Smooth Macro-Elements Based on Clough-Tocher Triangle Splits

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Abstract. Macro-elements of smoothness C^r on Clough-Tocher triangle splits are constructed for all $r \geq 0$. These new elements are improvements on elements constructed in [11] in that (disproving a conjecture made there) certain unneeded degrees of freedom have been removed. Numerical experiments on Hermite interpolation with the new elements are included.

§1. Introduction

A bivariate macro-element defined on a triangle T consists of a finite dimensional linear space $\mathcal{S}$ defined on T and a set Λ of linear functionals forming a basis for the dual of $\mathcal{S}$.

It is common to choose the space $\mathcal{S}$ to be a space of polynomials or a space of piecewise polynomials defined on some subtriangulation of T. The members of Λ, called the degrees of freedom, are usually taken to be point evaluations of derivatives, although here we will also work with sets of linear functionals which pick off certain spline coefficients.

A macro-element defines a local interpolation scheme. In particular, if f is a sufficiently smooth function, then we can define the corresponding interpolant as the unique function $s \in \mathcal{S}$ such that $\lambda s = \lambda f$ for all $\lambda \in \Lambda$. We say that a macro-element has smoothness C^r provided that if the element is used to construct an interpolating function locally on each triangle of a triangulation $\triangle$, then the resulting piecewise function is C^r continuous globally.

The first C^r macro-elements were constructed using polynomials of degree $4r+1$, see Remark 9.1. To get macro-elements using lower degree polynomials, it is necessary to split the triangle. Here we focus on the case where T is split into three subtriangles by connecting its vertices to some point v_T in the interior. We call the resulting triangulation T_{CT} the Clough-Tocher (CT) split of T. The standard choice for the interior vertex is the barycenter of T.

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The classical Clough-Tocher macro element [5] is based on the triangulation T_{CT} and the 12-dimensional space of C^1 cubic splines on T_{CT}. The 12 degrees of freedom are chosen to be the values and gradients at the three vertices of T, along with the first order perpendicular cross-boundary derivative at the midpoint of each edge of T.

Several authors have created smoother versions of this classical cubic CT-element, see [3,7,8,11,14–15] based on certain superspline spaces. The best results to date [11] used spaces with dimension

$$\dim := \begin{cases} \frac{43m^2+31m+6}{2}, & r = 2m, \\ \frac{43m^2+65m+24}{2}, & r = 2m+1. \end{cases} \tag{1.1}$$

which, of course, is also the number of degrees of freedom. It was conjectured there that C^r elements on the CT-split with a smaller number of degrees of freedom could not be constructed. Here we show that this is not the case by constructing C^r macro-element spaces with

$$\dim := \begin{cases} \frac{39m^2+33m+6}{2}, & r = 2m, \\ \frac{39m^2+63m+24}{2}, & r = 2m+1. \end{cases} \tag{1.2}$$

We accomplish this by working with spaces of supersplines defined by enforcing certain individual smoothness conditions across the interior edges of T_{CT}.

The paper is organized as follows. In Sect. 2 we review some well-known Bernstein-Bézier notation, and establish a basic smoothness lemma. In Sect. 3 we review minimal determining sets. The cases where r is even and odd are treated in Sects. 4 and 5, respectively. Sect. 6 shows how our degrees of freedom translate into nodal functionals, and Sect. 7 describes the application of our elements to Hermite interpolation of scattered data. Numerical experiments can be found in Sect. 8, and concluding remarks in Sect. 9.

§2. Preliminaries

Our starting point is [11], and we closely follow the notation used there. In this section we review only the most essential notation and concepts.

We will make extensive use of well-known Bernstein-Bézier techniques. In particular, we represent polynomials p of degree d on a triangle $T := \langle u_1, u_2, u_3 \rangle$ in their B-form

$$p = \sum_{i+j+k=d} c^T_{ijk} B^d_{ijk},$$

where B^d_{ijk} are the Bernstein polynomials of degree d associated with T. In particular, if (α, β, γ) are the barycentric coordinates of any point $u \in \mathbb{R}^2$ in terms of the triangle T, then

$$B^d_{ijk}(u) := \frac{d!}{i!\,j!\,k!} \alpha^i \beta^j \gamma^k, \qquad i+j+k=d. \tag{2.1}$$

As usual, we associate the coefficient c^T_{ijk} with the domain point

$$\xi^T_{ijk} := \frac{(iu_1 + ju_2 + ku_3)}{d}, \qquad i+j+k=d.$$

We will work with the usual rings and disks of domain points defined by

$$R^T_n(u_1) := \{\xi^T_{ijk} : \ i = d-n\},$$
$$D^T_n(u_1) := \{\xi^T_{ijk} : \ i \ge d-n\},$$

with similar definitions at the other vertices of T.

Suppose that $T := \langle u_1, u_2, u_3\rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2\rangle$ are two adjoining triangles which share the edge $e := \langle u_2, u_3\rangle$. Let p and $\tilde{p}$ be two polynomials of degree d with B-coefficients c_{ijk} and $\tilde{c}_{ijk}$ relative to T and $\widetilde{T}$, respectively. Then it is well known (cf. [4]) that p and $\tilde{p}$ join with C^r continuity across the edge e if and only if

$$\tilde{c}_{n,m-n,d-m} = \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \tag{2.2}$$

for $m = 0, \ldots, d-n$ and $n = 0, \ldots, r$. Here B^n_{ijk} are the Bernstein polynomials of degree n on the triangle T.

Assuming that the coefficients of p are known and that $\tilde{p}$ joins p with C^r continuity, the smoothness conditions (2.2) can be used to compute the coefficients $\tilde{c}_{n,m-n,d-m}$ of $\tilde{p}$ for $0 \le n \le r$. They can also be used in situations where some of the coefficients of both p and $\tilde{p}$ are known and others are unknown. We need the following lemma which shows how this works for computing coefficients on the ring $R^T_m(u_2) \cup R^{\widetilde{T}}_m(u_2)$.

