Test 3. The high score was 49 out of 50 points, achieved by one student. The average score in the class was about 75 percent. **Answers:**

(7 points) If $f(x) = \sin 2x$, find $f^{(8)}(x)$.

Solution. Compute:

$$\begin{aligned} f(x) &= \sin 2x \\ f'(x) &= 2\cos 2x \\ f''(x) &= -4\sin 2x \\ f^{(3)}(x) &= -8\cos 2x \\ f^{(4)}(x) &= 16\sin 2x \\ &\cdots \\ f^{(8)}(x) &= \boxed{2^8 \sin 2x} = \boxed{256\sin 2x}. \end{aligned}$$

(8 points) $\displaystyle\lim_{x\to\infty} \frac{5x+\sqrt{4x^2+5x}}{7x+3}$

Solution. $\displaystyle \lim_{x\to\infty} \frac{5x + x\sqrt{4+\frac{5}{x}}}{7x+3} = \lim_{x\to\infty} \frac{5+\sqrt{4+\frac{5}{x}}}{7+\frac{3}{x}} = \frac{5+\sqrt{4}}{7+0} = \boxed{1}.$

(8 points) $\displaystyle\lim_{x\to-\infty} \frac{5x+\sqrt{4x^2+5x}}{7x+3}$

Solution. $\displaystyle \lim_{x\to\infty} \frac{5x - x\sqrt{4+\frac{5}{x}}}{7x+3} = \lim_{x\to\infty} \frac{5-\sqrt{4+\frac{5}{x}}}{7+\frac{3}{x}} = \frac{5-\sqrt{4}}{7+0} = \boxed{\frac{3}{7}}.$
I gave only 3 points for an answer of 1.

For this problem and the next one, sketch a graph. Be sure to label any intercepts, local maxima, local minima, inflection points, vertical or horizontal tangents, cusps, vertical or horizontal asymptotes. **Label**

those points with their x-coordinates; in most cases you don't need to give the y-coordinates. However the y-intercept should be labeled with its y-coordinate, and any asymptote should be labeled with the equation of the line.

(9 points) $f(x) = x - x^{2/3} = x^{2/3}(x^{1/3} - 1)$.

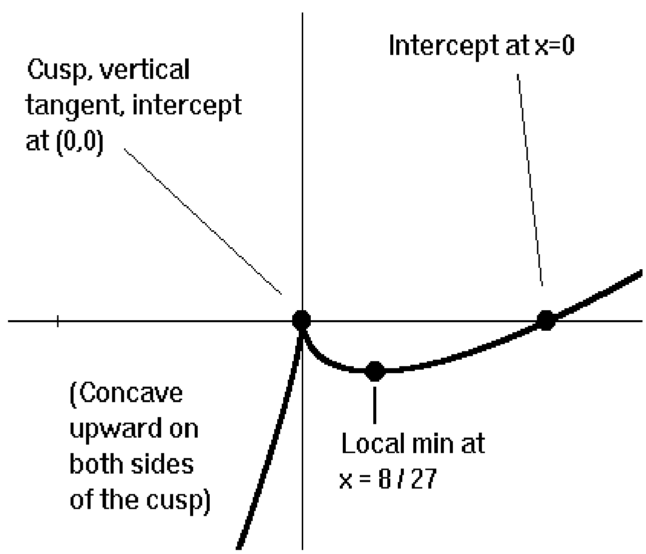
Solution. The domain is all real numbers. Some students confused $x^{2/3}$ with $x^{1/2}$, which has domain equal to just those $x \geq 0$. This led to omitting the left half of the graph, for which I deducted 3 points.

The formula $y = x^{2/3}(x^{1/3} - 1)$ is more convenient for figuring out where y itself is positive or negative, but the formula $y = x - x^{2/3}$ is more convenient for determining the derivative of y.

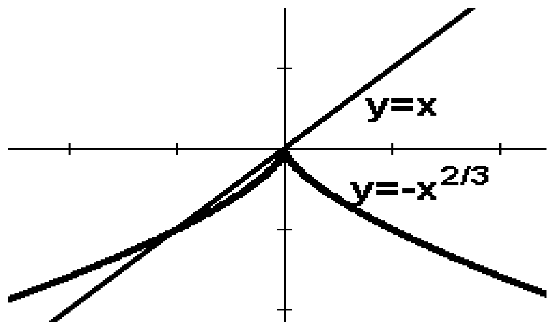
First you need to work out this chart:

x	.	0	.	8/27	.	1	.
$y = x^{2/3}(x^{1/3} - 1)$	$-$	0	$-$	$-$	$-$	0	$+$
$y' = 1 - \frac{2}{3}x^{-1/3}$	$+$	∞	$-$	0	$+$	$+$	$+$
$y'' = \frac{2}{9}x^{-4/3}$	$+$	∞	$+$	$+$	$+$	$+$	$+$

