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# Local lagrange interpolation with cubic $C^1$ splines on tetrahedral partitions

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## Abstract

We describe an algorithm for constructing a Lagrange interpolation pair based on  $C^1$  cubic splines defined on tetrahedral partitions. In particular, given a set of points  $\mathcal{V} \in \mathbb{R}^3$ , we construct a set  $P$  containing  $\mathcal{V}$  and a spline space  $\mathcal{S}_3^1(\Delta)$  based on a tetrahedral partition  $\Delta$  whose set of vertices include  $\mathcal{V}$  such that interpolation at the points of  $P$  is well-defined and unique. Earlier results are extended in two ways: (1) here we allow arbitrary sets  $\mathcal{V}$ , and (2) the method provides optimal approximation order of smooth functions. © 2008 Elsevier Inc. All rights reserved.

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## 1. Introduction

Let  $\mathcal{V} := \{\eta_i\}_{i=1}^m$  be a set of points in  $\mathbb{R}^3$ . In this paper we are interested in the following problem.

**Problem 1.1.** *Find a tetrahedral partition  $\Delta$  whose set of vertices includes  $\mathcal{V}$ , an  $N$ -dimensional space  $\mathcal{S}$  of  $C^1$  cubic splines defined on  $\Delta$ , and a set of additional points  $\{\eta_i\}_{i=m+1}^N$  such that for*

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every choice of real numbers  $\{r_i\}_{i=1}^N$ , there is a unique spline  $s \in \mathcal{S}_3^1(\Delta)$  satisfying

$$s(\eta_i) = r_i, \quad i = 1, \dots, N. \tag{1.1}$$

We call  $P := \{\eta_i\}_{i=1}^N$  and  $\mathcal{S}_3^1(\Delta)$  a Lagrange interpolation pair.

It is easy to solve this problem using  $C^0$  splines, see Remark 1. However, the situation is much more complicated if we want to use  $C^1$  splines. In [1] we solved the problem in the special case where the points  $\mathcal{V}$  lie on a rectangular grid and the partition is a Freudenthal partition, see Remark 3. In [7] we presented a  $C^1$  cubic spline method that works for arbitrary tetrahedral partitions, but with suboptimal approximation power. The aim of this paper is to describe a method which works for arbitrary tetrahedral partitions while giving optimal approximation order four.

To achieve this aim, we start with an arbitrary initial tetrahedral partition  $\Delta_0$  with vertex set  $\mathcal{V}$ , and then define an appropriate refinement  $\Delta$  and an associated set of points  $P$  such that  $P$  and the spline space  $\mathcal{S}_3^1(\Delta)$  form a Lagrange interpolation pair. The construction of both  $\Delta$  and  $P$  is based on special orderings of the tetrahedra in  $\Delta_0$  arising from two different decompositions of  $\Delta_0$ , but using different and more sophisticated techniques than in [7]. In addition, here we make use of the partial Worsey–Farin splits introduced in [1], rather than the simpler Worsey–Farin splits used in [7].

The paper is organized as follows. In the next section, we recall some basic notation and a few key concepts from the Bernstein–Bézier theory of splines, and present two useful results on bivariate splines. We discuss partial Worsey–Farin splits in Section 3. In Section 4 we give two algorithms for decomposing tetrahedral partitions. The construction of our Lagrange-interpolation pair follows from the algorithms of Sections 5 and 6, which show how to construct  $\Delta$  and  $P$ , respectively. The main result of the paper can be found in Section 7. In Section 8, we show that the resulting interpolation method is local and stable, and use these facts to establish error bounds for how well the interpolating spline approximates a given smooth function. We conclude with some remarks.

## 2. Preliminaries

We recall that for any given tetrahedral partition  $\Delta$ , the associated space of  $C^1$  cubic splines is

$$\mathcal{S}_3^1(\Delta) := \{s \in C^1(\Omega) : s|_T \in \mathcal{P}_3, \text{ for all } T \in \Delta\},$$

where  $\mathcal{P}_3$  is the 20-dimensional space of trivariate cubic polynomials. Throughout the paper we use standard Bernstein–Bézier techniques for dealing with trivariate splines. Here we recall only some of the most critical concepts and notation. For a detailed treatment and more on the notation, see the book [2] or our earlier papers [1,7].

Given a tetrahedron  $T = [v_1, v_2, v_3, v_4]$  with vertices  $v_1, v_2, v_3$ , and  $v_4$ , we write

$$\mathcal{D}_T := \left\{ \xi_{ijkl}^T := \frac{iv_1 + jv_2 + kv_3 + lv_4}{3} \right\}_{i+j+k+l=3}$$

for the associated set of domain points, and write  $\mathcal{D}_\Delta$  for the union of the sets  $\mathcal{D}_T$  over all  $T \in \Delta$ . The ball (of radius 1) around  $v_1$  is the set  $D^T(v_1) := \{\xi_{ijkl}^T : i \geq 2\}$ . Associated with an edge  $e := \langle v_1, v_2 \rangle$  of  $T$ , the tube (of radius 1) around  $e$  is the set  $t^T(e) := \{\xi_{ijkl}^T : k, \ell \leq 1\}$ , see [2, p. 439], where balls and tubes of arbitrary radius are defined. If  $\Delta$  is a tetrahedral partition, then the balls and tubes are defined as  $D(v) = \cup\{D^T(v) : T \text{ has a vertex at } v\}$ , and  $t(e) = \cup\{t^T(e) : T$

contains the edge  $e$ }. Associated with a tetrahedron  $T$  in a partition  $\Delta$ , we set  $\text{star}^0(T) := T$ , and for  $\ell \geq 1$ , define  $\text{star}^\ell(T)$  to be the union of the set of all tetrahedra in  $\Delta$  which touch a tetrahedron in  $\text{star}^{\ell-1}(T)$ .

We recall that a spline  $s$  is uniquely defined by its set  $\{c_\xi\}_{\xi \in \mathcal{D}_\Delta}$  of B-coefficients, and that a spline  $s$  belongs to  $C^1(\Omega)$  if and only these coefficients satisfy an appropriate set of simple linear side conditions, see [2, p. 454]. We also make use of the concepts of minimal determining sets and nodal minimal determining sets. Recall that a set  $\mathcal{M}$  of domain points of a spline space  $\mathcal{S}$  is called a **minimal determining set** for  $\mathcal{S}$  provided it is the smallest set of such points such that the corresponding coefficients  $\{c_\xi\}_{\xi \in \mathcal{M}}$  can be set independently, and all other coefficients of  $s$  can be consistently determined from smoothness conditions, i.e., in such a way that all smoothness conditions are satisfied, see [2, p. 485]. Suppose  $\mathcal{N} := \{\lambda_i\}_{i=1}^n$  is a set of linear functionals of the form  $\lambda_i := \varepsilon_{\eta_i} \sum_{|\alpha| \leq m_i} a_i^\alpha D^\alpha$ , where  $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$ , and  $\varepsilon_{\eta_i}$  denotes point-evaluation at the point  $\eta_i$  in  $\Omega$ . Then  $\mathcal{N}$  is called a **nodal determining set** for  $\mathcal{S}$  if  $s \in \mathcal{S}$  and  $\lambda s = 0$  for all  $\lambda \in \mathcal{N}$  implies  $s \equiv 0$ . If there is no smaller such set, then  $\mathcal{N}$  is called a **nodal minimal determining set**, see [2, p. 490].

