TEST 4. The high score was 48 points out of 50, achieved by one person. The class average was 74.88 percent, which was a big disappointment to me. The problem that the most students found difficult was the airplane problem; apparently we will need more review of word problems before the semester ends. — **ANSWERS** are below.

Reference section. You may find it helpful to refer to these.

$$\sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1) = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n$$
$$\sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^2(n+1)^2 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2$$

(7 points) $\displaystyle\sum_{j=1}^{100}(4j^3 + 2j - 1) =$

Solution. $= 4\left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right) + 2\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) - n = n^4 + 2n^3 + 2n^2$ evaluated at $n = 100$, yielding $100{,}000{,}000 + 2{,}000{,}000 + 20{,}000 = \boxed{102{,}020{,}000}$.

Only about half the class got full credit on this one. I deducted only 1 point if the student transformed this correctly into numbers — such as

$$102{,}010{,}000 + 10{,}100 - 100$$

— and then made an arithmetic error in the final step. Most errors were more substantial than that. I gave 5 points for the common erroneous answer of

$$\left[\sum_{j=1}^{100}(4j^3 + 2j)\right] - 1 = 102{,}010{,}000 + 10{,}100 - 1 = 102{,}020{,}099.$$

I gave 4 points for 102,029,900.

(5 points) $\displaystyle\sum_{j=3}^{6}\frac{2}{j} =$

Solution. This just requires knowing the definition. The answer is $\frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} = \frac{20+15+12+10}{30} = \frac{57}{30}$. I gave 4 points for stopping there, or for an answer of $\frac{38}{20}$ or $\frac{114}{60}$. For full credit, you need to simplify, which means an answer of $\boxed{\dfrac{19}{10}} = \boxed{1.9}$.

(7 points) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + \frac{5i}{n} - 3\right] =$

Solution. One method is to first hold n fixed, and compute

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + \frac{5i}{n} - 3\right] \quad &= \quad \frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{5}{n^2}\sum_{i=1}^{n} i - \frac{3}{n}\sum_{i=1}^{n} 1 \\ &= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{5}{n^2}\cdot\frac{n(n+1)}{2} - \frac{3}{n}\cdot n \\ &= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{5}{2}\left(1+\frac{1}{n}\right) - 3\end{aligned}$$

which, when $n \to \infty$, tends to a limit of $\frac{2}{6} + \frac{5}{2} - 3 = \frac{2+15-18}{6} = \boxed{\frac{-1}{6}}$.

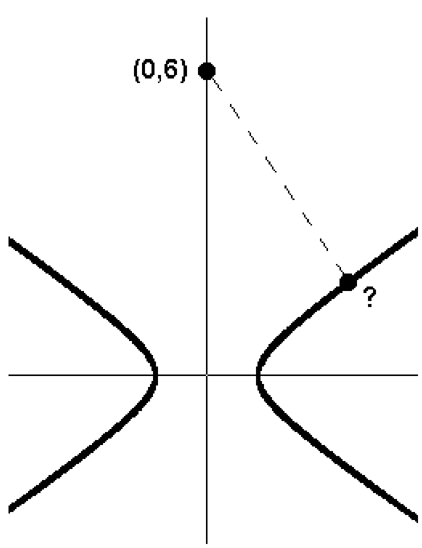
Another method is to recognize that the summation is just one form of the definition of the integral $\int_0^1 [x^2 + 5x - 3]\,dx$. That can be evaluated using antiderivatives; we obtain $\left[\frac{1}{3}x^3 + \frac{5}{2}x^2 - 3x\right]_0^1$, which simplifies to $\frac{1}{3} + \frac{5}{2} - 3 = \frac{-1}{6}$.

A little over half the class got this right. The most common error was to instead compute

$$\frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{5}{n^2}\sum_{i=1}^{n} i - \frac{3}{n}\cdot 1$$

— note that the last summation is missing, so an expression totalling n has been replaced by a 1. The resulting answer is then too high by an amount of $\frac{3}{n}(n-1)$, which tends to a limit of 3. The resulting answer is $\frac{-1}{6} + 3 = \frac{17}{6}$ (which is wrong).

(8 points) Find the point on the hyperbola $x^2 - 2y^2 = 1$ that lies in the right half-plane and that is closest to the point $(0,6)$. (See diagram.)



