Math 155a Test 5, Wednesday 5 December 2001, 4 pages, 50 points, 50 minutes.

Five students had perfect papers. The class average was 82.5 percent. **Answers** are below.

Reminder: Be sure to include "$+C$" in answers where appropriate. You will lose points for including too many C's or too few.

Page 1 problem 1. (6 points) $\int_1^{64} \frac{\sqrt{t} - \sqrt[3]{t}}{t^2} dt =$

Solution. The only problem here is to simplify before integrating. We have $\frac{\sqrt{t}-\sqrt[3]{t}}{t^2} = (t^{1/2} - t^{1/3})t^{-2} = t^{(1/2)-2} - t^{(1/3)-2} = t^{-3/2} - t^{-5/3}$. Thus the problem can be rewritten as $\int_1^{64} \left(t^{-3/2} - t^{-5/3}\right) dt$; that much is already worth 3 points. Rewriting now as $\left[-2t^{-1/2} + \frac{3}{2}t^{-2/3}\right]_1^{64}$ brings us up to 5 points. Finally, for full credit, that becomes

$$-2\left[\frac{1}{8} - 1\right] + \frac{3}{2}\left[\frac{1}{16} - 1\right] = \boxed{\frac{11}{32}} = \boxed{0.34375}.$$

About 2/3 of the class got this one right.

Page 1 problem 2. (6 points) $\int \frac{1 + \frac{1}{\sqrt{x}}}{\sqrt{x + 2\sqrt{x}}} dx =$

Solution. The idea is that $x + 2\sqrt{x}$ is the inside part of a composition $f(g(x))$ appearing in the integral, so it's natural to guess that we should substitute $u = x + 2\sqrt{x} = x + 2x^{1/2}$. Then $\frac{du}{dx} = 1 + \frac{1}{\sqrt{x}}$. Thus the given integral is equal to $\int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = 2u^{1/2} + C = \boxed{2\sqrt{x + 2\sqrt{x}} + C}$.

However, if you didn't guess that right away, there are other approaches to this problem. For instance, you might notice that most of the x's appearing in the integral appear in square root signs, so you might try to substitute $v = \sqrt{x}$. That yields $v^2 = x$ and $dx = 2vdv$. Then

$$\int \frac{1 + \frac{1}{\sqrt{x}}}{\sqrt{x + 2\sqrt{x}}} dx = \int \frac{1 + \frac{1}{v}}{\sqrt{v^2 + 2v}} \cdot 2vdv = \int \frac{2(v+1)}{\sqrt{v^2 + 2v}} dv.$$

That's perhaps better than what we started with, so we're making progress. Perhaps a second substitution will improve things still further. Now the inside of a composition is given by $u = v^2 + 2v$, so we make that substitution, with $du = 2(v+1)dv$. Thus the integral is $\int \frac{du}{\sqrt{u}}$, and now the problem is solved as in the first method described above.

Several students were off by a factor of 2, for which I deducted 1 point. A few students seemed to think that $\int \frac{u}{v}$ is equal to $\frac{\int u}{\int v}$, but it is **not**, and in fact I explicitly mentioned that it isn't, in class earlier this semester.

A little over half the class got this one right.

Page 2 —

Hint for both the problems on this page: Please look for the easy way to do these two problems; don't make them difficult.

(The problems on this page were more conceptual than computational. The problems were easy if you knew the concepts, and essentially impossible if you didn't. Don't expect much partial credit on problems like this.)

Page 2 problem 1. (5 points) $\int_{-3}^{3} \left(x^3 + \frac{\sin x}{\sqrt{x^2+4}} \right) dx =$

Solution. The integrand is an odd function, and the interval is symmetric, so the answer is $\boxed{0}$.

To see that the integrand is odd, note that x^2+4 is even, and any composition of the form $f \circ g$ (where g is even) is even. Also, $\sin x$ is odd, and any quotient of an odd function and an even function is odd. Finally, x^3 is odd, and the sum of two odd functions is odd.

Almost everyone got this right. I couldn't make any sense at all out of the few wrong answers, so I gave no partial credit.

Page 2 problem 2. (6 points) If $f(x) = \int_3^{\sin x} \sqrt{u^2+2}\, du$, find $f'(x)$.

Solution. We do *not* have a formula for $\int \sqrt{u^2+2}\, du$. And you won't find one even in somewhat more advanced calculus books, or in a table of integrals; the theory of $\int \sqrt{u^2+2} du$ is more advanced. If you began this problem by evaluating $\int_3^{\sin x} \sqrt{u^2+2}\, du$, then you've probably begun wrong.

