}$.

As usual, we define $\mathrm{star}^0(T) := T$, and for $\ell \ge 1$, define $\mathrm{star}^\ell(T)$ to be the union of the set of all tetrahedra from $\triangle$ which touch a tetrahedra in $\mathrm{star}^{\ell-1}(T)$. (Two tetrahedra from $\triangle$ are said to touch each other, if they have at least one common vertex.)

We recall that a subset $\mathcal{M}$ of $\mathcal{D}_{d,\triangle}$ is a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ provided that setting the B-coefficients $\{c_\xi\}_{\xi\in\mathcal{M}}$ of $s \in \mathcal{S}$ determines all other B-coefficients. It is called a minimal determining set (MDS) provided that there is no smaller determining set. It is well known that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$ if and only if setting the B-coefficient $\{c_\xi\}_{\xi\in\mathcal{M}}$ *uniquely* determines all B-coefficients of $s \in \mathcal{S}$. A MDS $\mathcal{M}$ is called ℓ-local provided that there exists an integer ℓ such that for every tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$ and every $\xi \in T \cap \mathcal{D}_{d,\triangle}$, the B-coefficient c_ξ of a spline in $\mathcal{S}$ can be computed from the B-coefficients $\{c_\eta\}_{\eta\in\Lambda_\xi}$, where $\Lambda_\xi \subseteq \mathcal{M} \cap \mathrm{star}^\ell(T)$. The MDS $\mathcal{M}$ is said to be stable provided that there exists a constant K depending only on the smallest angle of $\triangle$ such that

$$|c_\xi| \le K \max_{\eta\in\Lambda_\xi} |c_\eta|, \tag{5}$$

for all $\xi \in \mathcal{D}_{d,\triangle}$.

Finally, we say that an interpolation method based on a Lagrange interpolation pair P and $\mathcal{S}$ is ℓ-local provided there is an integer ℓ such that for every for every tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$ and every $\xi \in T \cap \mathcal{D}_{d,\triangle}$, the B-coefficient c_ξ of an interpolating spline in $\mathcal{S}$ can be computed from the values $\{z_\eta\}_{\eta\in\Gamma_\xi}$ for some $\Gamma_\xi \subseteq P \cap \mathrm{star}^\ell(T)$. The interpolation method is called stable provided there exists a constant C depending only on the smallest angle in $\triangle$ such that

$$|c_\xi| \le C \max_{\eta\in\Gamma_\xi} |z_\eta|, \tag{6}$$

for all $\xi \in \mathcal{D}_{d,\triangle}$.

For convenience, we say that the computation of the B-coefficients of a spline in $\mathcal{S}$ is a ℓ-local and stable process whenever $\mathcal{M}$ is a ℓ-local stable MDS for $\mathcal{S}$, or the coefficients are computed from an interpolation method based on a Lagrange interpolation pair P and $\mathcal{S}$ which is ℓ-local and stable.

§3. Classifying Tetrahedral Partitions

The key to our construction of Lagrange interpolating pairs giving a local and stable solution to Problem 1 is the following algorithm for separating the tetrahedra of a given tetrahedral partition $\triangle$ into classes $\mathcal{T}_0, \ldots, \mathcal{T}_4$. The algorithm also creates an ordering $T_1, \ldots, T_{n_0}$ of the tetrahedra of $\triangle$.

Algorithm 2.

1) *Let all vertices of $\triangle$ be unmarked,*
2) *for $i = 0$ to 3:*

   Repeat until no longer possible: choose a tetrahedron T such that exactly i of its vertices are marked. Put T in $\mathcal{T}_i$, and mark the remaining vertices of T,
3) *put all remaining tetrahedra in $\mathcal{T}_4$.*

Algorithm 2 defines an ordering $T_1, \ldots, T_{n_0}$ of the tetrahedra, where the tetrahedra are listed in the order depending on the number of marked vertices of $\triangle$. The algorithm is easy to program, and is efficient enough to decompose large tetrahedral partitions on a standard PC. Note that for a given tetrahedral partition, there may be many choices at each step, so obviously the decomposition is not unique.

The next lemma establishes some simple properties of the decomposition obtained from the above algorithm. These properties will be used to prove the locality of the spline interpolation methods described below.

Lemma 3. *Suppose $\mathcal{T}_0, \ldots, \mathcal{T}_4$ are the classes of tetrahedra created by Algorithm 2. Then*

1) *no two tetrahedra in the class $\mathcal{T}_0$ can touch each other,*
2) *if two tetrahedra in the same class $\mathcal{T}_i$ touch at a vertex v, then they must also touch a tetrahedra in one of the classes $\mathcal{T}_j$ with $0 \leq j \leq \min(3, i-1)$ at the same vertex v.*