Lemma 2.1. *Suppose T and $\widetilde{T}$ are as above, and that all coefficients c_{ijk} and $\tilde{c}_{ijk}$ of the polynomials p and $\tilde{p}$ are known except for*

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, \qquad && \nu = \ell+1, \ldots, q, \\ \tilde{c}_\nu &:= \tilde{c}_{\nu,m-\nu,d-m}, \qquad && \nu = \ell+1, \ldots, \tilde{q}, \end{aligned} \tag{2.3}$$

for some $\ell, m, q, \tilde{q}$ with $0 \le q, \tilde{q}$, $-1 \le \ell \le q, \tilde{q}$, and $q + \tilde{q} - \ell \le m \le d$. Then these coefficients are uniquely determined by the smoothness conditions

$$\tilde{c}_{n,m-n,d-m} = \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \qquad \ell+1 \le n \le q+\tilde{q}-\ell. \tag{2.4}$$

Proof: Let $c := (c_{\ell+1}, \ldots, c_q, \tilde{c}_{\ell+1}, \ldots, \tilde{c}_{\tilde{q}})^T$. Then (2.4) can be written in the form

$$Mc := b \tag{2.5}$$

with

$$M := \begin{bmatrix} A & -I \\ B & O \end{bmatrix},$$

where I is the $(\tilde{q}-\ell)\times(\tilde{q}-\ell)$ identity matrix, O is the $(q-\ell)\times(\tilde{q}-\ell)$ zero matrix,

$$A_{ij} := \binom{\ell+i}{\ell+j}\alpha^{\ell+j}\gamma^{i-j}, \qquad \begin{array}{l} i=1,\ldots,\tilde{q}-\ell, \\ j=1,\ldots,q-\ell, \end{array}$$

and

$$B_{ij} := \binom{\tilde{q}+i}{\ell+j}\alpha^{\ell+j}\gamma^{\tilde{q}-\ell+i-j}, \qquad i,j=1,\ldots,q-\ell.$$

Here α,β,γ are the barycentric coordinates of u_4 relative to T, *i.e.*, $u_4 = \alpha u_1 + \beta u_2 + \gamma u_3$. The right-hand side is given by

$$b_\nu = \begin{cases} \quad -a_\nu, & 1 \le \nu \le \tilde{q}-\ell, \\ \tilde{c}_{\ell+\nu} - a_\nu, & \tilde{q}-\ell+1 \le \nu \le q+\tilde{q}-2\ell \end{cases}$$

where

$$a_\nu := \sum_{i+j+k=\ell+\nu}{}^{\prime} c_{i,j+d-r+\nu+\ell-n,k+r-\nu-\ell} B^{\ell+\nu}_{ijk}(u_4).$$

Here the prime on the sum means that the sum is taken over all i,j,k such that $c_{i,j+d-r+\nu+\ell-n,k+r-\nu-\ell}$ is not one of the coefficients defined in (2.3).

By the block structure, to prove that M is nonsingular, it suffices to examine B. Let $\widetilde{B}$ be the matrix obtained by factoring $\alpha^{\ell+j}/(\ell+j)!$ from the j-th column of B for $j=1,\ldots,q-\ell$. By the Toeplitz nature of $\widetilde{B}$, we see that $\det(\widetilde{B}) = C\gamma^{(\tilde{q}-\ell)(q-\ell)}$ for some constant C. Since α,γ are nonzero, it remains to show that $C \ne 0$.

We note that the matrix $\widetilde{B}$ is the Gram matrix corresponding to the functions $\{x^{\tilde{q}+i}\}_{i=1}^{q-\ell}$, and the linear functionals $\{\delta_\gamma D^{\ell+j}\}_{j=1}^{q-\ell}$, where δ_γ is point evaluation at γ. Now if $\det(\widetilde{B})$ were zero, there would exist a nontrivial polynomial $f = \sum_{i=1}^{q-\ell} a_i x^{\tilde{q}+i}$ satisfying $D^{\ell+j}f(\gamma) = 0$ for $j=1,\ldots,q-\ell$. But then $g = D^{\ell+1}f$ would be a nontrivial polynomial of degree $q+\tilde{q}-2\ell-1$ which vanishes $\tilde{q}-\ell$ times at 0 and $q-\ell$ times at γ. This is impossible, and we conclude that C cannot be zero. □

Various weaker versions of Lemma 2.1 can be found in the literature – for example, see Lemma 3.3 of [6] in the special case where $q=\tilde{q}$. We now illustrate the lemma for the case $\ell=-1$, (which is the only case we use here).

Example 2.2. *Let* $d=10$, $m=8$, $q=2$, $\tilde{q}=3$, *and* $\ell=-1$ *in Lemma 2.1.*

Discussion: Fig. 1 shows the domain points of two adjoining triangles. We assume that we know the coefficients of p and $\tilde{p}$ corresponding to open disks, and that we want to compute the coefficients $c_{226}, c_{127}, c_{028}$ of p and $\tilde{c}_{352}, \tilde{c}_{262}, \tilde{c}_{172}, \tilde{c}_{082}$ of $\tilde{p}$ which correspond to the domain points marked with black boxes. By the lemma, these seven coefficients on $R_8^T(u_2) \cup R_8^{\widetilde{T}}(u_2)$ are uniquely determined by the $C^0,\ldots,C^6$ smoothness conditions listed in (2.4). □

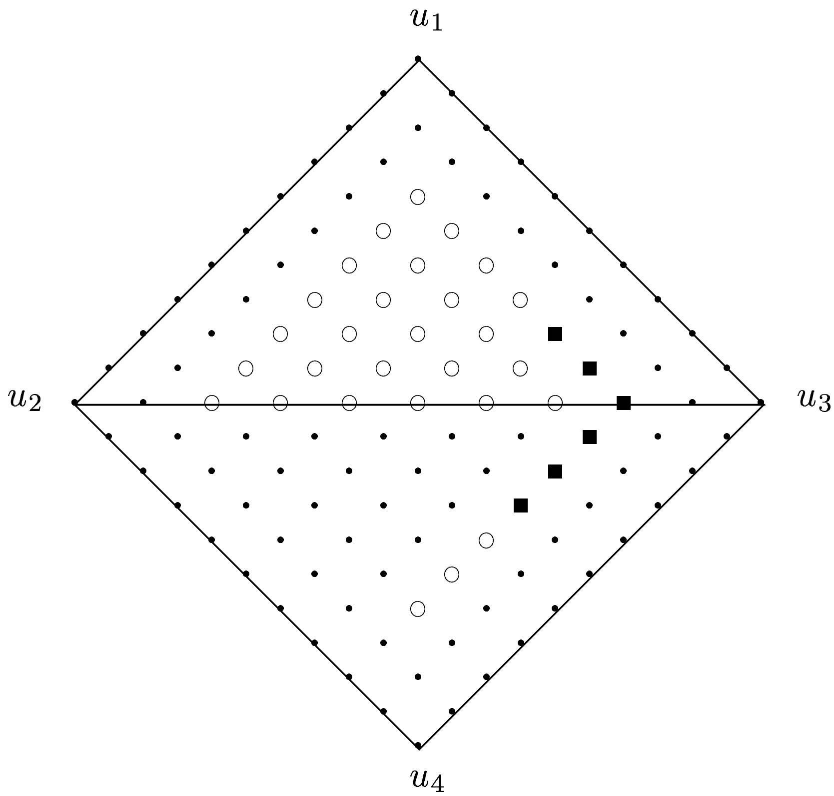

Fig. 1. The B-net for Example 2.2.