Here's the easy method: we have $f(x) = g(\sin x)$, where $g(w) = \int_3^{w} \sqrt{u^2+2}\, du$. Then $g'(w) = \sqrt{w^2+2}$ by the Fundamental Theorem of Calculus. By the Chain Rule, $f'(x) = g'(\sin x) \cdot \cos x = \boxed{\sqrt{\sin^2 x + 2} \cdot \cos x} = \boxed{(\cos x)\sqrt{2+\sin^2 x}}$.

Some common errors came from not including the term from the chain rule, or from not carrying out a needed substitution.

erroneous answer	points awarded
$-(\cos x)\sqrt{\sin^2 x+2}$	5
$(\cos x)\sqrt{\sin^2 x+2}\,dx$	5
$\sqrt{\sin^2 x+2}$	2
$\sqrt{\sin^2 x+2}$	2
$\sqrt{\sin^2 x+2}+C$	1
$\sqrt{x^2+2}$	1

Most other wrong answers got 1 point or 0 points.

Page 3 first problem. (5 points) $\int_5^{13} \sqrt{2x-1}dx =$

Solution. Substitute $u = 2x - 1$. Then $du = 2dx$, so $dx = \frac{1}{2}du$. Also, when x goes from 5 to 13, then $2x - 1$ goes from 9 to 25. Thus the given integral is equal to $\frac{1}{2}\int_9^{25} u^{1/2}du = \left[\frac{1}{3}u^{3/2}\right]_9^{25} = \frac{1}{3}(125-27) = \boxed{\dfrac{98}{3}} = \boxed{32.666\ldots} = \boxed{32\frac{2}{3}}$.

Common errors:

Several people misplaced a factor of 2 — presumably in dealing with $du = 2dx$. This leads to an answer of $\frac{196}{3} = 65\frac{1}{3}$ or else $\frac{392}{3} = 130\frac{2}{3}$; for either of those answers I gave 3 points.

I gave 2 points for reducing the problem to $\frac{1}{2}\int_9^{25}\sqrt{u}du$, even if subsequent work was done incorrectly; more points were possible if that step was followed by at least some correct reasoning. Note that $\frac{1}{2}\int_5^{13}\sqrt{u}du$ is **NOT** correct; I discussed that type of error in class.

A few people went correctly as far as $\frac{1}{2}\int_9^{25}\sqrt{u}\,du$, but then replaced $\int\sqrt{u}du$ with $\sqrt{u}+C$ instead of $\frac{2}{3}u^{3/2}+C$. That evaluation leads to $\left[\frac{1}{2}\sqrt{u}\right]_9^{25} = \frac{1}{2}(5-3) = 1$, for which I gave 3 points. That can also be written as $\left[\frac{1}{2}\sqrt{2x-1}\right]_5^{13} = \frac{1}{2}(5-3) = 1$. Some students followed this method but also made arithmetic errors, yielding an answer different from 1; for that I gave 2 points (if the method was clear enough).

Page 3 second problem. (8 points) Find the area between the curves $y = x^3 - x$ and $y = 1 - x^4$.

Solution. First find the intersections of those curves. When $x^3 - x = 1 - x^4$, then $x^4+x^3-x-1 = 0$. Factor that to obtain $(x+1)(x^3-1) = 0$, or $(x+1)(x-1)(x^2+x+1) = 0$. Thus the only roots are $x = \pm 1$.

On the interval $(-1, 1)$ we have $1 - x^4 > x^3 - x$. Then the height of the tall thin rectangles is $(1-x^4) - (x^3-x) = -x^4 - x^3 + x + 1$.

Finally, the area is $\int_{-1}^{1}(-x^4-x^3+x+1)\,dx = \int_{-1}^{1}(-x^4+1)dx + \int_{-1}^{1}(-x^3+x)dx$. The last integral is 0, since $-x^3+x$ is an odd function. Since $-x^4+1$ is an even function, the other integral simplifies slightly: $\int_{-1}^{1}(-x^4+1)dx = 2\int_0^1(-x^4+1)dx = 2\left[\frac{-1}{5}x^5+x\right]_0^1 = 2\left(\frac{-1}{5}+1\right) = \boxed{\frac{8}{5}} = \boxed{1.6} = \boxed{1\frac{3}{5}}$.