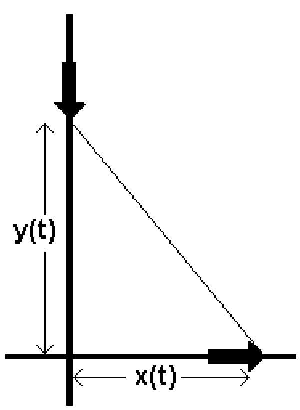
Solution. We wish to find (x, y) to minimize $s = \sqrt{(x-0)^2 + (y-6)^2}$, or equivalently to minimize $q = s^2 = x^2 + (y-6)^2$. Since $x^2 = 2y^2 + 1$, we can rewrite $q(y) = 2y^2 + 1 + (y-6)^2 = 3y^2 - 12y + 37$. Then $q'(y) = 6y - 12$ and $q''(y) = 6$. Since $q''(y) > 0$, we know that $q(y)$ has a unique minimum, which occurs where $q'(y) = 0$. There, we have $6y = 12$, so $y = 2$. Then $x = \pm\sqrt{2y^2 + 1}$, and we want it in the right half-plane, so that $\pm$ should be $+$. Thus $x = \sqrt{2 \cdot 2^2 + 1} = \sqrt{9} = 3$. Putting the coordinates together, $(x, y) = \boxed{(3,\ 2)}$.

About 3/4 of the class got full credit on this one.

A few students did this the hard way, or tried to: It's easier to compute everything in terms of y, but some students computed in terms of y instead. Here are the resulting computations, carried out correctly: We wish to find (x, y) to minimize $s = \sqrt{(x-0)^2 + (y-6)^2}$, or equivalently to minimize $q = s^2 = x^2 + (y-6)^2 = x^2 + y^2 - 12y + 36$. Since $y = \sqrt{(x^2-1)/2}$, we can rewrite $q(x) = x^2 + \frac{x^2-1}{2} - 12\sqrt{(x^2-1)/2} + 36 = \frac{3}{2}x^2 + 36 - \frac{1}{2} - 6\sqrt{2}\sqrt{x^2-1}$. Differentiating yields $q'(x) = 3x - 6\sqrt{2} \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot (2x) = 3x\left(1 - \frac{2\sqrt{2}}{\sqrt{x^2-1}}\right)$. This vanishes when $x = 0$ (an unfeasible answer) or when $1 - \frac{2\sqrt{2}}{\sqrt{x^2-1}} = 0$. That last equation simplifies to $\sqrt{x^2-1} = 2\sqrt{2}$, or $x^2 - 1 = 8$, or $x^2 = 9$, or $x = 3$. Finally, $y = \sqrt{(x^2-1)/2} = 2$.

(8 points) An airplane leaves Nashville at 1 pm, flying eastward (away from Nashville) at 300 miles per hour. At that same moment, another airplane is 400 miles north of Nashville, and is flying south (i.e., toward Nashville) at 400 miles per hour; thus it will arrive at 2 pm. Calculations show that the two planes are getting closer together for a while, and then moving apart after that. At what time are the two planes closest together, and how far apart are they at that moment?

Hint: Let t represent the amount of time that has passed (in hours) since 1 pm. Let $x(t)$ denote the distance between the eastbound plane and Nashville, in hundreds of miles. Let $y(t)$ denote the distance between the southbound plane and Nashville, in hundreds of miles. Then what?



Solution. We have $x(t) = 3t$ and $y(t) = 4(1-t)$. Then the distance between the two planes is $w = \sqrt{x^2+y^2}$, which we wish to minimize. That is minimized at the same instant as $q(t) = w^2 = x^2+y^2 = 9t^2+16(1-t)^2 = 9t^2+16-32t+16t^2 = 25t^2-32t+16$. Then $q'(t) = 50t-32$ and $q''(t) = 50$. Since $q''(t) > 0$, we see that $q(t)$ has a unique minimum, which occurs when $q'(t) = 0$. Then $50t-32 = 0$, so $t = \frac{32}{50} = \frac{16}{25} = \frac{64}{100} = \boxed{\text{0.64 hours after 1 pm}}$, or $\boxed{\text{about 1:38 pm}}$. (More precisely, 24 seconds after 1:38 pm.) At that time, we have $x = \frac{48}{25} = 4 \cdot \frac{12}{25}$ and $y = 4 \cdot \frac{9}{25} = \frac{36}{25} = 3 \cdot \frac{12}{25}$, hence $w = \frac{12}{25} \cdot \sqrt{3^2+4^2} = \frac{12 \cdot 5}{25} = \frac{12}{5} = \frac{24}{10} = 2.4$ hundreds of miles; that is, $\boxed{\text{240 miles}}$.