Proof: The first assertion is obvious. To establish 2), first note that after marking the vertices of tetrahedra in classes $\mathcal{T}_0, \ldots, \mathcal{T}_3$, all vertices of $\triangle$ are marked. This establishes the claim for $i = 4$. Now suppose two or more tetrahedra in $\mathcal{T}_1$ touch at a vertex v, and let $T, \tilde{T}$ be the first two marked by Algorithm 2. If v is not a vertex of some tetrahedra in $\mathcal{T}_0$, then before v was marked, $\tilde{T}$ would not have touched any marked tetrahedra, and so would have been put in class $\mathcal{T}_0$, since all its vertices would have been unmarked. A similar argument shows that 2) also holds for the classes $\mathcal{T}_2$ and $\mathcal{T}_3$. □

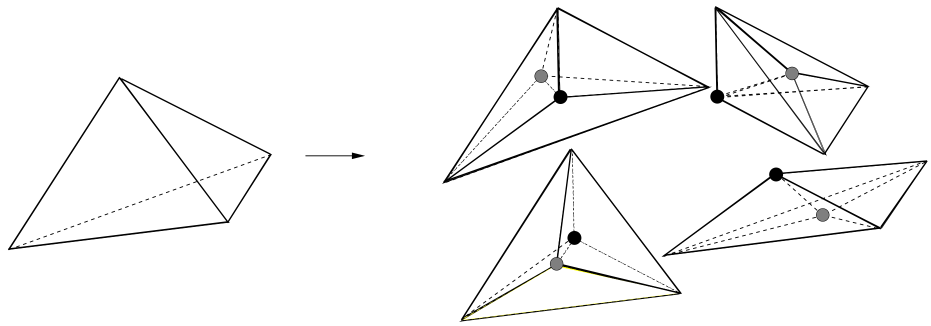

Fig. 1. The Worsey-Farin split subdivides a tetrahedron T into twelve subtetrahedra by connecting a point v_T from the interior of T with its vertices as well as with a point v_F in the interior of each triangular face of T.

§4. C^1 Cubic Splines on Worsey-Farin Refinements

Given an arbitrary tetrahedral partition $\triangle$, let $\triangle_{WF}$ be the tetrahedral partition obtained as follows:

1) for each tetrahedron $T \in \triangle$, choose a point v_T in the interior of T, and connect it to the four vertices of T,
2) for each triangular face F of $\triangle$, choose a point v_F in the interior of F and connect it to the three vertices of F and to the point v_T for each tetrahedron sharing the face F.

We call $\triangle_{WF}$ a generic Worsey-Farin refinement of $\triangle$. However, for our purposes, we cannot work with arbitrary generic Worsey-Farin refinements. We say that a generic Worsey-Farin refinement $\triangle_{WF}$ is an admissible Worsey-Farin refinement of $\triangle$ provided that if T and $\tilde{T}$ are two neighboring tetrahedra sharing a face F, then the split point v_F lying in the interior of F lies on the line joining v_T and $v_{\tilde{T}}$. The Worsey-Farin refinement for one tetrahedron is illustrated in Fig. 1, where v_T is shown (four times) as a black dot, while the four split points v_F are shown as grey dots.

In order to define the spline space of interest here, we need some additional notation. For each edge $e := \langle u, v\rangle$ of $\triangle$, let F_e^1 and F_e^2 be two consecutive faces of $\triangle_{WF}$ which contain e. Let $D_{e,1}$ be the directional derivative associated with the vector $\langle u, v_F^1\rangle$ where v_F^1 is the split point in the face F_e^1, and let $D_{e,2}$ be the analogous derivative associated with face F_e^2.

For an admissible Worsey-Farin refinement $\triangle_{WF}$ of $\triangle$, we define

$$\hat{\mathcal{S}}_3^1(\triangle_{WF}) := \{s \in \mathcal{S}_3^1(\triangle_{WF}) : D_{e,1}s|_e, D_{e,2}s|_e \in \mathcal{P}_1, \quad \text{for each edge } e \text{ of } \triangle\}. \tag{7}$$

Let n_V and n_E be the number of vertices and edges of $\triangle$, respectively. For each $v \in \mathcal{V}$, let T_v be a tetrahedron in $\triangle_{WF}$ with vertex at v.

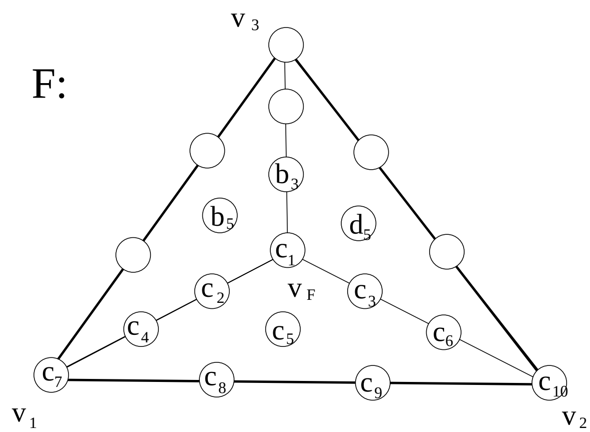

Fig. 2. Labelling of B-coefficients in the proof of Theorem 4.