The fact that y' blows up at $x = 0$ means that y has a vertical tangent at $x = 0$. I deducted 1 point for omitting the vertical tangent. In general, I gave points for graphs according to how much they resemble the correct graph:



One way to quickly get a rough idea of what the graph should look like is as follows: Presumably you already know how to graph $y = -x^{2/3}$ and how to graph $y = x$ (see below). Now just visually add them. Since $y = x$ is below the horizontal axis when you're in the left half of the plane, and above the horizontal axis when you're in the right half of the plane, the effect is roughly the same as taking the graph of $y = -x^{2/3}$ and tilting it, to make the left side go down and make the right side go up.



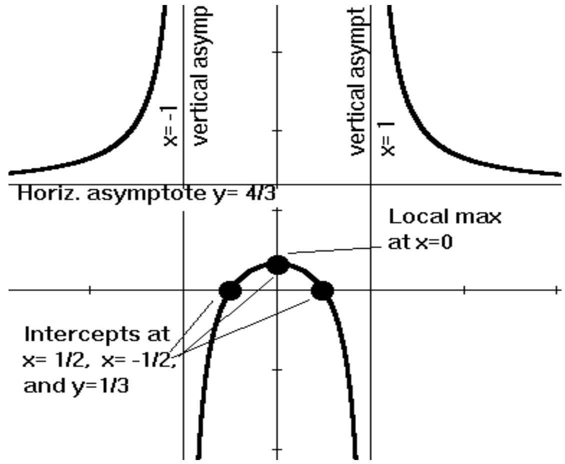
(9 points) $y = (4x^2 - 1)/(3x^2 - 3)$. *Hint*: For this problem I'll give you not only the function, but also its first two derivatives, to save you some work (but you need to know how to *use* those derivatives).

$$y = \frac{4x^2 - 1}{3(x^2 - 1)}, \qquad y' = \frac{-2x}{(x^2 - 1)^2}, \qquad y'' = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}.$$

Solution. First you need to work out this chart:

x	.	-1	.	$-1/2$	.	0	.	$1/2$	.	1	
y	$+$	∞	$-$	0	$+$	$+$	$+$	0	$-$	∞	$+$
y'	$+$	∞	$+$	$+$	$+$	0	$-$	$-$	$-$	∞	$-$
y''	$+$	∞	$-$	$-$	$-$	$-$	$-$	$-$	$-$	∞	$+$

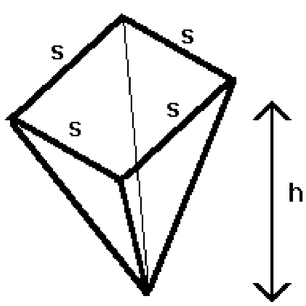
and then here is the answer:



I deducted 1 point if the picture was correct and the only error was to not compute numerically the location of the roots or the location of the horizontal asymptote.

(9 points) A water tank is in the shape of a square pyramid (with the point at the bottom and the square base at the top). It has a height of 10 meters, and each of the four sides of the square base is 10 meters long. It is being filled with water at a rate of 1 cubic meter per minute. How fast is the water level rising at the instant when it is 5 meters deep?

Hint: A square pyramid with height h and base edge s has volume $\frac{1}{3}hs^2$.



Solution. The filled portion of the water tank is also a square pyramid, with the same proportions as the entire tank but smaller. Thus its height is equal to the edge of its square base. Say that common value is h; then the filled portion has volume equal to $V = \frac{1}{3}h^3$ (since $s = h$). We are given that $\frac{dV}{dt} = 1$ (in cubic meters per minute). The problem is to find $\frac{dh}{dt}$, at the instant when $h = 5$.

Compute

$$1 = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}h^3\right) = h^2\frac{dh}{dt}$$

by the chain rule. When $h = 5$, then $h^2 = 25$, so $\frac{dh}{dt} =$

$$\boxed{\frac{1}{25} \text{ meters per minute}} = \boxed{0.04 \text{ meters per minute}}$$

$$= \boxed{4 \text{ centimeters per minute}}.$$

Partial credit: I gave 1 point for explicitly noting $\frac{dV}{dt} = 1$, and 3 points for the equation $V = \frac{1}{3}h^3$. Note that $\frac{dV}{dt}$ is *not* equal to h^2; students who wrote that down were missing the main point of this problem.