§3. Minimal determining sets

Let $\mathcal{S}_d^0(\triangle)$ be the space of continuous splines of degree d on the triangulation $\triangle$, and let $\mathcal{D}_{d,\triangle}$ be the union of the sets of domain points associated with each triangle of $\triangle$. Then it is well known that each spline in $\mathcal{S}_d^0(\triangle)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$, where the coefficients of the polynomial $s|_T$ are precisely $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}\cap T}$.

In this paper we are interested in subspaces $\mathcal{S}$ of $\mathcal{S}_d^0(\triangle)$ which satisfy additional smoothness conditions. In addition to the usual C^r smoothness conditions across edges and C^ρ smoothness conditions at vertices, we shall also make use of special individual smoothness conditions.

Suppose that $T := \langle u_1, u_2, u_3\rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2\rangle$ are two adjoining triangles which share the edge $e := \langle u_2, u_3\rangle$. Let c_{ijk} and $\tilde{c}_{ijk}$ be the coefficients of the B-representations of s_T and $s_{\widetilde{T}}$, respectively. Then for any $n \le m \le d$, let

$$\tau_{m,e}^n s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B_{ijk}^n(u_4), \tag{3.1}$$

where B_{ijk}^n are the Bernstein polynomials of degree n on the triangle T. In terms of these linear functionals, the conditions (2.2) for C^r smoothness across the edge

e can be restated as

$$\tau_{m,e}^n s = 0, \qquad 0 \le m \le d-n, \quad 0 \le n \le r.$$

If s is a spline in $\mathcal{S}_d^0(\triangle)$ which satisfies additional smoothness conditions beyond C^0 continuity, then clearly we cannot independently choose all of its coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. We recall that a **determining set** for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points $\mathcal{D}_{d,\triangle}$ such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $c_\xi = 0$ for all $\xi \in \mathcal{D}_{d,\triangle}$, *i.e.*, $s \equiv 0$. The set $\mathcal{M}$ is called a **minimal determining set (MDS)** for $\mathcal{S}$ if there is no smaller determining set. It is known that $\mathcal{M}$ is a MDS for $\mathcal{S}$ if and only if every spline $s \in \mathcal{S}$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$.

A MDS $\mathcal{M}$ is called **stable** provided that for each $\theta > 0$ there is a constant K_θ such that $\|s\|_\infty \le K_\theta \|c\|_\infty$, where $\|c\| := \max_{\xi \in \mathcal{M}} |c_\xi|$ whenever $\triangle$ is a triangulation whose smallest angle is at least θ.

We conclude this section with some additional notation. If v is a vertex of $\triangle$, the sets $R_m(v)$ and $D_m(v)$ are defined to be the unions of the rings $R_m^T(v)$ and disks $D_m^T(v)$, respectively, taken over all triangles T attached to v. Given a triangle T and a point v_T inside the triangle, we define the associated **Clough-Tocher split** to consist of the three triangles $T^{[i]} := \langle v_T, v_i, v_{i+1} \rangle$ for $i = 1, 2, 3$, where we identify $v_4 = v_1$. We write e_i for the edge $\langle v_i, v_T \rangle$ for $i = 1, 2, 3$.

§4. The case $r = 2m$

In this section we construct C^r macro-elements for even r associated with the Clough-Tocher split T_{CT} of a triangle T. Our starting point is the space of supersplines

$$\begin{aligned} \mathcal{S}_{6m+1}^{2m,3m,5m+1}(T_{CT}) := \{ s \in C^{2m}(T) :\ & s|_T \in \mathcal{P}_d \text{ all } T \in \triangle,\ s \in C^{3m}(v_i) \\ & \text{for } 1 \le i \le 3, \text{ and } s \in C^{5m+1}(v_T) \}, \end{aligned} \tag{4.1}$$

where v_T is the center point of the split and $\mathcal{P}_d$ is the space of polynomials of degree d. As usual, $C^\mu(v)$ means that all polynomials on triangles sharing the vertex v have common derivatives up to order μ at that vertex.

Theorem 4.1. *Fix $r = 2m$, and let $\mathcal{S}_r(T_{CT})$ be the linear subspace of all splines s in $\mathcal{S}_{6m+1}^{2m,3m,5m+1}(T_{CT})$ satisfying the following set of additional smoothness conditions:*

$$\tau_{3m+i+1,e_1}^{2m+1+i+j} s = 0, \qquad 1 \le j \le i, \quad 1 \le i \le m-1, \tag{4.2}$$

$$\tau_{3m+i+1,e_2}^{2m+1+i+j} s = 0, \qquad 1 \le j \le i, \quad 1 \le i \le m-1, \tag{4.3}$$

$$\tau_{4m+i,e_1}^{3m+i+j} s = 0, \qquad 1 \le j \le m-i+1, \quad 1 \le i \le m, \tag{4.4}$$

$$\tau_{4m+i,e_2}^{3m+i+j} s = 0, \qquad 1 \le j \le m-i, \quad 1 \le i \le m-1. \tag{4.5}$$

Then

$$\dim \mathcal{S}_r(T_{CT}) = \frac{39m^2+33m+6}{2}, \tag{4.6}$$

and the following set $\mathcal{M}_r$ of domain points is a stable MDS for $\mathcal{S}_r(T_{CT})$:

1) $D_{3m}^{T^{[i]}}(v_i)$ *for* $i=1,2,3$,
2) $\{\xi^{T^{[i]}}_{j,3m,3m-j+1},\ldots,\xi^{T^{[i]}}_{j,3m-j+1,3m}\}$ *for* $j=1,\ldots,2m$ *and* $i=1,2,3$.

Proof: First we show that $\mathcal{M}_r$ is a determining set. Suppose that we set the coefficients c_ξ of $s \in \mathcal{S}_r(T_{CT})$ to zero for all $\xi \in \mathcal{M}_r$. Then we claim that all other coefficients must be zero. We begin by using Lemma 2.1 to solve for the unset coefficients corresponding to domain points on the rings $R_{3m+i}(v_1)$ and $R_{3m+i}(v_2)$ for $i=1,\ldots,m$. On each ring this involves solving the nonsingular system of $2(m+i)-1$ homogeneous equations corresponding to the lemma. Note that the spline satisfies all of the smoothness conditions (2.4) required for the lemma, since either they are already implicit in the super-smoothness of the space, or have been explicitly enforced in the definition of $\mathcal{S}_r(T_{CT})$.