Common errors: A few students treated the function $1+x-x^3-x^4$ as though it were an even function — i.e., they thought that $\int_{-1}^{1}(1+x-x^3-x^4)\,dx$ is equal to $2\int_0^1(1+x-x^3-x^4)\,dx$. That leads to an answer of 2.1, for which I gave 5 points.

Page 4 first problem. (6 points) $\int_0^2 x^2\cos(x^3)dx =$

Solution. Substitute $u = x^3$. Then $du = 3x^2dx$, so $x^2dx = \frac{1}{3}du$. When x goes from 0 to 2, then u goes from 0 to 8. Hence the integral is $\frac{1}{3}\int_0^8\cos(u)du = \left[\frac{1}{3}\sin(u)\right]_{u=0}^{u=8} = \boxed{\frac{1}{3}\sin(8)} \approx \boxed{0.3297860822078\ldots}$. That could also be written as $\boxed{0.3298}$ or even as $\boxed{0.33}$, but it should *not* be written as 1/3.

About 2/3 the class got this right.

There were a wide variety of errors. A few students didn't know that the antiderivative of $\cos(u)$ is $\sin(u)+C$; by now you really ought to know that. Also, some students apparently used degrees on their calculators instead of radians, arriving at answers like $\frac{1}{3}\sin(8^o)$, even though I warned against this in class.

Note that if you really must insist on using degrees instead of radians (e.g., if you think that the math teachers are full of nonsense, and you want to follow the approach of the engineering teachers), then the antiderivative of $\cos(u)$ is *not* equal to $\sin(u)+C$, and so $\frac{1}{3}\sin(8^o)$ still is not the right answer. Just in case you've forgotten, here is how that computation goes (I discussed this in class some weeks ago): π radians equals 180 degrees, so 1 degree is equal to $\frac{\pi}{180}$ radians. So when you use degrees,

$$\frac{d}{dx}\sin(x^o) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{180}x\text{ rad}\right)\right] = \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\text{ rad}\right) = \frac{\pi}{180}\cos(x^o)$$

(by the chain rule). Then the antiderivative of $\cos(x^o)$ is $\frac{180}{\pi}\sin(x^o)$, or about $57.3\sin(x^o)$. That's a lot more complicated, which is why mathematicians prefer to use radians. If you really want to interpret all your angles in degrees, then the answer to this problem would be $\frac{1}{3}\cdot\frac{180}{\pi}\cdot\sin(8^o)$, which is about 2.658, but no students gave that answer.

I deducted 2 points for using degrees instead of radians; 2 points for a sign error; 2 points for arriving at $\frac{1}{2}$ instead of $\frac{1}{3}$ for the coefficient; 2 points for arriving at cosine instead of sine. If you combined two or more of those errors, I combined the penalties. Here are some of the most common erroneous answers, and the numbers of points that I gave for those answers:

$\frac{-1}{3}\sin(8) = -0.3298$	$4pts$	$\frac{-1}{2}\sin(8) = -0.49468$	$2pts$
$\frac{1}{3}\sin(8^o) = 0.04639$	$4pts$	$\frac{-1}{3}\sin(2) = -0.303$	$2pts$
$\frac{1}{2}\sin(8) = 0.49468$	$4pts$	$\frac{\cos(8)-1}{3} = -0.3818$	$2pts$
$\frac{\sin(8)-1}{3} = -0.003547$	$4pts$	$\frac{1}{2}\sin(8^o) = 0.06949$	$2pts$
$\frac{1}{3}\sin(2) = 0.303$	$4pts$		

Page 4, 2nd problem. (8 points) Find $\int_0^3 |x^2-1|dx$. (*Hint*; First sketch a graph of $y = x^2-1$.)

Solution. The parabola $y = x^2 - 1$ crosses the x-axis at $x = -1$ and $x = 1$. Thus $x^2 - 1$ is zero at ± 1, negative when $-1 < x < 1$, and positive elsewhere. Therefore $\int_0^3 |x^2 - 1|dx = \int_0^1 (1 - x^2)dx + \int_1^3 (x^2 - 1)dx = \left[x - \frac{1}{3}x^3\right]_0^1 + \left[\frac{1}{3}x^3 - x\right]_1^3 = (1 - \frac{1}{3}) + \frac{1}{3}(27 - 1) - (3 - 1) = \frac{2}{3} + \frac{26}{3} - \frac{6}{3} = \boxed{\frac{22}{3}} = \boxed{7.333\ldots} = \boxed{7\frac{1}{3}}$. About 2/3 of the class got this one right.