For later use, we now present two results on bivariate interpolation. Let  $F := \langle v_1, v_2, v_3 \rangle$  be an arbitrary triangle. The first result is a minor extension of Lemma 5.1 in [1].

**Lemma 2.1.** *Suppose that we are given all of the coefficients  $\{c_{ijk}^F\}_{i+j+k=3}$  of a bivariate cubic polynomial  $p$  except for  $c_{111}^F$ . Then for any given real number  $r$ , there exists a unique  $c_{111}^F$  so that  $p(\xi_{111}^F) = r$ . Moreover, the computation of  $c_{111}^F$  is a stable process in the sense that*

$$|c_{111}^F| \leq C \left( |r| + \max_{(i,j,k) \neq (1,1,1)} |c_{ijk}^F| \right),$$

where  $C = \frac{9}{2}$ .

**Proof.** The interpolation condition gives

$$c_{111}^F B_{111}^F(\xi_{111}^F) = r - \sum_{(i,j,k) \neq (1,1,1)} c_{ijk}^F B_{ijk}^F(\xi_{111}^F),$$

where  $B_{ijk}^F$  are the bivariate Bernstein-basis polynomials associated with the triangle  $F$ . The result then follows from the fact that  $B_{111}^F(\xi_{111}^F) = 2/9$  and the fact that the  $B_{ijk}^F$  form a partition of unity.  $\square$

Now suppose  $F_{CT}$  is the well-known Clough–Tocher split of  $F$  into three subtriangles  $F_i := \langle v_F, v_i, v_{i+1} \rangle$ ,  $i = 1, 2, 3$ , where  $v_F$  is a point in the interior of  $F$ . The following result is similar to Lemma 5.2 in [1].

**Lemma 2.2.** *Suppose that we are given all of the B-coefficients of a  $C^1$  cubic bivariate spline  $s$  defined on  $F_{CT}$  except for  $c_{300}^{F_1}, c_{210}^{F_1}, c_{201}^{F_1}, c_{111}^{F_1}$ . Then for any given real number  $r$ , there exists a unique choice of these coefficients so that  $s(\xi_{111}^{F_1}) = r$ . The computation of these coefficients is stable in the sense that there is an absolute constant  $C$  such that*

$$|c_{ijk}^{F_1}| \leq C \left( |r| + \max_{c_\xi \in \mathcal{K}} |c_\xi| \right), \quad (ijk) \in \{(300), (210), (201), (111)\},$$

where  $\mathcal{K}$  is the set of known coefficients of  $s$ .

**Proof.** The interpolation condition  $s(\zeta_{111}^{F_1}) = r$  leads to the equation

$$c_{300}^{F_1} + 3c_{210}^{F_1} + 3c_{201}^{F_1} + 6c_{111}^{F_1} = R_1,$$

where  $R_1$  is a combination of  $r$  and the known B-coefficients of  $s$ . Suppose  $b_1, b_2, b_3$  are the barycentric coordinates of  $v_F$  relative to  $F$ . Writing down the  $C^1$  smoothness conditions across the interior edges of  $F_1$ , we are led to the linear system

$$\begin{pmatrix} 1 & 3 & 3 & 6 \\ 1 & -b_1 & -b_2 & 0 \\ 0 & 1 & 0 & -b_2 \\ 0 & 0 & 1 & -b_1 \end{pmatrix} \begin{pmatrix} c_{300}^{F_1} \\ c_{210}^{F_1} \\ c_{201}^{F_1} \\ c_{111}^{F_1} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix}, \tag{2.1}$$

where  $R_2, R_3, R_4$  are combinations of  $r$  and the coefficients in  $\mathcal{H}$ . The determinant of this matrix is  $D := -6 - 3b_1 - 3b_2 - 2b_1b_2$ . It is easy to see that as we vary  $v_F$  in  $T$ ,  $(b_1, b_2)$  runs over the set  $\{(b_1, b_2) : 0 \leq b_1, b_2 \leq 1, b_1 + b_2 \leq 1\}$ . This gives  $|D| \geq 6$  for any choice of  $v_F$ , which implies both the existence of a solution and the stability of its computation.  $\square$

### 3. Partial Worsley–Farin splits

The following definition is taken from [1].

**Definition 3.1.** Let  $T$  be a tetrahedron, and let  $v_T$  be a point in its interior. Given an integer  $0 \leq m \leq 4$ , let  $F_1, \dots, F_m$  be distinct faces of  $T$ , and for each  $i = 1, \dots, m$ , let  $v_{F_i}$  be a point in the interior of  $F_i$ . Then we define the corresponding  $m$ -th order partial Worsley–Farin split  $\Delta_{WF}^m$  of  $T$  to be the tetrahedral partition obtained by the following steps:

- (1) connect  $v_T$  to each of the four vertices of  $T$ ,
- (2) connect  $v_T$  to the points  $v_{F_i}$  for  $i = 1, \dots, m$ ,
- (3) connect  $v_{F_i}$  to the three vertices of  $F_i$  for  $i = 1, \dots, m$ .

The  $m$ -th order partial Worsley–Farin split of a tetrahedron results in  $4 + 2m$  subtetrahedra, see Fig. 1. The splits  $\Delta_{WF}^0$  and  $\Delta_{WF}^4$  are the well-known Alfeld and Worsley–Farin splits, respectively, see [2, Section 16.7]. We now recall an important fact about the space  $\mathcal{S}_3^1(\Delta_{WF}^m)$ , where  $\Delta_{WF}^m$  is an  $m$ -th order partial Worsley–Farin split of a tetrahedron  $T := \langle v_1, v_2, v_3, v_4 \rangle$ .

**Theorem 3.2.** ([1, Theorem 6.3]). Fix  $0 \leq m \leq 4$ . Let  $\mathcal{M}_m$  be the union of the following sets of domain points in  $\mathcal{D}_{\Delta_{WF}^m}$ :

- (1) for each  $i = 1, \dots, 4$ ,  $D(v_i) \cap T_i$  for some tetrahedron  $T_i \in \Delta_{WF}^m$  containing  $v_i$ ;
- (2) for each face  $F$  of  $T$  that is not split, the point  $\zeta_{111}^F$ ;
- (3) for each face  $F$  of  $T$  that has been subjected to a Clough–Tocher split, the points  $\{\zeta_{111}^{F_i}\}_{i=1}^3$ , where  $F_1, F_2, F_3$  are the subfaces of  $F$ .