We next compute unset coefficients on the ring $R_{4m+1}(v_1)$. This involves solving a $(4m+1)\times(4m+1)$ system with zero right-hand side. Then we do the ring $R_{4m+1}(v_2)$ which involves solving a $4m\times 4m$ system since $R_{4m+1}(v_1)$ and $R_{4m+1}(v_2)$ overlap in one point. We continue alternating between rings around v_1 and v_2. In particular, for each $i=2,\ldots,m$ we do the ring $R_{4m+i}(v_1)$ followed by the ring $R_{4m+i}(v_2)$. The first of these involves solving a $(4m+1)\times(4m+1)$ system, and the second involves solving a $4m\times 4m$ system.

Next we successively compute undetermined coefficients on each of the rings $R_{3m+i}(v_3)$ for $i=1,\ldots,3m+1$. Each of these involves solving a $(2m+1)\times(2m+1)$ system with zero right-hand side. Finally, the remaining coefficients in $T^{[1]}$ can be computed from the smoothness conditions across the edge $\langle v_1, v_T\rangle$. We have shown that all coefficients of s must be zero, and thus that $\mathcal{M}_r$ is a determining set for $\mathcal{S}_r(T_{CT})$.

To show that $\mathcal{M}_r$ is a minimal determining set, we show that its cardinality is equal to the dimension of $\mathcal{S}_r(T_{CT})$. It is easy to check that $\#\mathcal{M}_r$ is equal to the number in (4.6). Now consider the superspline space $\mathcal{S}^{2m,5m+1}_{6m+1}(T_{CT})$. By Theorem 2.2 in [16], the dimension of this space is $(46m^2+34m+6)/2$. Our space $\mathcal{S}_r(T_{CT})$ is the subspace which satisfies the $2m^2-m$ special conditions (4.2)–(4.5) and the supersmoothness $C^{4m}(v_i)$ for $i=1,2,3$. Enforcing this supersmoothness requires an additional $3(m^2+m)/2$ conditions, and thus

$$\frac{46m^2+34m+6}{2} - \frac{4m^2-2m}{2} - \frac{3(m^2+m)}{2} \le \dim \mathcal{S}_r(T_{CT}) \le \frac{39m^2+33m+6}{2}.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_r(T_{CT})$. This proves $\mathcal{M}_r$ is a MDS.

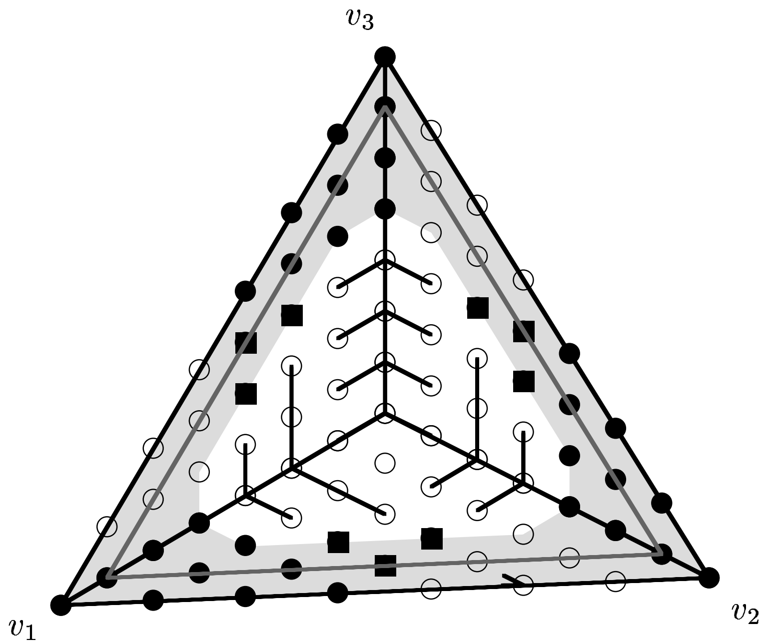

Fig. 2. The C^2 macro-element $\mathcal{S}_2(T_{CT})$.

Finally, we claim that the MDS $\mathcal{M}_r$ is stable. This follows from the fact that once we set the coefficients of a spline $s \in \mathcal{S}_r(T_{CT})$, the remaining unset coefficients can be computed in the order described above either directly from smoothness conditions (known to be a stable process) or from Lemma 2.1. The latter computation involves solving a non-singular linear system whose determinant (and thus the constant K of stability) depends only the barycentric coordinates of v_T. □

The solution of systems of linear equations required in the above construction can be simplified by precomputing the inverses of the relevant matrices. These explicit inverses can be used to give explicit formulae for all of the computed coefficients in terms of the coefficients which have been set. However, in practice, it is generally easier to compute the right-hand sides of the systems using the de Casteljau algorithm, and then multiply by the precomputed inverses.

Example 4.2. *Let $r = 2m$ with $m = 1$. In this case the macro-element space $\mathcal{S}_2(T_{CT})$ is the subspace of $\mathcal{S}_7^{2,3,6}(T_{CT})$ which satisfies the additional smoothness condition τ_{5,e_1}^5.*

Discussion: By Theorem 4.1, the dimension of $\mathcal{S}_2(T_{CT})$ is 39, and a MDS $\mathcal{M}_r$ is given by

1) $D_3^{T^{[i]}}(v_i)$ for $i = 1, 2, 3$,
2) $\xi_{1,3,3}^{T^{[i]}}$, $\xi_{2,3,2}^{T^{[i]}}$, and $\xi_{2,2,3}^{T^{[i]}}$, for $i = 1, 2, 3$.