Theorem 4. *The dimension of* $\hat{\mathcal{S}}_3^1(\triangle_{WF})$ *is* $4n_V$, *and* $\mathcal{M} := \bigcup_{v\in\mathcal{V}} D_1^{T_v}(v)$ *is a 1-local stable minimal determining set.*

Proof: First we show that $\mathcal{M}$ is a determining set. Suppose $s \in \hat{\mathcal{S}}_3^1(\triangle_{WF})$, and set the B-coefficients of s to zero corresponding to all domain points in $\mathcal{M}$. This implies that s and its derivatives of order 1 must vanish at all vertices of $\triangle$. Now for each edge $e := \langle u, v\rangle$ of $\triangle$, the fact that the univariate linear polynomials $g_e^1 := D_{e,1}s|_e$ and $g_e^2 := D_{e,2}s|_e$ vanish at both u and v implies that they vanish everywhere on e. It follows from [25] that $s \equiv 0$. This shows that $\mathcal{M}$ is a determining set for $\hat{\mathcal{S}}_3^1(\triangle_{WF})$, and $\dim \hat{\mathcal{S}}_3^1(\triangle_{WF}) \le \#\mathcal{M} = 4n_V$.

It was shown in [25] (see also [12]) that the dimension of $\mathcal{S}_3^1(\triangle_{WF})$ is $4n_V + 2n_E$. Our space $\hat{\mathcal{S}}_3^1(\triangle_{WF})$ is the subspace of splines s where g_e^1 and g_e^2 are linear polynomials for each edge e of $\triangle$. We claim that enforcing this condition requires exactly two linear constraints on the B-coefficients of s for each edge e. To verify the claim, suppose the coefficients of s in face F_e^1 are as in Fig. 2. Then for all $0 \le t \le 1$,

$$g_e^1(t) = (1-t)^2(c_4 - c_7) + 2t(1-t)(c_5 - c_8) + t^2(c_6 - c_9). \tag{8}$$

This implies g_e^1 is linear if and only if

$$c_5 = \frac{c_4 - c_7 + c_6 - c_9 + 2c_8}{2}. \tag{9}$$

The condition for g_e^2 to be linear is similar. We have now shown that $\dim \hat{\mathcal{S}}_3^1(\triangle_{WF}) \ge 4n_V + 2n_E - 2n_E = 4n_V$, and combining this with the upper bound established above, we conclude that $\dim \hat{\mathcal{S}}_3^1(\triangle_{WF}) = 4n_V$. This implies $\mathcal{M}$ is not just a determining set, but is in fact a minimal determining set.

We now show that $\mathcal{M}$ is stable and 1-local. Clearly, the B-coefficients of s corresponding to domain points in the balls $D_1(v)$ can be computed directly from B-coefficients in $\mathcal{M} \cap D_1^{T_v}(v)$ using smoothness conditions. This is a stable local process. Now let T be a tetrahedron in $\triangle$. We begin by examining the computation of the B-coefficients of s corresponding to the remaining domain points on $R_3(v_T)$, i.e., on the faces of T. Let $F := \langle v_1, v_2, v_3 \rangle$ be such a face. Then referring to Fig. 2, we need to compute the B-coefficients labelled $c_1, \ldots, c_6$ and b_3, b_5, d_5. We first discuss coefficient c_5. Let $e = \langle v_1, v_2 \rangle$. The requirement that $D_{e,1}s|_e$ and $D_{e,2}s|_e$ be linear polynomials implies that $D_{\langle v_1, v_F \rangle}s|_e$ is also linear, where v_F is the split point of F. But then coefficient c_5 can be computed directly from (9). Clearly, this is a stable local process. The B-coefficients b_5 and d_5 can be computed in a similar way. Now suppose that $(\lambda_1, \lambda_2, \lambda_3)$ are the barycentric coordinates of v_F relative to F. Then $c_2 = \lambda_1 c_4 + \lambda_2 c_5 + \lambda_3 b_5$, with similar formulae for the remaining B-coefficients associated with domain points on F.

Next we consider B-coefficients corresponding to domain points on the shell $R_2(v_T)$. All of these B-coefficients can be stably computed from B-coefficients on $R_3(v_T)$ using C^1 smoothness. Repeating this process, we stably compute the B-coefficients corresponding to domain points on $R_1(v_T)$, and finally the B-coefficient corresponding to the domain point v_T. □

We are now ready to define a point set P_3 to go with the space $\hat{\mathcal{S}}_3^1(\triangle_{WF})$. Suppose that $T_1, \ldots, T_{n_0}$ is a the ordering of the tetrahedra induced by Algorithm 2.

Algorithm 5.

1) *Put all vertices of* $\triangle$ *in* P_3,
2) *for* $i = 1, \ldots, n_0$:

   For each edge $e := \langle u, v \rangle$ *of* T_i:

   a) *if neither* u *nor* v *are vertices of some tetrahedron* T_j *with* $j < i$, *put the points* $\frac{2u+v}{3}$ *and* $\frac{u+2v}{3}$ *into* P_3,
   b) *if* u *is a vertex of some tetrahedron* T_j *with* $j < i$ *but* v *is not, put the point* $\frac{u+2v}{3}$ *into* P_3,
   c) *if* v *is a vertex of some tetrahedron* T_j *with* $j < i$ *but* u *is not, put the point* $\frac{2u+v}{3}$ *into* P_3.