Then  $\mathcal{M}_m$  is a minimal determining set for  $\mathcal{S}_3^1(\Delta_{WF}^m)$ .

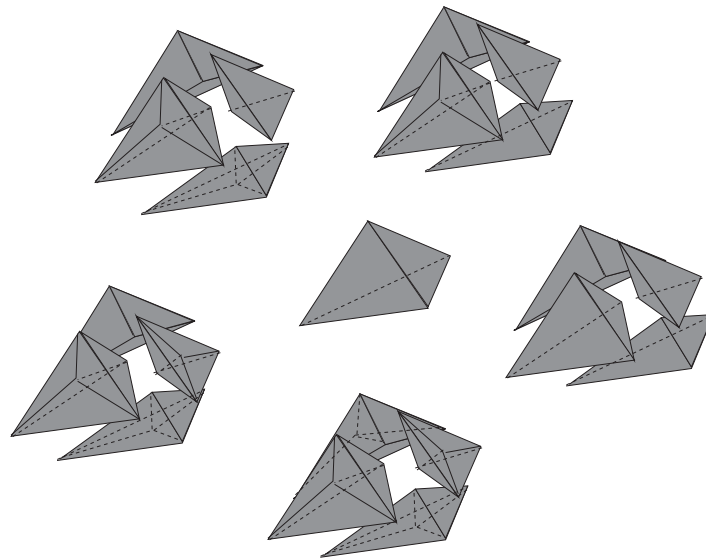


Fig. 1. Partial Worsey–Farin splits subdividing a given tetrahedron into 4, 6, 8, 10, or 12 subtetrahedra.

#### 4. Decomposition of tetrahedral partitions

In this section we describe two algorithms for decomposing a given tetrahedral partition  $\Delta_0$  into classes of tetrahedra. These decompositions will be used to define our Lagrange interpolation pair.

**Algorithm 4.1.** For  $i = 0$  to 4, repeat until no longer possible: choose a tetrahedron  $T$  such that exactly  $i$  vertices of  $T$  belong to tetrahedra that have been chosen earlier. We call these vertices the marked vertices of  $T$ . Put  $T$  into the class  $\mathcal{A}_i$ .

This algorithm decomposes the partition  $\Delta_0$  into five classes  $\mathcal{A}_0, \dots, \mathcal{A}_4$ . It also produces an ordering  $T_1, \dots, T_n$  of the tetrahedra of  $\Delta_0$ . Clearly, if  $T$  is a tetrahedron in the class  $\mathcal{A}_0$ , then all of its vertices are unmarked.

**Lemma 4.2.** Suppose  $v$  is a marked vertex of a tetrahedron  $T_n$  of class  $\mathcal{A}_j$  with  $1 \leq j \leq 4$ . Then there exists a tetrahedron  $T_m$  of class  $\mathcal{A}_i$  with  $i < j$  such that  $v$  is a vertex of  $T_m$ .

**Proof.** Let  $v$  be a marked vertex of  $T_n$ , and let  $T_m$  be the first of the tetrahedra  $T_1, \dots, T_{n-1}$  to contain  $v$ . Then  $T_m$  must be in some class  $\mathcal{A}_i$  with  $0 \leq i \leq j$ . We claim that it cannot be in class  $\mathcal{A}_j$ , since at the time  $T_m$  was chosen,  $T_n$  would have had at most  $j - 1$  vertices in common with the tetrahedra  $T_1, \dots, T_{m-1}$ , and thus would have been chosen before  $T_m$  unless  $T_m$  is in  $\mathcal{A}_i$  for some  $i < j$ .  $\square$

The following algorithm produces a different decomposition based on shared edges.

**Algorithm 4.3.** For  $i = 0$  to 6, repeat until no longer possible: choose a tetrahedron  $T$  such that exactly  $i$  edges of  $T$  belong to tetrahedra that have been chosen earlier. We call these edges marked edges of  $T$ . Put  $T$  into the class  $\mathcal{B}_i$ .

This algorithm partitions  $\Delta_0$  into seven classes  $\mathcal{B}_0, \dots, \mathcal{B}_6$ . It also produces an ordering  $\tilde{T}_1, \dots, \tilde{T}_n$  of the tetrahedra of  $\Delta_0$ . It is easy to see that  $\mathcal{A}_0 \subseteq \mathcal{B}_0$ , and if  $\tilde{T}$  is a tetrahedron in the class  $\mathcal{B}_0$ , then all of its edges are unmarked.

**Lemma 4.4.** *Suppose  $e$  is a marked edge of a tetrahedron  $\tilde{T}_n$  of class  $\mathcal{B}_j$  with  $1 \leq j \leq 6$ . Then there exists a tetrahedron  $\tilde{T}_m$  of class  $\mathcal{B}_i$  with  $i < j$  such that  $e$  is an edge of  $\tilde{T}_m$ .*

**Proof.** Let  $e$  be a marked edge of  $\tilde{T}_n$ , and let  $\tilde{T}_m$  be the first of the tetrahedra  $\tilde{T}_1, \dots, \tilde{T}_{n-1}$  to contain  $e$ . Then  $\tilde{T}_m$  must be in some class  $\mathcal{B}_i$  with  $0 \leq i \leq j$ . We claim that it cannot be in class  $\mathcal{B}_j$ , since at the time  $\tilde{T}_m$  was chosen,  $\tilde{T}_n$  would have had at most  $j - 1$  edges in common with the tetrahedra  $\tilde{T}_1, \dots, \tilde{T}_{m-1}$ , and thus would have been chosen before  $\tilde{T}_m$ .  $\square$

## 5. Construction of $\Delta$

As a first step towards creating a Lagrange interpolating pair solving Problem 1.1, in this section we describe an algorithm for constructing a suitable tetrahedral partition  $\Delta$ . As a starting point, let  $\Delta_0$  be an arbitrary tetrahedral partition with vertices at the points  $\mathcal{V}$ . We shall construct  $\Delta$  by splitting certain of the tetrahedra of  $\Delta_0$ .

The following algorithm proceeds in two steps. In the first step we apply Clough–Tocher splits to some of the triangular faces of  $\Delta_0$ . The choice of which faces to split is controlled by the ordering  $\tilde{T}_1, \dots, \tilde{T}_n$  of the tetrahedra in  $\Delta_0$  produced by Algorithm 4.3. The Clough–Tocher split of a face involves inserting a point  $v_F$  in the interior of  $F$  and connecting it to each of the three vertices of  $F$ . In the second step we apply partial Worsey–Farin splits to the tetrahedra of  $\Delta_0$  that have one or more split faces.