This MDS is illustrated in Fig. 2, where the points in 1) are marked with black disks while those in 2) are marked with black squares. The tip of the smoothness

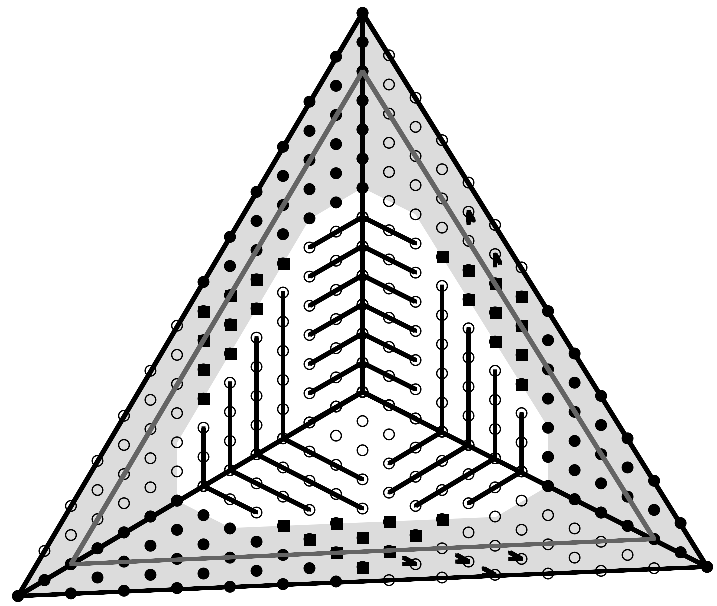

Fig. 3. The C^4 macro-element $\mathcal{S}_4(T_{CT})$.

condition τ^5_{5,e_1} is marked with a bracket. Note that the smoothness conditions $\tau^3_{5,e_1} s = 0$ and $\tau^4_{5,e_1} s = 0$ are automatically satisfied due to the supersmoothness $C^6(v_T)$. The steps of the computation are indicated by straight lines drawn through domain points in Fig. 2. First we compute the marked coefficients on $R_4(v_1)$, then on the rings $R_4(v_2)$, $R_5(v_1)$, and $R_5(v_2)$. Next the coefficients on rings $R_4(v_3), R_5(v_3), R_6(v_3)$ are computed, followed by the last remaining coefficient in the bottom triangle. □

The analogous diagram for our C^4 macro-element is shown in Fig. 3.

§5. The case $r = 2m + 1$

In this section we construct C^r macro-elements on T_{CT} for odd r. Here we start with the space of supersplines

$$\mathcal{S}^{2m+1,3m+1,5m+2}_{6m+3}(T_{CT}) := \{s \in C^{2m+1}(T) : \; s|_T \in \mathcal{P}_d \text{ all } T \in \triangle, \; s \in C^{3m+1}(v_i) \text{ for } 1 \le i \le 3, \text{ and } s \in C^{5m+2}(v_T)\}, \tag{5.1}$$

where v_T is the center point of the split.

Theorem 5.1. *Fix $r = 2m+1$, and let $\mathcal{S}_r(T_{CT})$ be the linear subspace of all splines s in $\mathcal{S}^{2m+1,3m+1,5m+2}_{6m+3}(T_{CT})$ that satisfy the following set of additional smoothness conditions:*

$$\tau^{2m+1+i+j}_{3m+i+2,e_1} s = 0, \qquad 1 \le j \le i, \quad 1 \le i \le m, \tag{5.2}$$

$$\tau^{2m+1+i+j}_{3m+i+2,e_2} s = 0, \qquad 1 \le j \le i, \quad 1 \le i \le m, \tag{5.3}$$

$$\tau^{3m+i+j+1}_{4m+i+2,e_1} s = 0, \qquad 1 \le j \le m-i+1, \quad 1 \le i \le m, \tag{5.4}$$

$$\tau^{3m+i+j+1}_{4m+i+2,e_2} s = 0, \qquad 1 \le j \le m-i, \quad 1 \le i \le m-1. \tag{5.5}$$

Then

$$\dim \mathcal{S}_r(T_{CT}) = \frac{39m^2+63m+24}{2}. \tag{5.6}$$

Moreover, the following set $\mathcal{M}_r$ of domain points is a stable MDS:

1) $D^{T^{[i]}}_{3m+1}(v_i)$ *for* $i = 1,2,3$,

2) $\{\xi^{T^{[i]}}_{j,3m+1,3m-j+2}, \ldots, \xi^{T^{[i]}}_{j,3m-j+2,3m+1}\}$ *for* $j = 1,\ldots,2m+1$ *and* $i = 1,2,3$.

Proof: First we show that $\mathcal{M}_r$ is a determining set. Suppose that we set the coefficients c_ξ of $s \in \mathcal{S}_r(T_{CT})$ to zero for all $\xi \in \mathcal{M}_r$. Then we claim that all other coefficients must be zero. First we use Lemma 2.1 to solve for the unset coefficients corresponding to domain points on the rings $R_{3m+i+2}(v_1)$ and $R_{3m+i+2}(v_2)$ for $i = 0,\ldots,m$. Each step involves solving a $(2m+2i+1) \times (2m+2i+1)$ system with zero right-hand side.

We now use the lemma to find the unset coefficients on the ring $R_{4m+3}(v_1)$. This involves solving a $(4m+2) \times (4m+2)$ system. Then we use the lemma to compute the unset coefficients on the ring $R_{4m+3}(v_2)$ by solving a $(4m+1)\times(4m+1)$ system. We continue alternating between rings around v_1 and v_2. In particular, for each $i = 2,\ldots,m$ we use the lemma on ring $R_{4m+i+2}(v_1)$ and then on the ring $R_{4m+i+2}(v_2)$. The first of these involves solving a $(4m+2) \times (4m+2)$ nonsingular system, and the second involves solving a $(4m+1) \times (4m+1)$ nonsingular system.

Now we successively compute undetermined coefficients on each of the rings $R_{3m+i+1}(v_3)$ for $i = 1,\ldots,3m+2$. Each of these involves solving a $(2m+1) \times (2m+1)$ system. Finally, the remaining coefficients in $T^{[1]}$ can be computed from the smoothness conditions across the edge $\langle v_1, v_T\rangle$. We have shown that all coefficients of s must be zero, and thus $\mathcal{M}_r$ is a determining set.

To show that $\mathcal{M}_r$ is a minimal determining set, we now show that its cardinality is equal to the dimension of $\mathcal{S}_r(T_{CT})$. It is easy to check that $\#\mathcal{M}_r$ is equal to the number in (5.6). Now consider the superspline space $\mathcal{S}^{2m+1,5m+2}_{6m+3}(T_{CT})$. By Theorem 2.2 in [16], the dimension of this space is $(46m^2+68m+24)/2$. Our space $\mathcal{S}_r(T_{CT})$ is the subspace which satisfies the $2m^2+m$ special conditions (5.2)–(5.5) and the supersmoothness $C^{4m+1}(v_i)$ for $i = 1,2,3$. Enforcing this supersmoothness requires an additional $3(m^2+m)/2$ conditions, and thus

$$\frac{46m^2+68m+24}{2} - \frac{4m^2+2m}{2} - \frac{3(m^2+m)}{2} \le \dim \mathcal{S}_r(T_{CT}) \le \frac{39m^2+63m+24}{2}.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_r(T_{CT})$, and $\mathcal{M}_r$ is a MDS. Its stability follows exactly as in the even case. $\square$