The cardinality of P_3 is $4n_V$, where n_V is the number of vertices of $\triangle$. We are now ready to prove the main result of this section, namely that P_3 and $\hat{\mathcal{S}}_3^1(\triangle_{WF})$ form a Lagrange interpolation pair. At the same time we show that the corresponding interpolation method is 4-local and stable as defined in Sect. 2.

Theorem 6. *Given any real numbers $\{z_\eta\}_{\eta\in P_3}$, there exists a unique spline $s \in \hat{\mathcal{S}}_3^1(\triangle_{WF})$ such that $s(\eta) = z_\eta$ for all $\eta \in P_3$. Moreover, the computation of the B-coefficients of s is a 4-local and stable process.*

Proof: Suppose we are given $Z := \{z_\eta\}_{\eta\in P_3}$. We begin by showing that the B-coefficients of s associated with the points in the balls $D_1(v)$ are uniquely determined by Z for all vertices v of $\triangle$. We do this by considering one tetrahedron at a time, where we go through the tetrahedra $T_1, \ldots, T_{n_0}$ of $\triangle$ in the order defined by Algorithm 2.

Let $\mathcal{T}_0, \ldots, \mathcal{T}_4$ be the classes of tetrahedra created by the algorithm. We say that a vertex of $\triangle$ is a type-k vertex if it is a vertex of a tetrahedron in $\mathcal{T}_k$, but not a vertex of any tetrahedron in $\mathcal{T}_j$ with $0 \le j < k$. Note that every vertex of $\triangle$ must be of type 0, 1, 2 or 3.

We first consider the tetrahedron $T := T_1 \in \mathcal{T}_0$. According to Algorithm 5, for each edge $e := \langle u, v\rangle$ of T, P_3 contains the points $u, \frac{2u+v}{3}, \frac{u+2v}{3}, v$. But since s reduces to a univariate polynomial on e, it follows that the B-coefficients of s corresponding to the four domain points on e are uniquely determined by interpolation at these points. This is a stable computation since the matrix of the system depends only on barycentric coordinates, and is the same for all edges. Similarly, s is uniquely defined on the edges of each of the other tetrahedra in the class $\mathcal{T}_0$, since by Lemma 3, two different tetrahedra in class $\mathcal{T}_0$ cannot touch each other.

Since $s \in C^1(v)$ for every vertex v of $\triangle$, it follows that all of the B-coefficients c_ξ of s corresponding to domain points ξ in the balls $D_1(v)$ are uniquely determined for all type-0 vertices v of $\triangle$. If ξ is one of these domain points and lies in the tetrahedron T, then (6) holds with $\Gamma_\xi \subseteq \mathrm{star}(T)$.

Now suppose we have completed the computation of c_ξ for all domain points ξ in the balls $D_1(v)$, where v is a vertex of the tetrahedra $T_1, \ldots, T_{i-1}$, and let $T := T_i \in \mathcal{T}_1$. Then by the definition of $\mathcal{T}_1$, there must be a vertex u of T where T touches at least one of the tetrahedra in $\{T_1, \ldots, T_{i-1}\}$, and does not touch any of these tetrahedra anywhere else. Let T_u be the first such tetrahedron, which by the ordering must be in $\mathcal{T}_0 \cup \mathcal{T}_1$. Statement 2 of Lemma 3 implies $T_u \in \mathcal{T}_0$. Thus, u must be a type-0 vertex, and the B-coefficients $\{c_\xi\}_{\xi\in D_1^T(u)}$ are already known. We emphasize here that T can touch other tetrahedra $\tilde{T}$ in class $\mathcal{T}_1$ at u, but even if it does, the B-coefficients of s corresponding to $\xi \in D_1^T(u)$ do not depend on those associated with domain points in $D_1^{\tilde{T}}(u)$ since u is a type-0 vertex. If $e = \langle v, w\rangle$ is an edge of T which does not have u as an endpoint, then according to Algorithm 5, P_3 contains the points $v, \frac{2v+w}{3}, \frac{v+2w}{3}, w$. For these edges of T, the above argument shows that the B-coefficients of s associated with the domain points on e are uniquely determined by interpolation at these points. We now consider edges of the form $e = \langle u, v\rangle$. Then Algorithm 5 shows that P_3 contains the point $\frac{u+2v}{3}$, and no other points in the interior of e. Now we already know three of the B-coefficients of the univariate polynomial $s|_e$, and the fourth B-coefficient can be computed from the interpolation condition at $\frac{u+2v}{3}$. This is easily seen to be a stable computation. Similarly, s is uniquely defined on the edges of each of the other tetrahedra in the class $\mathcal{T}_1$.