**Algorithm 5.1** (Construct  $\Delta$ ).

- (1) For  $i = 1, \dots, n$ , if  $F$  is a face of  $\tilde{T}_i$  that is not shared with any tetrahedron  $\tilde{T}_j$  with  $j < i$ , apply a Clough–Tocher split to  $F$  when either two or three of the edges of  $F$  are marked edges of  $\tilde{T}_i$ .
- (2) For each tetrahedron  $T \in \Delta_0$ , let  $m$  be the number of its faces that have been split in step 1. If  $m > 0$ , apply an  $m$ -th order partial Worsey–Farin split to  $T$  using its incenter.

In Algorithm 5.1 we did not specify how to choose the split points  $v_F$  to be used in creating the Clough–Tocher splits of faces. For our purposes, these points must be chosen in a special way:

- (1) if  $F$  is a boundary face of  $\Delta_0$ , choose  $v_F$  to be the barycenter of  $F$ ,
- (2) if  $F$  is an interior face of  $\Delta_0$ , choose  $v_F$  to be the intersection of  $F$  with the line connecting the incenters of the two tetrahedra sharing  $F$ .

It is easy to see that none of the tetrahedra in the classes  $\mathcal{B}_0$  or  $\mathcal{B}_1$  is split. Tetrahedra in class  $\mathcal{B}_2$  are either not split, or are subjected to a partial-Worsey–Farin split of order  $m = 1$ . The types of splits that may be applied to tetrahedra in the various classes are shown in Table 1, where  $m$ -WF stands for the  $m$ -th order Worsey–Farin split. In this table the symbol “–” indicates that the corresponding split does not occur. The symbol “o” identifies cases where a tetrahedron in a class  $\mathcal{B}_j$  may be given the split in the indicated column because the tetrahedron shares one or more faces with tetrahedra in lower classes. The symbol “x” identifies cases where a tetrahedron in  $\mathcal{B}_j$  has no neighbor in a lower class.

Table 1  
Possible splits for the different classes .

Class	No split	1-WF	2-WF	3-WF	4-WF
$\mathcal{B}_0$	×	–	–	–	–
$\mathcal{B}_1$	×	–	–	–	–
$\mathcal{B}_2$	×	×	–	–	–
$\mathcal{B}_3$	○	× ○	×	×	–
$\mathcal{B}_4$	–	–	○	× ○	×
$\mathcal{B}_5$	–	–	○	○	× ○
$\mathcal{B}_6$	–	–	○	○	× ○

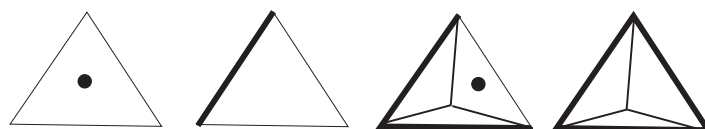


Fig. 2. Points inserted on faces (black dots) by Algorithm 6.1.

### 6. Construction of $P$

In this section we present an algorithm for constructing a set  $P$  of interpolation points which together with the spline space  $\mathcal{S}_3^1(\Delta)$  based on the tetrahedral partition  $\Delta$  of the previous section will form a Lagrange interpolation pair. The algorithm makes use of both of the ordered lists  $T_1, \dots, T_n$  and  $\tilde{T}_1, \dots, \tilde{T}_n$  created by Algorithms 4.1 and 4.3.

**Algorithm 6.1** (Construction of  $P$ ). For  $i = 1, \dots, n$ ,

- (1) for each edge  $\langle u, v \rangle$  of  $T_i$ ;
  - (a) choose the points  $v$  and  $(u + 2v)/3$  if  $v$  is not a marked vertex of  $T_i$ ;
  - (b) choose the points  $u$  and  $(2u + v)/3$  if  $u$  is not a marked vertex of  $T_i$ .
- (2) for each face  $F$  of  $\tilde{T}_i$  that is not a face of a tetrahedron  $\tilde{T}_j$  with  $j < i$ ;
  - (a) choose the barycenter of  $F$  if  $F$  has no marked edges of  $\tilde{T}_i$ ,
  - (b) choose the barycenter of the subtriangle of  $F$  with no marked edge if  $F$  has exactly two marked edges of  $\tilde{T}_i$ .

We emphasize that the interpolation points chosen by this algorithm belong to  $\mathcal{D}_\Delta$ , and all lie on edges and faces of  $\Delta_0$ . In particular,  $P$  contains all of the points in  $\mathcal{V}$ , i.e., the vertices of  $\Delta_0$ . Interpolation points are chosen on faces of tetrahedra in  $\Delta_0$  when the face is not split and contains no marked edges, or when the face is split but contains exactly two marked edges, see Fig. 2, where the thicker lines indicate marked edges, i.e. edges in common with tetrahedra appearing earlier in the list  $\tilde{T}_1, \dots, \tilde{T}_n$ .

### 7. The main result

In this section we prove the main result of this paper, Theorem 7.3 below, which states that  $P$  and  $\mathcal{S}_3^1(\Delta)$  form a Lagrange interpolating pair. We begin by showing that  $P$  is a minimal determining set for  $\mathcal{S}_3^1(\Delta)$ .



**Theorem 7.1.** *The set  $\mathcal{M} := P$  produced by Algorithm 6.1 is a minimal determining set for  $\mathcal{S}_3^1(\Delta)$ .*

**Proof.** Suppose that for each  $\xi \in \mathcal{M}$ , we fix the corresponding coefficient of a spline  $s \in \mathcal{S}_3^0(\Delta)$ . We now show that all other coefficients of  $s$  are consistently determined, i.e., in such a way that all  $C^1$  smoothness conditions are satisfied so that  $s$  lies in  $\mathcal{S}_3^1(\Delta)$ .

*Step 1:* It is easy to see by the construction of  $P$  that for each vertex  $v$  of  $\Delta_0$ , there is a tetrahedron  $T$  in  $\Delta_0$  with vertex at  $v$  such that  $P$  contains the four domain points in  $D^T(v)$ . Then the coefficients associated with the remaining domain points in  $D(v)$  are consistently determined by the  $C^1$  smoothness at  $v$ . The consistency is insured by the fact that these balls do not overlap.