Here is another, slightly different way to do the problem. Compute $x(t) = 3t$ and $y(t) = 4(1-t)$, as above, and $q = x^2+y^2$. We're interested in the point where $q' = 0$. We have $q' = 2xx'+2yy' = 2(3t)(3)+2(4(1-t))(-4)$. Set that equal to 0 and simplify. Dividing out 2 yields $9t-16(1-t) = 0$, or $25t = 16$, so $t = 16/25$. Then find $\sqrt{q(16/25)}$, etc.

Only 4 students got full credit for this problem. Most other students got rather little partial credit.

Partial credit: Nearly everyone noticed that $x^2+y^2 =$something; I gave 1 point for that. A consequence is that $2xx'+2yy' =$something; I gave 3 points for that (total — i.e., that includes the 1 point already mentioned). Relatively few students had the right idea about the right hand side of that equation — the something in the first equation should *not* be a constant.

Very few students noted that $x(t) = 3t$ and $y(t) = 4(1-t)$ (or, if you used miles instead of hundreds of miles, $x(t) = 300t$ and $y(t) = 400(1-t)$). I gave 4 points for those two equations. Far more students wrote $x(t) = 3t$ and $y(t) = 4t$, which was worth no points.

(5 points) If $f''(x) = \sin x$ and $f(0) = f'(0) = 0$, find $f(x)$.

Solution. Integrating yields $f'(x) = -\cos x + C_1$, and then integrating again yields $f(x) = -\sin x + C_1 x + C_2$. Now plug in $0 = f(0) = 0+0+C_2$ and $0 = f'(0) = -1+C_1$. Thus we obtain $C_1 = 1$ and $C_2 = 0$. Hence $\boxed{f(x) = -\sin x + x}$. Almost everyone got this right.

(5 points) $\int_0^{\pi/2} (2\sin x + x^3)\,dx =$

Solution. $= \left[-2\cos x + \frac{1}{4}x^4\right]_0^{\pi/2} = -2\cos\frac{\pi}{2} + 2\cos 0 + \frac{1}{4}\left(\frac{\pi}{2}\right)^4 - \frac{1}{4}(0)^4 = \boxed{2+\dfrac{\pi^4}{64}} \approx \boxed{3.522}$. About 2/3 of the class got this right.

The most common erroneous answer was $\frac{\pi^4}{64} \approx 1.522$, for which I gave only 2 points. That's because this answer arose from a conceptual error — either thinking that $\cos\frac{\pi}{2}$ is 1, or thinking that $\cos 0$ is 0, or assuming that $\int_0^{\pi/2} F'(x)dx = F(\pi/2)$.

I was more lenient with several errors that were mere arithmetic errors — e.g., copying numbers wrong from one step to the next.

(5 points) Let $y = \int_0^{\cos(x)} (t^2 - 1)dt$. Find $\dfrac{dy}{dx}$.

Solution. We have $y = \int_0^u (t^2 - 1)dt$ where $u = \cos(x)$. Then $\frac{dy}{du} = u^2 - 1 = \cos^2 x - 1 = -\sin^2 x$ and $\frac{du}{dx} = -\sin x$; hence $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \boxed{\sin^3 x}$. I reluctantly gave full credit also for $\boxed{-(\sin x)(\cos^2 x - 1)}$, but that answer really is not simplified enough. I gave 1 point for either of the factors of the answer — i.e., for $-\sin x$ or for $\cos^2 x - 1$ or for $u^2 - 1$.

Some students used a harder method that I had not anticipated: We know that $\int (t^2 - 1)dt = \frac{1}{3}t^3 - t + C$, hence $\int_a^b (t^2 - 1)dt = F(b) - F(a)$ where $F(t) = \frac{1}{3}t^3 - t$. In particular, $\int_0^{\cos x} (t^2 - 1)dt = F(\cos x) - F(0) = \frac{1}{3}\cos^3 x - \cos x$. Some students stopped there, and thought that's the answer. I gave 2 points for that answer. But the problem asks for the *derivative* of that function; it turns out to be $\sin^3 x$.

I gave 1 point for mentioning the chain rule, if nothing else was done right.