Since $s \in C^1(v)$ for every vertex v of $\triangle$, it follows that all of the B-coefficients c_ξ of s corresponding to domain points ξ in the balls $D_1(v)$ are uniquely determined for all type-1 vertices v of $\triangle$. If ξ is one of these points and lies in the tetrahedron T, then (6) holds with $\Gamma_\xi \subseteq \text{star}^2(T)$.

Now suppose we have completed the computation of c_ξ for all the domain points in the balls $D_1(v)$, where v is a vertex of the tetrahedra $T_1, \ldots, T_{i-1}$, and let $T := T_i := \langle v_1, v_2, v_3, v_4\rangle \in \mathcal{T}_2$. Then by the definition of $\mathcal{T}_2$, there must be two vertices, say v_1, v_2, where T touches some tetrahedra T_{v_1} and T_{v_2} in $\{T_1, \ldots, T_{i-1}\}$, and T does not touch these tetrahedra anywhere else. We may suppose T_{v_1} and T_{v_2} are the first such tetrahedra. Then statement 2 of Lemma 3 implies that these tetrahedra must be in $\mathcal{T}_0 \cup \mathcal{T}_1$, and thus v_1 and v_2 are either type-0 or type-1 vertices. This means that the B-coefficients of s corresponding to domain points in $D_1^T(v_1) \cup D_1^T(v_2)$ are already known. Thus, the B-coefficients of $s|_{\langle v_1, v_2\rangle}$ are already determined. Now by construction, the set P_3 contains exactly four points on the edge $e := \langle v_3, v_4\rangle$, and as before the B-coefficients of $s|_e$ are uniquely determined. We now deal with the four remaining edges of T. Each such edge e contains either v_1 or v_2, and $s|_e$ has just one undetermined B-coefficient, which is uniquely determined by interpolation at the one point in P_3 lying in the interior of e. Similarly, s is uniquely defined on the edges of each of the other tetrahedra in the class $\mathcal{T}_2$, and it follows that all of the B-coefficients c_ξ of s corresponding to domain points ξ in the balls $D_1(w)$ are uniquely determined for all type-2 vertices w of $\triangle$. For these ξ, we see that (6) holds with $\Gamma_\xi \subseteq \text{star}^3(T)$, where T is the tetrahedron containing ξ.

A similar proof shows that the B-coefficients in the balls $D_1(v)$ surrounding vertices of tetrahedra $T := T_i \in \mathcal{T}_3$ are uniquely determined by interpolation at the points of P_3. If $\xi \in T$ is in one of these balls, it follows that $\Gamma_\xi \subseteq \text{star}^4(T)$ and (6) holds. This completes the proof that all B-coefficients of s corresponding to domain points in balls $D_1(v)$ surrounding vertices v of $\triangle$ are uniquely determined in a local stable way from the Lagrange data. Now by Theorem 4, all remaining B-coefficients of s can be stably computed from these B-coefficients. In particular, if $\xi \in \mathcal{D}_{3,\triangle_{WF}}$ lies in a tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$, then c_ξ can be computed from the B-coefficients in $\bigcup_{i=1}^4 D_1^T(v_i)$, and it follows that (6) holds with $\Gamma_\xi \subseteq \text{star}^4(T)$. $\square$

The proof of Theorem 6 shows that in the worst case, a B-coefficient

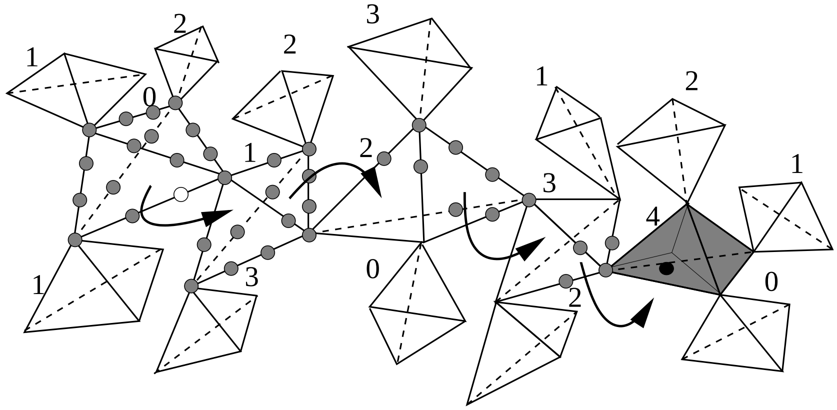

Fig. 3. An example where Γ_ξ can be as large as $\mathrm{star}^4(T)$.

c_ξ with $\xi \in \mathcal{D}_{3,\triangle_{WF}}$ in a tetrahedron T of $\triangle$ depends only on the values $\{z_\eta\}_{\eta\in\Gamma_\xi}$, where $\Gamma_\xi \subseteq \mathrm{star}^4(T)$. In Fig. 3 we show a part of a tetrahedral partition to illustrate that this worst case behavior can occur. The numbers on top of each of the tetrahedra indicate the classes to which they belong. Suppose ξ is the domain point marked with a black dot in the grey tetrahedron in class $\mathcal{T}_4$ on the right. We claim that the value of c_ξ depends on the value of z_η, where η is a point in P_3 marked with a white dot lying in the tetrahedron on the left which is numbered zero. The arrows indicate the direction of propagation, and the interpolation points in P_3 on the edges of these tetrahedra are marked with grey dots. As is clear from the proof of Theorem 6, the worst case of $\mathrm{star}^4(T)$ only appears in very particular constellations, and for most ξ, the set Γ_ξ is much smaller.