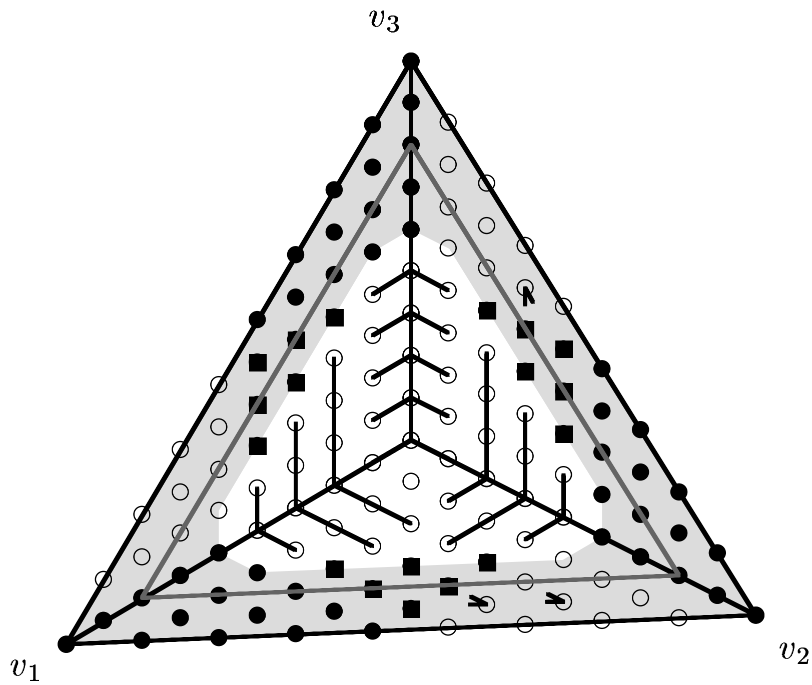

Fig. 4. The C^3 macro-element $\mathcal{S}_3(T_{CT})$.

As in the even case, we can get explicit formulae for the computed coefficients in terms of the coefficients which have been set by finding the explicit inverses of the matrices arising in Lemma 2.1. Alternatively, we can solve the linear systems by computing the needed right-hand sides using the de Casteljau algorithm and then multiplying by the inverses.

Example 5.2. *Let $r = 2m+1$ with $m = 1$. Then the macro-element space $\mathcal{S}_3(T_{CT})$ is the subspace of $\mathcal{S}_9^{3,4,7}(T_{CT})$ which satisfies the additional smoothness conditions τ^5_{6,e_1}, τ^5_{6,e_2}, and τ^6_{7,e_1}.*

Discussion: By Theorem 5.1, the dimension of $\mathcal{S}_3(T_{CT})$ is 66, and the set $\mathcal{M}_r$ consisting of the points

1) $D_4^{T^{[i]}}(v_i)$ for $i = 1, 2, 3$,

2) $\xi^{T^{[i]}}_{1,4,4}$, $\xi^{T^{[i]}}_{2,4,3}$, $\xi^{T^{[i]}}_{2,3,4}$, $\xi^{T^{[i]}}_{3,4,2}$, $\xi^{T^{[i]}}_{3,3,3}$, $\xi^{T^{[i]}}_{3,2,4}$, for $i = 1, 2, 3$

is a MDS. This set is illustrated in Fig. 4, where as in the even case, the points in 1) are marked with black disks, and points in 2) are marked with black squares. The tips of the special smoothness conditions are marked with brackets. □

The analogous diagram for our C^5 macro-element is shown in Fig. 5.

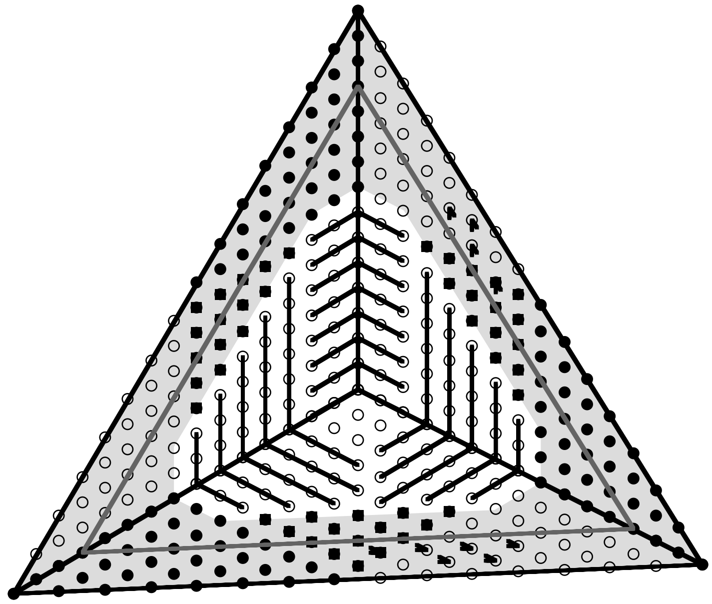

Fig. 5. The C^5 macro-element $\mathcal{S}_5(T_{CT})$.

§6. Nodal degrees of freedom

It is common in the finite-element literature to describe the degrees of freedom of macro-elements in terms of derivatives. In this section we show that there is a natural way to do this for the macro elements introduced in Sects. 4 and 5. Let D_x and D_y be the usual partial derivatives. Let δ_t be point evaluation at t. If e is one of the edges of T, we denote the derivative normal to that edge by D_e. Let

$$\eta_{k,i}^j := \frac{(j+1-i)v_k + iv_{k+1}}{j+1}, \qquad i = 1,\ldots,j, \quad k = 1,2,3, \tag{6.1}$$

and

$$\rho_r := \begin{cases} 3m, & r = 2m, \\ 3m+1, & r = 2m+1. \end{cases} \tag{6.2}$$

Theorem 6.1. *Let $\mathcal{S}_r(T_{CT})$ be the spline space defined in Theorems 4.1 and 5.1 for r even and odd, respectively. Then any spline $s \in \mathcal{S}_r(T_{CT})$ is uniquely determined by the following set of data:*

1) $\{\delta_{v_i} D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le \rho_r}$, *for* $i = 1,2,3$,
2) $\{\delta_{\eta_{k,i}^j} D_{\langle v_k, v_{k+1}\rangle}\}_{i=1}^j$, *for* $j = 1,\ldots,r$ *and* $k = 1,2,3$.

