The Integral: An Easy Approach after Kurzweil and Henstock.
By Lee Peng Yee and Rudolf Výborný.
Cambridge University Press, 2000, xii + 311 pp., $39.95 softcover.

Reviewed by **J. Alan Alewine** and **Eric Schechter**.

A simple definition. Riemann's integral of 1867 can be summarized as

$$\int f(t)dt = \lim \sum f(\tau_i)(t_i - t_{i-1}).$$

This summary conceals some of the complexity—for example, the limit is of a net, not a sequence—but it displays what we wish to emphasize: The integral is formed by combining the values $f(\tau_i)$ in a very direct fashion.

The values of f are used less directly in Lebesgue's integral (1902), which can be described as $\lim_{n\to\infty} \int_a^b g_n(t)dt$. The approximating functions g_n must be chosen carefully, using deep, abstract notions of measure theory. Simpler definitions are possible—for example, functional analysts might consider the metric completion of $C[0,1]$ using the L^1 norm—but such a definition does not give us easy access to the Lebesgue integral's simple and powerful properties such as the Monotone Convergence Theorem. We generally think in terms of those simple properties, rather than the various complicated definitions, when we actually use the Lebesgue integral.

The KH integral (also known as the gauge integral, the generalized Riemann integral, etc.) was discovered or invented independently by Kurzweil and Henstock in the 1950's; it has attracted growing interest in recent years. It offers the best of both worlds: a powerful integral with a simple definition. In fact, its definition is nearly identical to that of the Riemann integral, as we now show. For any *tagged partition*

$$P: \qquad a = t_0 < t_1 < t_2 < \cdots < t_n = b; \qquad \tau_i \in [t_{i-1}, t_i]$$

of the interval $[a,b]$, let us abbreviate $f(P) = \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1})$. For any given function δ on $[a,b]$, we say that P is *δ-fine* if $t_i - t_{i-1} < \delta(\tau_i)$ for all i. Finally, a number v is the *Riemann integral* (respectively, the *KH integral*) of a function $f : [a,b] \to \mathbb{R}$ if

> for each constant $\varepsilon > 0$ there exists a constant $\delta > 0$ (respectively, a function $\delta : [a,b] \to (0, +\infty)$) such that whenever P is a δ-fine tagged partition of $[a,b]$, then $|v - f(P)| < \varepsilon$.

We emphasize that the function $\delta(\tau)$, though positive at each τ, need not be bounded below by a positive constant. In applications we generally take $\delta(\tau)$ to be extra small at locations τ where f behaves erratically, so that those locations have less effect on the summation $\sum f(\tau_i)(t_i - t_{i-1})$. Think of approximating the region under a curve as a union of thin rectangles, as in a calculus course: The Riemann integral simply requires that all the rectangles be narrower than a certain constant width, but the KH integral uses more sophisticated spacing.

For example, let $1_{\mathbb{Q}}$ denote the characteristic function of the rationals. This function is discontinuous everywhere; it is a standard example of a bounded function that is not Riemann integrable on $[0,1]$. Nevertheless, the KH integral $\int_0^1 1_{\mathbb{Q}}(t)dt$ exists and equals zero. Indeed, let $p_1, p_2, p_3, \ldots$ be any enumeration of the rationals, and let

$$\delta(\tau) \quad = \quad \begin{cases} 2^{-j-1}\varepsilon & \text{if} \quad \tau = p_j, \\ \quad 1 & \text{if} \quad \tau \notin \mathbb{Q}; \end{cases}$$

the proof follows easily. It is less elementary to produce a bounded function that is not KH integrable on $[0,1]$; that requires the Axiom of Choice.

For a second example, consider the function $f(t)$ equal to $t^{-1}\sin(t^{-2})$ on $(0,1]$, and vanishing at $t=0$. This function is neither Riemann nor Lebesgue integrable, but it is KH integrable; that can be shown using

$$\delta(\tau) \quad = \quad \begin{cases} \sqrt{\varepsilon} & \text{if } \tau = 0, \\ \min\{\tau/2,\ \varepsilon\tau^4/24\} & \text{if } 0 < \tau \leq 1. \end{cases}$$

Finding and working with such functions $\delta(\tau)$ may require skill and effort, but it does not require a deep, abstract theory.