§5. C^1 Quadratic Splines on Worsey-Piper Refinements

Given an arbitrary tetrahedral partition $\triangle$, let $\triangle_{WP}$ be a tetrahedral partition obtained from $\triangle$ by the following procedure:

1) for each tetrahedron T, choose a point v_T in the interior of T and connect it to the four vertices of T,
2) for each triangular face F of $\triangle$, choose a point v_F in the interior of the face and connect it to the three vertices of F as well as to the point v_T for each tetrahedron T which shares the face F,
3) for each edge e of $\triangle$, choose a point v_e in the interior of e and connect it to the point v_F for all faces F sharing the edge e, and the point v_T for each tetrahedron T sharing e.

We call $\triangle_{WP}$ a generic Worsey-Piper refinement of $\triangle$. The Worsey-Piper refinement of one tetrahedron results in 24 subtetrahedra. It is illustrated in Fig. 4, where v_T is shown (four times) as a black dot, the four (face)

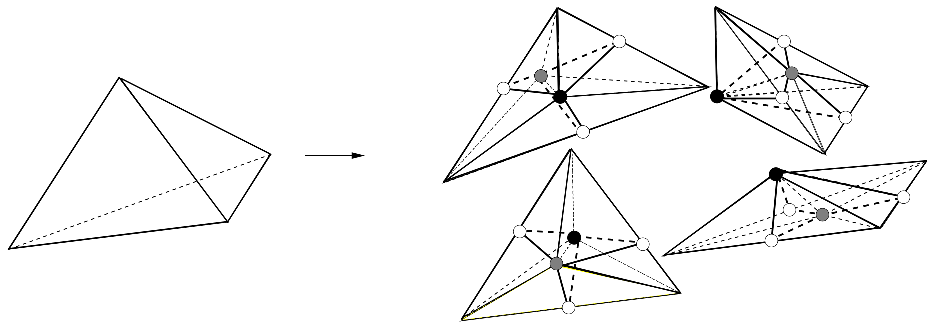

Fig. 4. The Worsey-Piper split subdivides a tetrahedron T into 24 subtetrahedra. First, the Worsey-Farin split is applied to T. Then, the resulting partition is further subdivided by connecting a point v_e from the interior of each edge e of T with the splitting point v_T as well as with the face splitting points v_F of the faces sharing e.

split points v_F are shown as grey dots, and the six (edge) split points v_e are shown as white dots. Note that with this split, each face of T has been subjected to the well-known Powell-Sabin split into 6 triangles.

For our purposes, we cannot work with arbitrary generic Worsey-Piper refinements. We say that a refinement $\triangle_{WP}$ of $\triangle$ is an admissible Worsey-Piper split of $\triangle$ provided that:

1) for each edge e of $\triangle$, the point v_e is coplanar with the points v_T for all tetrahedra T sharing the edge e, and also coplanar with the points v_F for all faces F of $\triangle$ containing e,
2) for each interior face F of $\triangle$, the point v_F lies on the line joining the points v_T and $v_{\tilde{T}}$ associated with the two tetrahedra $T, \tilde{T}$ sharing the face F.

Throughout the remainder of this section we assume that $\triangle_{WP}$ is an admissible Worsey-Piper refinement of $\triangle$. Worsey-Piper splits were introduced in [26] as a means for constructing C^1 quadratic tetrahedral macro-elements. Here we use the spline space $\mathcal{S}_2^1(\triangle_{WP})$ to construct a Lagrange interpolation pair solving Problem 1. It is known [26] that the dimension of $\mathcal{S}_2^1(\triangle_{WP})$ is $4n_V$, where n_V is the number of vertices of $\triangle$. In addition, it is shown in [26] that each spline $s \in \mathcal{S}_2^1(\triangle_{WP})$ is uniquely determined by the values and the first derivatives at the vertices of $\triangle$. As is well known in Bernstein-Bézier theory, setting the value and the first derivatives of a C^1 spline at vertex v of $\triangle$ is equivalent to setting the B-coefficients associated with the domain points in $D_1^T(v)$ for some tetrahedron T_v in $\triangle_{WP}$ with vertex v. It follows that the set $\mathcal{M} := \bigcup_{v \in \mathcal{V}} D_1^{T_v}(v)$ is a stable 1-local minimal determining set for $\mathcal{S}_2^1(\triangle_{WP})$ (see [12]).