*Step 2:* To show that all remaining coefficients are consistently determined, we consider the tetrahedra in the order  $\tilde{T}_1, \dots, \tilde{T}_n$  of Algorithm 4.3, and apply Theorem 3.2. We begin with  $\tilde{T}_1$ . We already have determined the coefficients associated with the balls around the vertices of  $\tilde{T}_1$ . Since  $\tilde{T}_1$  is in the class  $\mathcal{B}_0$ , none of its faces is split and  $P$  contains the point  $\xi_{111}^F$  for each face  $F$ , which means that the corresponding coefficient has been fixed. Theorem 3.2 then implies that all remaining coefficients corresponding to domain points in  $\tilde{T}_1$  are consistently determined. The  $C^1$  conditions imply that all coefficients associated with domain points in the tubes around the edges of  $\tilde{T}_1$  are also consistently determined. Now suppose we have determined the coefficients corresponding to all of the domain points in the tetrahedra  $\tilde{T}_1, \dots, \tilde{T}_{j-1}$ , and consider  $\tilde{T}_j$ . Using  $C^1$  smoothness, we get all coefficients of  $s$  corresponding to domain points in the balls around the vertices of  $\tilde{T}_j$ , as well as in the tube around each marked edge of  $\tilde{T}_j$ . To apply Theorem 3.2, we now show that we have already determined the coefficients corresponding to the domain points in items (2) and (3) of the theorem. Consider a face  $F$  of  $\tilde{T}_j$ . If  $F$  is shared with a tetrahedron  $\tilde{T}_i$  with  $i < j$ , then all coefficients corresponding to domain points on  $F$  are already known. Suppose now that  $F$  is not shared with such a tetrahedron. There are four cases:

- (a)  $F$  has no marked edges. Then  $F$  is not split, and  $P$  contains the point  $\xi_{111}^F$ . We have already fixed the corresponding coefficient.
- (b)  $F$  has one marked edge  $e$ . Then  $F$  is not split, and we already know the coefficient corresponding to  $\xi_{111}^F$  since it lies in the tube around  $e$ .
- (c)  $F$  has two marked edges. Then  $F$  has been subjected to a CT split. In this case the coefficients corresponding to the barycenters of the three subtriangles are known since the corresponding domain points are either in a tube around a marked edge, or the point is in  $P$ .
- (d)  $F$  has three marked edges. Then  $F$  has been subjected to a CT split. In this case the coefficients corresponding to the barycenters of the three subtriangles are known since the corresponding domain points are in tubes around the marked edges.

We have shown that the coefficients of  $s$  corresponding to domain points in the minimal determining set of Theorem 3.2 are determined, and we conclude from the theorem that the coefficients of  $s$  corresponding to all remaining domain points in  $\tilde{T}_j$  are consistently determined. Continuing with the remaining tetrahedra, we get all coefficients of  $s$ .

There is one subtle point concerning consistency that remains to be discussed. It concerns the  $C^1$  smoothness conditions across faces of  $\Delta_0$ . Suppose  $F$  is a face of  $\Delta_0$  that is shared by two tetrahedra  $T_m$  and  $T_n$ , and suppose  $F$  has been subjected to a Clough–Tocher split. Then a necessary and sufficient condition that all  $C^1$  smoothness conditions across  $F$  are satisfied is that the split point  $v_F$  lie on the straight line between the interior split points  $v_{T_m}$  and  $v_{T_n}$ , see the proof of Theorem 18.11 in [2]. But we have made sure that this is the case in our construction of  $\Delta$ .  $\square$

The fact that  $P$  is a minimal determining set for  $\mathcal{S}_3^1(\Delta)$  implies that the dimension of  $\mathcal{S}_3^1(\Delta)$  is equal to the number  $N$  of points in the set  $P$ , see [2, Section 17.3]. Clearly,  $N = 4n_V + n_F$ , where  $n_V = \#\mathcal{V}$  is the number of vertices of  $\Delta_0$ , and  $n_F$  is the number of faces where an interpolation point was added to  $P$  in step 2 of Algorithm 6.1. To state our next theorem, we recall that  $\varepsilon_\eta$  denotes point-evaluation at the point  $\eta$ .

**Theorem 7.2.** *Let  $P$  be the set of domain points chosen by Algorithm 6.1. Then the set  $\mathcal{N} := \{\varepsilon_\eta\}_{\eta \in P}$  is a nodal minimal determining set for  $\mathcal{S}_3^1(\Delta)$ .*

**Proof.** We have already observed that the dimension of  $\mathcal{S}_3^1(\Delta)$  is equal to  $\#P = \#\mathcal{N}$ . Thus, to prove  $\mathcal{N}$  is a nodal minimal determining set for  $\mathcal{S}_3^1(\Delta)$ , it suffices to show that if we fix the values of  $s(\eta)$  for all  $\eta \in P$ , then all B-coefficients of  $s$  are determined.

*Step 1:* We show that all coefficients of  $s$  associated with domain points on the edges of  $\Delta_0$  and in the balls around the vertices of  $\Delta_0$  are determined. To this end, we examine the tetrahedra of  $\Delta_0$  in the order  $T_1, \dots, T_n$  defined by Algorithm 4.1. Suppose we have established the assertion for the edges and vertices of all of the tetrahedra  $T_1, \dots, T_{j-1}$ , and consider  $T_j$ . We have already determined the coefficients of  $s$  in the balls around marked vertices of  $T_j$ . Let  $e := \langle u, v \rangle$  be an edge of  $T_j$ . Suppose first that  $u, v$  are both unmarked vertices of  $T_j$ . Then  $P$  contains all four domain points on  $e$ , call them  $\eta_1, \dots, \eta_4$ . We are given the values of the univariate cubic polynomial  $s|_e$  at these four points. This leads to a linear system of four equations for the four B-coefficients of  $s$  associated with  $\eta_1, \dots, \eta_4$ . It is easy to see that this system is nonsingular with determinant equal to  $\frac{12}{81}$ , independent of the length or orientation of the edge  $e$ . Now suppose that one vertex of  $e$  is marked, say  $u$ . Then we know the coefficients associated with  $D(u)$ , but not those associated with  $D(v)$ . But now we can use the interpolation conditions at the two points  $(u + 2v)/3$  and  $v$  to get a linear system of 2 equations for the B-coefficients associated with these two domain points. This system is nonsingular with determinant equal to  $\frac{12}{81}$ , independent of the length or orientation of the edge  $e$ . We have shown how to compute the coefficients of  $s$  associated with all domain points on the edges of  $T_j$ . We then use these to determine the coefficients associated with domain points in the ball  $D(v)$  around each unmarked vertex of  $T_j$ .