A more general integral. Generalizing the constant δ to a function $\delta(\tau)$ obviously yields a wider class of integrands, but it is surprising just how much wider. It turns out that

$$\left\{ \begin{array}{l} \text{Riemann} \\ \text{integrable} \\ \text{functions} \end{array} \right\} \subsetneqq \left\{ \begin{array}{l} \text{Lebesgue} \\ \text{integrable} \\ \text{functions} \end{array} \right\} \subsetneqq \left\{ \begin{array}{l} \text{KH} \\ \text{integrable} \\ \text{functions} \end{array} \right\}.$$

The classes of KH integrable functions and Lebesgue integrable functions are closely related. Indeed, it can be shown that

(A) a function f is Lebesgue integrable if and only if both f and $|f|$ are KH integrable.

If f is KH integrable on $[a,b]$, then f is also KH integrable on every subinterval, but not necessarily on every measurable subset. (We say that f is KH integrable on $S \subseteq [a,b]$ if $f1_S$ is KH integrable on $[a,b]$, where 1_S is the characteristic function of S.) In fact,

(B) $\quad f$ is Lebesgue integrable if and only if f is KH integrable on every measurable subset of $[a,b]$.

For motivation, we suggest an analogy: KH integrable functions are like convergent series, whereas Lebesgue integrable functions are like *absolutely* convergent series (motivation for (A)); any subseries of an absolutely convergent series is also convergent (motivation for (B)).

Kurzweil was led to Riemann-like integrals by his investigations of differential equations $u'(t) = f(t, u(t))$. To see the connection between the two topics, note that some initial value problems can be restated as integral equations:

$$u(t) = u_0 + \int_0^t f(s, u(s))ds.$$

For some kinds of applications the integrals $\int_S f(s, u(s))ds$, over arbitrary measurable sets S, are not relevant; we need to consider only integrals $\int_a^b f(s, u(s))ds$ over intervals. Thus we may use KH integrals and KH techniques; see [6] or [14].

The KH integral brings wider applicability to differential equations, to Fourier analysis (mentioned later in this review), and to some other branches of analysis, because the KH integral can integrate more functions. However, $t^{-1}\sin(t^{-2})$ is typical of the new functions: They are erratic, more often cited for pathological counterexamples than for useful applications. Thus, the chief benefit of the KH theory may not be its wider applicability, but rather its concrete and elementary formulations of ideas that we already know in the Lebesgue theory. For example, a set $S \subseteq [a,b]$ is Lebesgue measurable if and only if its characteristic function 1_S is KH-integrable, in which case its Lebesgue measure is $\int_a^b 1_S(t)dt$.

For brevity, we have defined the KH integral only on a compact interval, but the basic ideas extend easily to bounded or unbounded regions in finite-dimensional Euclidean space, and to measures other than Lebesgue measure. For example, it is shown in Theorem 4.1.1 of [12] or Theorem 24.35 of [13] that the measures μ on the Borel subsets of an interval $[a,b]$ can be expressed as KH-Stieltjes integrals:

$$\mu_\varphi(S) \;=\; \int_a^b 1_S(t)d\varphi(t) \;=\; \lim_P \sum_{i=1}^n 1_S(\tau_i)\,[\varphi(t_i) - \varphi(t_{i-1})]$$

for a suitable function φ of bounded variation.

The preceding characterization of measures makes good use of the special properties of intervals in $\mathbb{R}$, admittedly a rather special setting. The KH integral can be extended to a more abstract and general setting of "division spaces" (see, for example, [9]), and the resulting theory is applied to the Wiener and Feynman integrals in [11], but this theory is more complicated. Still, it is a continuation of the ideas of the Riemann integral; we do not have to start over with a whole new approach involving σ-algebras. This may make the KH approach attractive to scientists and engineers.

Teaching the KH integral. Where, if at all, does the KH integral belong in our standard analysis curriculum? Bartle [1] suggests that it could replace the Lebesgue integral, while Gordon [8] says that it should not. Perhaps the difference in their opinions reflects different audiences. For example, the KH approach permits us to avoid, or at least postpone, the notion of σ-algebras. Such abstract notions are insightful and valuable for a mathematically advanced audience, but may be less accessible for undergraduates or for scientists and engineers.