Proof: It is well known (see [9] for explicit formulae) that setting this nodal data is equivalent to setting the B-coefficients listed in Theorems 4.1 and 5.1. □

§7. Hermite interpolation of scattered data

In this section we briefly examine the use of our macro-elements for interpolation of Hermite data at a set of scattered points $\mathcal{V} := \{(x_i, y_i)\}_{i=1}^{V}$. Our aim is to construct a C^r spline which interpolates this data.

We begin by triangulating the data points. Let $\triangle$ be a triangulation with vertices at the points of V. For many applications, this might be the Delaunay triangulation. Let $\triangle_{CT}$ be the triangulation obtained from $\triangle$ by splitting each triangle in $\triangle$ about its centroid, and let

$$\mathcal{S}_r(\triangle_{CT}) := \{s \in C^r(\Omega) : \; s|_T \in \mathcal{S}_r(T_{CT}) \text{ for all } T \in \triangle\}, \tag{7.1}$$

where Ω is the union of the triangles of $\triangle$.

Theorem 7.1. *For all $m \geq 0$,*

$$\dim \mathcal{S}_r(\triangle_{CT}) = \begin{cases} \binom{3m+2}{2} V + \binom{2m+1}{2} E, & r = 2m, \\ \binom{3m+3}{2} V + \binom{2m+2}{2} E, & r = 2m+1, \end{cases} \tag{7.2}$$

where V and E are the number of vertices and edges of $\triangle$, respectively.

Proof: We consider only the case $r = 2m$ as the case $r = 2m+1$ is similar. Let $\mathcal{M}_r$ be the following set of domain points:

1) for each vertex v of $\triangle$, choose a triangle T of $\triangle_{CT}$ attached to v and include $D_{3m}^T(v)$,
2) for each edge $e = \langle v_1, v_2 \rangle$ of $\triangle$, let $T = \langle v, v_1, v_2 \rangle$ be a triangle of $\triangle_{CT}$ containing the edge e. Then include the points $\{\xi^T_{j,3m,3m-j+1}, \ldots, \xi^T_{j,3m-j+1,3m}\}$ for $j = 1, \ldots, 2m$ and $i = 1, 2, 3$.

The cardinality of $\mathcal{M}_r$ is precisely the number in (7.2). Now setting the coefficients c_ξ of s for $\xi \in \mathcal{M}_r$, we can use the smoothness conditions to uniquely determine all remaining coefficients in the disks $D_{3m+1}(v_i)$. Then the remaining coefficients in each triangle can be uniquely computed as in the proofs of Theorem 4.1 and 5.1. This shows that $\mathcal{M}_r$ is a MDS, and the proof is complete. $\square$

We are now ready to solve the Hermite interpolation problem.

Theorem 7.2. *For any function f which is sufficiently smooth so that the needed derivatives exist, there is a unique spline $s \in \mathcal{S}_r(\triangle_{CT})$ such that*

$$D_x^\nu D_y^\mu s(x_i, y_i) = D_x^\nu D_y^\mu f(x_i, y_i), \qquad 0 \leq \nu + \mu \leq \rho_r, \qquad i = 1, \ldots, n,$$

and

$$D_e^j s(\eta^j_{e,i}) = D_e^j f(\eta^j_{e,i}), \qquad 1 \leq i \leq j, \quad 1 \leq j \leq r,$$

for all edges e of $\triangle$, where $\eta^j_{e,i}$ and ρ_r are defined in (6.1), (6.2).

Proof: For each triangle T of $\triangle$, the interpolant s can be constructed locally since the given data uniquely determines the nodal data of Theorem 6.1. $\square$

The Hermite interpolant of Theorem 7.2 is exact for polynomials of degree

$$d_r := \begin{cases} 6m+1, & r = 2m, \\ 6m+3, & r = 2m+1. \end{cases} \tag{7.3}$$

Coupling this with the stability of the construction and using the Bramble-Hilbert lemma as in [11], it is easy to establish the following error bound which shows that for sufficiently smooth functions, the Hermite interpolant provides optimal order approximation.

Theorem 7.3. *Suppose f lies in the Sobolev space $W_\infty^{k+1}(\Omega)$ for some $\rho_r \le k \le d_r$, and let s be the interpolating spline of Theorem 7.2. Then*

$$\|D_x^\alpha D_y^\beta (f - s)\|_\infty \le K |\triangle|^{k+1-\alpha-\beta} |f|_{k+1,\infty} \tag{7.4}$$

for $0 \le \alpha + \beta \le k$, where $|\triangle|$ is the mesh size of $\triangle$ (i.e., the diameter of the largest triangle), $|f|_{k+1,p}$ is the usual Sobolev semi-norm, and ρ_r is defined in (6.2). If Ω is convex, then the constant K depends only on r and on the smallest angle $\theta_\triangle$ in $\triangle$. If Ω is nonconvex, it also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω.

Although we did not need a basis to solve the Hermite interpolation problem, for other applications it is useful to observe that the space $\mathcal{S}_r(\triangle_{CT})$ has a convenient stable local basis. For each $\xi \in \mathcal{M}_r$, let B_ξ be the unique spline in $\mathcal{S}_r(\triangle_{CT})$ such that

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \eta \in \mathcal{M}_r, \tag{7.5}$$

where λ_η is the linear functional which picks off the B-coefficient corresponding to the domain point η. In view of (7.5), the splines $\{B_\xi\}_{\xi \in \mathcal{M}_r}$ are linearly independent, and thus form a basis for $\mathcal{S}_r(\triangle_{CT})$. It is easy to see that the B_ξ have local support. In particular,

1) If ξ is a point as in item 1) of Theorem 7.1, then $\text{supp}(B_\xi)$ is contained in the union of all triangles of $\triangle$ sharing the vertex v.
2) If ξ is a point as in item 2) of Theorem 7.1, then $\text{supp}(B_\xi)$ is contained in $T \cup \widetilde{T}$, where e is the edge between T and $\widetilde{T}$. (If e is a boundary edge of a triangle T, then the support is simply T). By the stability of the construction, there is a constant K depending only on the smallest angle in $\triangle$ such that $\|B_\xi\|_\infty \le K$ for all $\xi \in \mathcal{M}_r$.

§8. Numerical results

To test the performance of our higher smoothness macro-elements on Clough-Tocher splits, we used them to perform Hermite interpolation of some given functions on scattered data. Here we report just one suite of simple tests which examines the convergence order of the methods.