*Step 2:* We now show that all remaining coefficients of  $s$  can be computed from the interpolation conditions. To this end, we consider the tetrahedra in the order  $\tilde{T}_1, \dots, \tilde{T}_n$  defined by Algorithm 4.3. Suppose we have computed all coefficients of  $s$  associated with domain points in the tetrahedra  $\tilde{T}_1, \dots, \tilde{T}_{j-1}$ , and consider  $\tilde{T}_j$ . It suffices to show how to compute the coefficients associated with the minimal determining set  $\mathcal{M}_m$  of Theorem 3.2. We already have the coefficients associated with domain points in the balls around the vertices of  $\tilde{T}_j$  and in the tubes around its marked edges. We now deal with the coefficients corresponding to the remaining points in  $\mathcal{M}_m$ . These lie on faces of  $\tilde{T}_j$ . Let  $F$  be a face of  $\tilde{T}_j$ . If  $F$  is shared with a tetrahedron  $\tilde{T}_i$  with  $i < j$ , then we have already computed the coefficients of  $s$  associated with all domain points on  $F$ . Thus, it suffices to consider faces  $F$  of  $\tilde{T}_j$  that are not shared with a tetrahedron  $\tilde{T}_i$  with  $i < j$ .

Suppose that  $F$  is not split. We need to show how to compute the coefficient associated with  $\xi_{111}^F$ . There are two subcases. If  $F$  contains no marked edges of  $\tilde{T}_j$ , then  $P$  contains the point  $\xi_{111}^F$ , and we can use the interpolation condition at this point to compute the corresponding coefficient, see Lemma 2.1. If  $F$  contains one marked edge of  $\tilde{T}_j$ , then  $\xi_{111}^F$  lies in the tube around that edge, and the corresponding coefficient is already known.

Now suppose  $F$  has been subjected to a Clough–Tocher split. We have to show how to compute the coefficients associated with the domain points  $\xi_{111}^{F_i}$ , where  $F_1, F_2, F_3$  are the subfaces of  $F$ .

If  $F_i$  contains a marked edge of  $\tilde{T}_j$ , then the coefficient corresponding to  $\xi_{111}^{F_i}$  is already known since this point lies in the tube around the marked edge. Thus, if all three edges of  $F$  are marked edges of  $\tilde{T}_j$ , we are done. Now suppose only two of the edges of  $F$  are marked edges of  $\tilde{T}_j$ , say those contained in  $F_2, F_3$ . Then  $P$  contains the barycenter  $\xi_{111}^{F_1}$  of  $F_1$ , and we can apply Lemma 2.2 to compute the corresponding coefficient  $c_{111}^{F_1}$ . We now apply Theorem 3.2 to determine the coefficients of  $s$  corresponding to all remaining domain points in  $\tilde{T}_j$ .  $\square$

We are ready to establish the main result of the paper.

**Theorem 7.3.** *The set  $P := \{\eta_i\}_{i=1}^N$  together with the spline space  $\mathcal{S}_3^1(\Delta)$  form a Lagrange interpolation pair.*

**Proof.** The proof of Theorem 7.2 shows that for any set of real numbers  $r_1, \dots, r_N$ , there is a unique spline  $s \in \mathcal{S}_3^1(\Delta)$  satisfying  $s(\eta_i) = r_i$  for  $i = 1, \dots, N$ .  $\square$

### 8. Bounds on the error of the interpolant

Suppose that  $P := \{\eta_i\}_{i=1}^N$  and  $\mathcal{S}_3^1(\Delta)$  are the Lagrange interpolation pair constructed above from an initial tetrahedral partition  $\Delta_0$  of a domain  $\Omega \subset \mathbb{R}^3$ . Then for every  $f \in C(\Omega)$ , there is a unique spline  $s_f \in \mathcal{S}_3^1(\Delta)$  such that

$$s_f(\eta_i) = f(\eta_i), \quad i = 1, \dots, N.$$

This interpolation process defines a linear projector mapping  $C(\Omega)$  onto  $\mathcal{S}_3^1(\Delta)$ . Our goal in this section is to provide a bound on  $\|f - s\|_\Omega$  for smooth functions, where the error is measured in the maximum norm on  $\Omega$ . To accomplish this goal, we will apply Theorem 17.22 of [2]. To do so, we need to show that nodal minimal determining set  $P$  of Theorem 7.2 is local and stable, see Definition 17.21 of [2]. For each  $\xi \in \mathcal{D}_\Delta$ , let

$$A_\xi := \{\eta \in P : c_\xi \text{ depends on } s(\eta)\},$$

where  $\{c_\xi\}_{\xi \in \mathcal{D}_\Delta}$  are the B-coefficients of  $s_f$ .

**Theorem 8.1.** *The nodal minimal determining set  $P$  for  $\mathcal{S}_3^1(\Delta)$  is local in the sense that for all  $\xi \in \mathcal{D}_\Delta$ ,*

$$A_\xi \subseteq \text{star}^\ell(T_\xi), \tag{8.1}$$

with  $\ell = 10$ , where  $T_\xi$  is a tetrahedron in  $\Delta_0$  that contains  $\xi$ . It is also stable in the sense that there exists a constant  $C$  depending only on the smallest solid and face angles in  $\Delta_0$  such that for all  $\xi \in \mathcal{D}_\Delta$ ,

$$|c_\xi| \leq C \max_{\eta \in A_\xi} |f(\eta)|. \tag{8.2}$$

**Proof.** For the definition of solid and face angles of a tetrahedron, see [2, p. 462]. We consider three cases.

*Case 1:* ( $\xi$  lies on an edge of a tetrahedron  $T_\xi$  in  $\Delta_0$ ). Let  $e := \langle u, v \rangle$  be the edge of  $T_\xi$  containing  $\xi$ . If neither  $u$  nor  $v$  is a marked vertex of  $T_\xi$ , then  $P$  contains all four domain points on  $e$ , and the

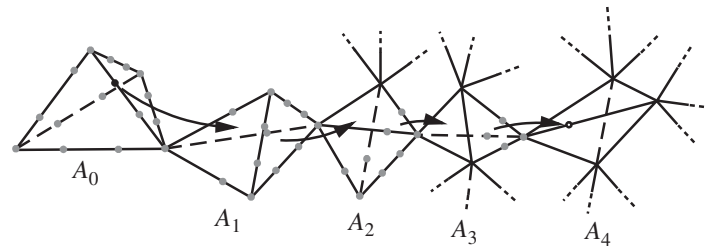


Fig. 3. Longest chain of influence of an interpolation value along the edges of  $\Delta_0$ . Interpolation points are shown as gray and black dots, and the direction of propagation (of the value  $f_\eta$  at the black dot) into neighboring tetrahedra of higher classes  $\mathcal{A}_i$  is indicated by the arrows. The value  $f_\eta$  at the black dot can influence the B-coefficient indicated as an open circle, which is associated with a domain point in a tetrahedron from class  $\mathcal{A}_4$ .