At many American universities today, the standard analysis curriculum is in three stages:

1. freshman calculus, introducing the Riemann integral but omitting most proofs;

2. an advanced undergraduate course, typically titled "Introduction to Real Analysis", which includes (among other things) those omitted proofs; and

3. a graduate course on the Lebesgue integral.

We consider each of these stages.

(1) *Freshman calculus.* The KH integral simplifies and strengthens some classical results. For example, one half of the Fundamental Theorem of Calculus says

> If $G : [a, b] \to \mathbb{R}$ is differentiable [and G' is continuous], then G' is integrable and $\int_a^b G'(t)dt = G(b) - G(a)$.

The continuity assumption, or some other assumption like it, is needed for Riemann integrability; that assumption can be omitted entirely if we use the KH integral.

Another improvement on calculus is Hake's Theorem (Theorem 2.8.3 in the Lee-Výborný book):

> The KH integral $\int_a^b f$ exists if and only if $\lim_{r \downarrow a} \int_r^b f$ exists, in which case they are equal.

In effect, this says that we do not need to define an "improper" KH integral, analogous to the improper Riemann integral of calculus; any improper KH integral is also a proper KH integral. (Hake's Theorem is not valid for Lebesgue integrals: $\lim_{r \downarrow 0} \int_r^1 t^{-1} \sin(t^{-2})dt = 0.312\ldots$, but $\int_0^1 t^{-1} \sin(t^{-2})dt$ does not exist as a Lebesgue integral.)

For the sake of improvements such as these, new calculus teachers might be tempted to introduce the KH integral, but experienced teachers may be less optimistic about their students' abilities. Most of our calculus students can learn computations, but lack the mathematical maturity for proofs. For them, the Fundamental Theorem of Calculus is simply the equation $\int_a^b G'(t)dt = G(b) - G(a)$, and discussions about the integrability of $1_{\mathbb{Q}}$ are just gibberish. The value of the calculus course to these students does not lie chiefly in its proofs.

(2) *The advanced undergraduate course.* A general goal of this course is to enable students to understand analysis proofs. That is accomplished more specifically by practicing techniques of ε-δ, convergent sequences, limsups, and the like. As it happens, those techniques are also the main tools in the KH theory. Thus, adding the KH integral to the advanced undergraduate course would require only small alterations in that course.

At least two textbooks are already available that support such a course: DePree and Swartz [5] and the third edition of Bartle and Sherbert [4]. Each of these books covers the material of a conventional undergraduate analysis course, but then adds a chapter on the KH integral. The additional chapter appears late enough in the book so that it is not crucial; thus teachers who are hesitant about the KH integral can adopt it at their own pace.

(3) *The graduate course on Lebesgue integration.* This course traditionally commits a large amount of time to plowing through the terminology and

lemmas of set theory, σ-algebras, measurable functions, inner and outer measure, etc., eventually arriving at $L^1[0,1]$ and related theory. Introducing the KH theory into this course, either in addition to or instead of the Lebesgue theory, would change the course substantially, for the two theories are based on very different methods—for example, ε-δ inequalities versus σ-algebras.

Nevertheless, the *results* of the two theories are closely related, as we noted in (A) and (B). Consequently, results from either theory can be used as tools in the development of the other theory. Gordon's book [7] develops the Lebesgue theory first, and then uses some of its results in developing the KH theory. The first few chapters of this book would fit the traditional graduate course with few alterations; the book lies somewhere between textbook and research monograph.

The book under review. The book of Lee and Výborný goes in the other direction, developing the KH theory first and then using it to develop the Lebesgue theory. It is written to be used on several different levels, as explained in its preface: Chapters 1–3 and 6–7 might replace the standard courses that we have called "Stage 2" and "Stage 3". Even Chapter 1, on the Riemann integral, can be read on different levels, according as one includes or omits the material marked "optional". Chapters 4–5 are more advanced reading intended for specialists in integration theory.

The book is rich in examples and applications. We were fascinated by Example 1.4.5, a construction over a page long producing an everywhere differentiable function whose derivative is bounded but not Riemann integrable. The applications include things such as Corollary 7.5.3, which states that the Fourier series of a KH integrable function is Abel summable almost everywhere to that function. Also included are some of the latest discoveries. For instance, in 1993 Výborný formulated the notion of "negligible variation", which Bartle used in 1997 to characterize indefinite KH-integrals; that characterization is Theorem 3.9.1.