In order to solve the Lagrange interpolation problem, we now define a point set P_2 to go with $\mathcal{S}_2^1(\triangle_{WP})$. Suppose that $T_1, \ldots, T_{n_0}$ is the ordering of the tetrahedra induced by Algorithm 2. For each edge e of $\triangle$, let v_e be the split point on e.

Algorithm 7.

1) *Put all vertices of* $\triangle$ *in* P_2,
2) *for* $i = 1, \ldots, n_0$:

   For each edge $e := \langle u, v \rangle$ *of* T_i:

   a) *if neither* u *nor* v *are vertices of some tetrahedron* T_j *with* $j < i$, *put the points* $\frac{u+v_e}{2}$ *and* $\frac{v_e+v}{2}$ *into* P_2,
   b) *if* u *is a vertex of some tetrahedron* T_j *with* $j < i$ *but* v *is not, put the point* $\frac{v_e+v}{2}$ *into* P_2,
   c) *if* v *is a vertex of some tetrahedron* T_j *with* $j < i$ *but* u *is not, put the point* $\frac{u+v_e}{2}$ *into* P_2.

The cardinality of the set P_2 produced by Algorithm 7 is $4n_V$, where n_V is the number of vertices of $\triangle$.

We are now ready to prove that P_2 and $\mathcal{S}_2^1(\triangle_{WP})$ form a Lagrange interpolation pair, and that the associated interpolation method is 4-local and stable.

Theorem 8. *Given any real numbers* $\{z_\eta\}_{\eta \in P_2}$, *there exists a unique spline* $s \in \mathcal{S}_2^1(\triangle_{WP})$ *such that* $s(\eta) = z_\eta$ *for all* $\eta \in P_2$. *Moreover, the computation of the B-coefficients of* s *is a 4-local and stable process.*

Proof: The proof is nearly the same as the proof of Theorem 6, except that here the univariate interpolation schemes on the edges are slightly different. For each edge e of $\triangle$, a spline $s \in \mathcal{S}_2^1(\triangle_{WP})$ reduces to a univariate quadratic C^1 spline on e which is defined by five B-coefficients associated with the domain points on e. As we go through the tetrahedra of $\triangle$ in the order defined by Algorithm 2, we have to solve three different types of interpolation problems involving B-coefficients associated with domain points on an edge e, namely, where 1) all five B-coefficients are unknown, 2) three B-coefficients corresponding to domain points on a subinterval of e are known, and 3) only the B-coefficient corresponding to the domain point v_e is unknown, where v_e is the split point of e. Since the matrices of the systems resulting from these interpolation processes depend only on the barycentric coordinates of the split point v_e, this is a stable process. Once all the B-coefficients corresponding to domain points in the balls $D_1(v)$ are uniquely and locally determined for all vertices v of $\triangle$, the remaining B-coefficients are also stably determined, which follows from the methods described in [12,23,26]. Arguing as in the proof of Theorem 6, it follows that the computations are 4-local. The example given in Fig. 3 shows that worst case can occur, i.e., we cannot assert the method is 3-local. □

§6. Bounds on the Error of Lagrange Interpolation

Let $\triangle$ be some initial tetrahedral partition of a volumetric domain Ω, and suppose P and $\mathcal{S}$ is a Lagrange interpolation pair with $\mathcal{S} \subseteq \mathcal{S}_d^1(\tilde{\triangle})$, where $\tilde{\triangle}$ is a refinement of $\triangle$. Then for every $f \in C(\Omega)$, there is a unique spline $\mathcal{I}f \in \mathcal{S}$ such that

$$\mathcal{I}f(\eta) = f(\eta), \qquad \eta \in P.$$

Clearly, this defines a linear projector $\mathcal{I}$ mapping $C(\Omega)$ onto $\mathcal{S}$. We now give an error bound for $f - \mathcal{I}f$, under the assumption that the interpolation method $\mathcal{I}$ corresponding to P and $\mathcal{S}$ is stable and ℓ-local as defined in Sect. 2.

Let $W_\infty^m(\Omega)$ be the classical Sobolev space, and let

$$|f|_{m,\infty,B} := \sum_{|\alpha|=m} \|D^\alpha f\|_B \tag{10}$$

for any compact subset B of Ω, where $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$, and $\|.\|_B$ denotes the infinity norm on B. Let $|\triangle|$ be the mesh size of $\triangle$, i.e. the maximum diameter of the tetrahedra in $\triangle$.

The hypotheses of the following result hold for the Lagrange interpolation pairs described in Sects. 4 and 5 with $\ell = 4$ and $\bar{m} = 2$, where $d \in \{2, 3\}$.

Theorem 9. *Suppose $\mathcal{I}$ is the interpolation projector associated with a Lagrange interpolation pair P and $\mathcal{S}$ which is ℓ-local and stable for some ℓ. Suppose $\mathcal{I}p = p$ for all $p \in \mathcal{P}_{\bar{m}}$ with $\bar{m} \le d$. Then there exists a constant K depending only on ℓ and the smallest angle in $\triangle$ such that for any $0 \le m \le \bar{m}$,*

$$\|f - \mathcal{I}f\|_\Omega \le K |\triangle|^{m+1} |f|_{m+1,\Omega}, \tag{11}$$

for all f in the Sobolev space $W_\infty^{m+1}(\Omega)$.