corresponding coefficients depend only on the values of  $s$  at these four points, see the proof of Theorem 7.2. For these coefficients, (8.1) holds with  $\ell = 0$ . This argument applies to all edges of tetrahedra  $T_\xi$  of class  $\mathcal{A}_0$ . If  $T_\xi$  lies in a higher class, the situation is more complicated, and there is a certain propagation of influence. We claim that if  $T_\xi$  is of class  $\mathcal{A}_j$ , then (8.1) holds with  $\ell = j$ . We have already established this for  $j = 0$ , and can now proceed by induction. Suppose we have established the claim for classes  $\mathcal{A}_0, \dots, \mathcal{A}_{j-1}$ , and consider  $T_\xi \in \mathcal{A}_j$ . If neither end of  $e$  is marked, then as before we get (8.1) with  $\ell = 0$ . Now suppose  $u$  is marked. Let  $\xi \in D(u)$ . By Lemma 4.2, there exists a tetrahedron  $T_n$  in class  $\mathcal{A}_{j-1}$  sharing the vertex  $u$ , and we can compute  $c_\xi$  as a combination of coefficients associated with  $D^{T_n}(u)$ . By the induction hypothesis, these can be computed from data at points lying in  $\text{star}^{j-1}(T_n)$ , and we conclude that (8.1) holds with  $\ell = j$ . It remains to consider the case  $\xi \in D(v)$ , where  $u$  is marked but  $v$  is not. In this case  $c_\xi$  is computed by solving a  $2 \times 2$  linear system to find the coefficients associated with the domain points  $(u + 2v)/2$  and  $v$ . But the right-hand side of this system includes coefficients associated with domain points in the ball  $D(u)$ . It follows that  $A_\xi \subseteq \text{star}^j(T_\xi)$ . We conclude that in all cases, if  $\xi$  lies on any edge of a tetrahedron  $T_\xi$ , then (8.1) holds with  $\ell = 4$ . The worst case is illustrated in Fig. 3.

We now verify the stability condition (8.2) when  $\xi$  lies on an edge of  $T_\xi$ . By the proof of Theorem 7.2, coefficients with domain points on edges of  $\Delta_0$  are only computed in one of three ways:

- (1) by solving a  $2 \times 2$  linear system whose determinant is  $\frac{4}{9}$ ;
- (2) by solving a  $4 \times 4$  linear system whose determinant is  $\frac{12}{81}$ ;
- (3) by employing the  $C^1$  smoothness at a vertex to compute a coefficient from previously computed coefficients.

The constant of stability in (3) depends on the smallest solid and face angles of  $\Delta_0$ .

*Case 2:* ( $\xi$  lies on a face  $F$  of a tetrahedron  $T_\xi$  in  $\Delta_0$ , but not on an edge). We claim that if  $T_\xi$  is of class  $\mathcal{B}_j$ , then (8.1) holds with  $\ell = 4 + j$ . This is immediate for  $j = 0$ , since when  $T_\xi$  is of class  $\mathcal{B}_0$ , then  $c_\xi$  is computed from Lemma 2.1, which shows that  $c_\xi$  depends on the value of  $s(\xi)$ , but also on the coefficients corresponding to domain points in the balls around the vertices of  $F$ . We showed in Case 1 that such coefficients depend only on data at points in  $\text{star}^4(T_\xi)$ . Suppose now that we have established the claim for all tetrahedra in the classes  $\mathcal{B}_0, \dots, \mathcal{B}_{j-1}$ , and consider  $T_\xi \in \mathcal{B}_j$ . Now  $c_\xi$  will be computed from a  $C^1$  smoothness condition around a marked edge  $e$ , or by applying Lemma 2.1 or 2.2. Thus,  $c_\xi$  may depend on the value of  $s$  at a point  $\eta$  in  $F$ , but more importantly can also depend on coefficients associated with points in another tetrahedron sharing the marked edge  $e$ , which by Lemma 4.4 is in a lower class. Then, applying the induction

hypothesis, we conclude that  $A_\xi \subseteq \text{star}^{4+j}(T_\xi)$ . The worst case is for  $j = 6$ , and we conclude that (8.1) holds with  $\ell = 10$  for all  $\xi$ .

We now verify the stability condition (8.2) when  $\xi$  lies on a face  $F$  of a tetrahedron  $T_\xi$  in  $\Delta_0$ , but not on an edge. By the proof of Theorem 7.2, coefficients with domain points in the interior of faces of  $\Delta_0$  are only computed in one of three ways:

- (1) by enforcing a  $C^1$  smoothness condition around an edge,
- (2) by Lemma 2.1,
- (3) by Lemma 2.2.

All three of these computations are stable, where the constant of stability in (1) depends on the smallest solid and face angles of  $\Delta_0$ .

Case 3: ( $\xi$  lies in the interior of a tetrahedron  $T_\xi$  in  $\Delta_0$ ). In this case  $c_\xi$  is computed from known coefficients on the faces of  $T_\xi$ , cf. Theorem 3.2, by enforcing  $C^1$  smoothness conditions. Thus, (8.1) holds with  $\ell = 10$ . These computations are stable with a constant depending on the largest solid and face angles in  $\Delta_0$ .  $\square$

Given a compact set  $B \subseteq \Omega$  and an integer  $m > 0$ , let

$$|f|_{m,B} := \sum_{|\alpha|=m} \|D^\alpha f\|_B \tag{8.3}$$

be the usual Sobolev seminorm, where  $\|\cdot\|_B$  denotes the infinity norm on  $B$ . For any tetrahedron  $T$  in  $\Delta_0$ , let  $|T|$  be the length of its longest edge. The following result follows immediately from Theorem 17.22 of [2].

**Theorem 8.2.** *Given  $T \in \Delta_0$ , let  $\Omega_T := \text{star}^{10}(T)$  in  $\Delta_0$ . Then for every  $f \in C^{m+1}(\Omega_T)$  with  $0 \leq m \leq 3$ ,*

$$\|D^\alpha(f - s_f)\|_T \leq K |T|^{m+1-|\alpha|} |f|_{m+1,\Omega_T}, \tag{8.4}$$

for all  $|\alpha| \leq m$ . If  $\Omega_T$  is convex, the constant  $K$  depends only on the smallest solid and face angles of  $\Omega_T$  as a subpartition of  $\Delta$ . If  $\Omega_T$  is not convex,  $K$  also depends on the Lipschitz constant of the boundary of  $\Omega_T$ .

The global version of this theorem also holds with  $T$  and  $\Omega_T$  replaced by  $\Omega$ , and with  $|T|$  replaced by the mesh size  $|\Delta_0|$  of  $\Delta_0$ , i.e., the length of the longest edge in  $\Delta_0$ .