The book covers not only the KH integral but also an assortment of "other" integrals—notably, the Denjoy integral (1912), the Perron integral (1914), and the SL integral (formulated by Lee and Výborný in 1993 based on the strong Lusin condition). Actually, these "other" integrals all turn out to be equivalent to the KH integral—that is, they yield the same classes of integrable functions and the same numerical values for the integrals. Though the KH definition is already quite enough for beginners, the alternate definitions yield additional insights that may be helpful in certain kinds of research. For example, the value of $\int_a^b f(t)dt$ is not affected if we alter f on a set of measure zero; that fact follows only indirectly from the KH definition, but very directly from the Denjoy or SL definition. Thus either of those definitions, and its associated ideas and techniques, might be useful in investigations that involve discarding null sets.

One more topic that deserves mention is that of convergence theorems—that is, sufficient conditions for $\int_a^b f_n \to \int_a^b f$. The Dominated Convergence Theorem for Lebesgue integrals is a special case of the Vitali Convergence Theorem, which can be generalized to the KH setting as follows (Theorem 3.7.5):

> *Equiintegrability Theorem.* Suppose (f_n) is a sequence of KH integrable functions on $[a, b]$, convergent pointwise to some function f. Suppose that (f_n) is *KH equiintegrable*, in the sense that
>
> > for each $\varepsilon > 0$ there exists a function $\delta : [a, b] \to (0, +\infty)$, such that whenever P is a δ-fine tagged partition, then $\sup_n \left| f_n(P) - \int_a^b f_n \right| < \varepsilon$.
>
> Then f is KH integrable and $\int_a^b f_n \to \int_a^b f$.

That theorem is probably general enough for a first course on integration, and for anyone except a specialist in integration theory, but some deeper results with weaker but more complicated hypotheses are included in the advanced chapters.

The theory of the KH integral has not yet settled down to a classical formulation, but already it is worthy of a place in our standard curriculum. The book of Lee and Výborný serves well as an introduction and reference for anyone interested in this topic. Other good sources are Gordon [7], which covers much of the same material from a somewhat different perspective, and the forthcoming book of Bartle [3].

References

[1] R. G. Bartle, Return to the Riemann integral, this MONTHLY **103** (1996) 625–632.

[2] R. G. Bartle, The concept of “negligible variation”, *Real Anal. Exch.* **23** (1997) 47–48.

[3] R. G. Bartle, *A Modern Theory of Integration*, Graduate Studies in Mathematics, American Mathematical Society, Providence, 2001.

[4] R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis* (3rd ed.), John Wiley, New York, 2000.

[5] J. DePree and C. Swartz, *Introduction to Real Analysis*, John Wiley, New York, 1988.

[6] V.V. Filippov, *Basic Topological Structures of Ordinary Differential Equations*, Kluwer, Dordrecht, 1998.

[7] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics 4, American Mathematical Society, Providence, 1994.

[8] R. A. Gordon, Is nonabsolute integration worth doing?, *Real Anal. Exch.* **22** (1996) 23–33.

[9] Lee P. Y., *Lanzhou Lectures on Henstock Integration*, Series in Real Analysis 2, World Scientific, Singapore, 1989.

[10] Lee P. Y. and R. Výborný, Kurzweil-Henstock integration and the strong Lusin condition, *Boll. Un. Mat. Ital. B* (7) **7** (1993) 761-773.

[11] P. Muldowney, *A General Theory of Integration in Function Spaces*, Pitman Research Notes in Mathematics 153, Longman, Essex, and John Wiley, New York, 1987.

[12] W. F. Pfeffer, *The Riemann Approach to Integration: Local Geometric Theory*, Cambridge University Press, Cambridge, 1993.

[13] E. Schechter, *Handbook of Analysis and its Foundations*, Academic Press, San Diego, 1997.

[14] S. Schwabik, *Generalized Ordinary Differential Equations*, Series in Real Analysis 5, World Scientific, Singapore, 1992.

Vanderbilt University, Nashville, TN 37240
aalewine@math.vanderbilt.edu
schectex@math.vanderbilt.edu