All tests are performed on the unit square $\Omega := [0,1] \times [0,1]$. For data points we choose the sets

$$D_\nu := \{(i/n_\nu, j/n_\nu),\ 0 \le i,j \le n_\nu\},$$

where $n_\nu = 2^\nu$. This is a set of $(n_\nu + 1)^2$ gridded data points. We choose $\triangle_\nu$ to be the associated type-I triangulation (also called the three-direction mesh). It is a Delaunay triangulation of these data, and it is easy to see that $|\triangle_\nu| = 2^{1/2}2^{-\nu}$. As a test function we take the standard Franke function

$$f(x,y) := \frac{3}{4}\big[e^{-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4}} + e^{-\frac{(9x+1)^2}{49} - \frac{(9y+1)}{10}}\big] + \frac{1}{2}e^{-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4}} - \frac{1}{5}e^{-(9x-4)^2-(9y-7)^2}$$

on the unit square. For each choice of $r = 1,2,3,4$ we perform Hermite interpolation at the points of D_ν for $\nu = 1,\ldots,5$. The corresponding tables show the number ND of data values used, the maximum error measured on approximately 10^6 uniformly distributed points in Ω, and the estimated rates of convergence. All computations were performed in quadruple precision on an SGI. Table 4 stops at $\nu = 4$ since for the C^4 element, the error is already smaller than quadruple precision (10^{-32}).

ND	Error	Rate
43	5.191826E-01	
131	7.864189E-02	2.722872
451	2.000073E-02	1.975244
1667	1.982802E-03	3.334439
6403	1.403019E-04	3.820934
25091	9.574896E-06	3.873134

Tab. 1. Performance of the C^1 element.

ND	Error	Rate
138	1.809348E-01	
418	3.853874E-02	2.231089
1434	4.398925E-04	6.453015
5290	4.005142E-06	6.779153
20298	1.885962E-08	7.730409
79498	7.824640E-11	7.913061

Tab. 2. Performance of the C^2 element.

The results confirm the theory. In particular, for an element constructed from polynomials of degree d we expect the convergence rate to be $d+1$. Since the C^1, C^2, C^3 and C^4 elements are of degrees 3, 7, 9, and 13, we should get rates of 4, 8,

ND	Error	Rate
231	1.367869E-01	
711	1.365725E-02	3.324191
2463	4.412828E-05	8.273748
9135	1.159296E-07	8.572311
35151	1.357903E-10	9.737653
13787	1.542725E-13	9.781683

Tab. 3. Performance of the C^3 element.

ND	Error	Rate
412	2.833822E-01	
1260	1.451828E-03	7.608735
4348	9.271893E-07	10.612712
16092	1.323868E-10	12.773889
61852	1.106321E-14	13.546701
242460	4.077150E-19	14.727850

Tab. 4. Performance of the C^4 element.

10, and 14. Note that for a given number of data, the higher smoothness elements give significantly smaller errors. For example, with about 20,000 data, errors are on the order 10^{-8}, 10^{-14}, 10^{-18}, and 10^{-22}.

In connection with this experiment, we note that although the Franke function looks quite harmless, its higher order derivatives are very large at some points in the unit square. For example, the sixth derivative $D_x^6 f$ which is needed for the C^4 element exceeds 10^7 at some points.

§9. Remarks

Remark 9.1. Macro-elements based on polynomials were constructed in [18–21]. As observed in [17], when used globally, they correspond to the superspline spaces $\mathcal{S}_{4r+1}^{r,2r}(\triangle)$.

Remark 9.2. The java code of the first author for examining determining sets for superspline spaces was the key tool in discovering the macro-elements described in this paper. The code is described in [1], and can be used or downloaded from `http://www.math.utah.edu/~alfeld`. That web site also contains code that can be used to generate colored versions of our figures for any r, as well as a detailed documentation of the MDS code.

Remark 9.3. As shown in [11], it is not possible to construct C^r macro-elements on the CT-split using splines of lower degree than those considered here, and it is not possible to enforce lower supersmoothness at the vertices of T.

Remark 9.4. It is of course possible to construct smooth macro-elements using lower degree splines provided we work with more complicated triangle splits. For macro-elements on Powell-Sabin splits, see [2,12].

Remark 9.5. It is also possible to create macro-elements with even fewer degrees of freedom by the process of condensation. This amounts to further restricting the spline space (usually by forcing certain cross-derivatives along edges of the triangle T to be of lower degree than they naturally are). The main problem with this strategy is that it produces elements which no longer have the capability of reproducing the full polynomial space, and thus have reduced approximation power.

Remark 9.6. As an aid to comparing our macro-elements to each other and to other elements in the literature, we list their essential properties for $r = 1, \ldots, 10$. In particular, for each value of r, we tabulate the corresponding polynomial degree d, the highest derivative D needed to construct the element, the number of degrees of freedom N of the element, and the number of B-coefficients n of the element.

r	d	D	N	n
1	3	1	12	19
2	7	3	39	85
3	9	4	63	136
4	13	6	114	274
5	15	7	153	361
6	19	9	228	571
7	21	10	282	694
8	25	12	381	1489
9	27	13	450	1135
10	31	15	573	2110

Remark 9.7. The construction described here is not unique in the sense that there are other choices of the extra smoothness conditions which also lead to macro-elements based on the degrees of freedom used here.

Remark 9.8. When $m = 0$, Theorem 5.1 describes the classical C^1 Clough-Tocher macro element first constructed in [5]. As observed by various people, it is automatically C^2 at v_T, even though this was not enforced in the original construction.

Remark 9.9. The java code mentioned in Remark 9.2 not only checks whether a given set of domain points is a MDS, but also produces the equations needed to compute all unset coefficients from those that have been set. This allows the custom design of macro-elements on arbitrary splits. We have written a program to produce FORTRAN code for any such custom-designed element, and are currently conducting tests to compare various possible elements with a given smoothness.

Remark 9.10. Frequently in practice one has to interpolate given values at scattered data points where no derivative information is provided. In this case, macro-element methods can still be applied, but the needed derivatives (or the equivalent set of B-coefficients) have to be estimated from the data. A number of ad hoc methods are available in the literature for first derivatives; we are currently examining the problem for higher derivatives.

Remark 9.11. Theorem 7.3 describes the approximation power of the spaces $\mathcal{S}_r(\triangle_{CT})$ measured in the uniform norm. Analogous results hold for the p-norms, and can be proved using the quasi-interpolation operators Q_k defined by

$$Q_k f := \sum_{\xi \in \mathcal{M}_r} \lambda_{\xi,k} f \, B_\xi,$$

where B_ξ are the dual basis splines of Sect. 6 and $\lambda_{\xi,k}$ are the linear functionals defined in Sect. 10 of [10].

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