## 9. Remarks

**Remark 1.** Suppose  $\mathcal{V}$  is an arbitrary set of points in  $\mathbb{R}^3$ , and that  $\Delta$  is a tetrahedral partition with vertices  $\mathcal{V}$ . Set  $P := \mathcal{V}$ , and let  $\mathcal{S} := \mathcal{S}_1^0(\Delta)$  be the space of continuous linear splines. Then clearly  $P$  and  $\mathcal{S}$  form a Lagrange interpolation pair. It is also straightforward to create a Lagrange interpolation pair using  $C^0$  splines of higher degree, provided we add an appropriate set of additional interpolation points.

**Remark 2.** There is a fairly extensive history of Lagrange interpolation with bivariate splines, although even in this case, it is quite difficult to construct Lagrange interpolating pairs using  $C^1$  splines. See [3–6,8,9].

**Remark 3.** Much less is known about Lagrange interpolation with  $C^1$  trivariate splines. The first local Lagrange method was constructed in [10] using quintic splines, but only for the case where the points of  $\mathcal{V}$  lie on a grid. With this same restriction on  $\mathcal{V}$ , we recently constructed a Lagrange interpolation pair using cubic splines, see [1]. In [7] we constructed Lagrange interpolation pairs for an arbitrary initial point set  $\mathcal{V}$  using both quadratic and cubic splines. However, as pointed out in the introduction, that cubic spline method did not provide optimal approximation order.

**Remark 4.** For given  $\mathcal{V}$  and initial tetrahedral partition  $\Delta_0$ , there are generally many different associated Lagrange interpolation pairs. They arise by different choices of the decompositions in Section 4.

**Remark 5.** Our construction shows that the minimal determining set  $\mathcal{M}$  of Theorem 7.1 is stable and 1-local in the sense of Definition 17.11 of [2]. But then Theorem 17.14 of [2] applies to show that  $\mathcal{S}_3^1(\Delta)$  approximates functions in the Sobolev spaces  $W_q^{m+1}(\Omega)$  with  $0 \leq m \leq 3$  and  $1 \leq q \leq \infty$  to order  $|\Delta_0|^{m+1}$ . In particular, with  $m = 3$ , this shows that the space has full approximation power in all of the  $q$ -norms.

**Remark 6.** With a little additional work, it can be shown that (8.1) in Theorem 8.1 holds with  $\ell = 9$ . This is based on the fact that if  $T_\xi$  is in class  $\mathcal{B}_1$ , then none of its faces is split, and so coefficients associated with a tube around one edge do not interact via  $C^1$  smoothness conditions with coefficients associated with a tube around its opposite edge. This implies that if  $T_\xi$  is in class  $\mathcal{B}_2$ , then (8.1) actually holds with  $\ell = 5$ , and induction then gives a worst case of  $\ell = 9$ . We note that for most tetrahedral partitions,  $\ell$  will be smaller than 9.

**Remark 7.** It should be emphasized that care is needed in the choice of the interior split points  $v_T$  in Step 2 of Algorithm 5.1 for constructing the refined partition  $\Delta$  from  $\Delta_0$ . To insure  $C^1$  smoothness across interior faces of  $\Delta_0$ , we need to choose the split point on the face to lie on the line joining the interior split points of the neighboring tetrahedra. To accomplish this, we have chosen the incenters of these tetrahedra, rather than the more natural barycenters, since then the line joining them always intersects the common face  $F$  at an interior point, see Lemma 16.24 in [2].

**Remark 8.** It was shown in Corollary 6.2 of [1] that a  $C^1$  cubic spline defined on a partial-Worsey–Farin split of a tetrahedron is actually  $C^2$  at the interior split point  $v_T$ . It follows that the splines in our space  $\mathcal{S}_3^1(\Delta)$  enjoy this property at all such vertices of  $\Delta$ .

**Remark 9.** Given a set of  $n$  vertices  $\mathcal{V}$ , an associated initial triangulation  $\Delta_0$  can be computed with standard algorithms. Once we have  $\Delta_0$ , the operation count for constructing our Lagrange interpolation pair is  $\mathcal{O}(n)$ . Finding the coefficients of an interpolating spline is also of linear complexity in  $n$ .

**Remark 10.** In step 2 of Algorithm 5.1 we apply a partial-Worsey–Farin split to  $T$  whenever the number of CT-split faces  $m$  is positive. We could have also applied a 0-th order split (also called the Alfeld split) in the case when  $m = 0$ , but this would lead to a refinement  $\Delta$  with more tetrahedra, so instead we do not split the tetrahedron at all when  $m = 0$ .

**Remark 11.** Trivariate local Lagrange interpolation methods are useful for the construction and reconstruction of volumetric models, as well as for scattered data fitting problems. A major

advantage is that they only require function values, but no derivatives. For example, the method here could be used in a two-stage process, where in the first stage we construct a  $C^0$  linear spline that interpolates at the vertices of a very fine tetrahedral partition. In the second stage, we construct a  $C^1$  cubic spline on a coarser tetrahedral subpartition  $\Delta$ , where the data needed for interpolation are taken directly from the linear spline. Because of the optimal approximation order of the interpolation methods, it is natural to choose the partitions such that  $|\Delta_1|^2$  is about  $|\Delta_0|^4$ . For numerical results based on this idea in the bivariate setting, see [9].

**Remark 12.** As shown in Theorem 17.17 of [2], there is a natural stable local basis  $\{\psi_\xi\}_{\xi \in \mathcal{M}}$  for  $\mathcal{S}_3^1(\Delta)$  associated with the minimal determining set  $\mathcal{M}$  of Theorem 7.1. Since  $\mathcal{M}$  is 1-local, each  $\psi_\xi$  has support at most  $\text{star}(T_\xi)$ , where  $T_\xi$  is a tetrahedron of  $\Delta_0$  containing  $\xi$ .

**Remark 13.** There is a different stable local basis  $\{\phi_\xi\}_{\xi \in \mathcal{N}}$  for  $\mathcal{S}_3^1(\Delta)$  associated with the nodal minimal determining set  $\mathcal{N}$  of Theorem 7.2. Since  $\mathcal{N}$  is just the set of point-evaluation functionals at the points of  $P$ , the  $\phi_\xi$  are Lagrange basis splines in the sense that  $\phi_\xi(\eta) = \delta_{\xi,\eta}$  for all  $\xi, \eta \in P$ . Theorem 8.1 implies that each  $\phi_\xi$  has support at most  $\text{star}^{10}(T_\xi)$ , where  $T_\xi$  is a tetrahedron of  $\Delta_0$  containing  $\xi$ . For typical tetrahedral partitions, most if not all of these basis functions will have much smaller supports.

**Remark 14.** In the theory of trivariate splines, nodal determining sets are usually constructed using point-evaluation of both function values and derivatives. Here we have restricted ourselves to the use of function values only in order to get a Lagrange interpolation method rather than a Hermite interpolation method.

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