Proof: Since the proof is similar to the proof of Theorem 6.2 in [21] (see also [13,14] for similar arguments in the bivariate case), we can be brief. Let $f \in W_\infty^{m+1}(\Omega)$, fix $\tilde{T} \in \tilde{\triangle}$ and choose $T \in \triangle$ such that $\tilde{T} \subset T$. By Lemma 4.3.8 of [7], there exists a polynomial $q := q_{f,T} \in \mathcal{P}_m$ such that

$$\|f - q\|_{\Omega_T} \le K_1 |\Omega_T|^{m+1} |f|_{m+1,\infty,\Omega_T}, \tag{12}$$

where Ω_T is the union of the tetrahedra in $\mathrm{star}^\ell(T)$. Since $\mathcal{I}p = p$ for all $p \in \mathcal{P}_m$,

$$\|f - \mathcal{I}f\|_{\tilde{T}} \le \|f - q\|_{\tilde{T}} + \|\mathcal{I}(f - q)\|_{\tilde{T}}.$$

It suffices to estimate the second quantity. The fact that the Bernstein basis polynomials form a partition of unity coupled with the locality and stability of $\mathcal{I}$ gives

$$\|\mathcal{I}(f-q)\|_{\tilde{T}} \le \max_{\xi \in \mathcal{D}_{\tilde{T},d}} |c_\xi^{\tilde{T}}| \le K_2 \|f-q\|_{\Omega_T},$$

where $c_\xi^{\tilde{T}}$ are the B-coefficients of the polynomial $\mathcal{I}(f-q)|_{\tilde{T}}$. Combining this with (12) and the fact that $|\Omega_T| \le (2\ell+1)|\triangle|$, we have

$$\|\mathcal{I}(f-q)\|_{\tilde{T}} \le K_3 |\triangle|^{m+1} |f|_{m+1,\Omega_T},$$

and hence

$$\|f - \mathcal{I}f\|_{\tilde{T}} \le K_4 |\triangle|^{m+1} |f|_{m+1,\Omega_T}.$$

Taking the maximum over all tetrahedra $\tilde{T}$ in $\tilde{\triangle}$ gives (11). □

§7. Remarks

Remark 1. Lagrange interpolation using bivariate polynomial splines on triangulations has been considered in a series of papers, see [13–19]. Our most recent paper [14] deals with the general case of arbitrary smoothness and initial triangulations.

Remark 2. The trivariate Lagrange interpolation problem can be easily solved if we are willing to use C^0 splines, since all we need is to provide enough Lagrange data to uniquely determine a spline on each tetrahedron while insuring all polynomial pieces sharing a vertex v, an edge e and a triangular face F of $\triangle$ have the same B-coefficients associated with domain points on v, e and F, respectively. However, it is much harder to solve the Lagrange interpolation problem using splines with higher smoothness.

Remark 3. The first C^1 Lagrange interpolation method based on trivariate splines on tetrahedral partitions can be found in [21]. It is based on quintic C^1 splines on certain uniform type tetrahedral partitions.

Remark 4. Trivariate splines have attracted considerable attention recently. For results on the structure of trivariate splines, see [5,6,8,9,12]. For constructions of macro-element methods, see [1–4,10,12,21–26].

Remark 5. Our interpolation methods can be extended to splines on admissible partitions of domains in $\mathbb{R}^k$. This can be done by coupling appropriate generalizations of our algorithms with macro-element results in [23–25]. In this case, it turns out that the B-coefficients c_ξ of the quadratic and cubic splines are bounded by the values which are interpolated at points in $\mathrm{star}^{k+1}(T)$, where T is the $k+1$ simplex of the given partition which contains the associated domain point ξ.

Remark 6. Due to their small polynomial degree, quadratic and cubic trivariate splines are important for contouring three-dimensional data, also called iso-surfacing. Standard methods in volume visualization are based on trilinear approaches (tensor product splines), i.e. continuous piecewise cubics. On the other hand, it is often important to involve smoothness conditions for higher visual quality of the reconstructed objects (e.g. iso-surfaces). This topic has been discussed in [15,20].

Remark 7. It is known that constructing well-behaved tetrahedral partitions of given points in $\mathbb{R}^3$ is non-trivial. Currently available methods in computational geometry are able to solve this problem with algorithmic complexity $\mathcal{O}(n^2 \log(n))$, where n is the number of given data points. Once a tetrahedral partition $\triangle$ has been constructed, the proposed interpolation methods have linear algorithmic complexity.

Remark 8. As noted in [26], the construction of admissible Worsey-Piper refinements of a given initial tetrahedral partition $\triangle$ seems to be a quite difficult problem. For this reason, in practice we expect the method discussed in Sect. 4 is likely to be more useful.

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