Quiz 4. Three people got full credit on everything. The class average was about 81 percent. Answers:

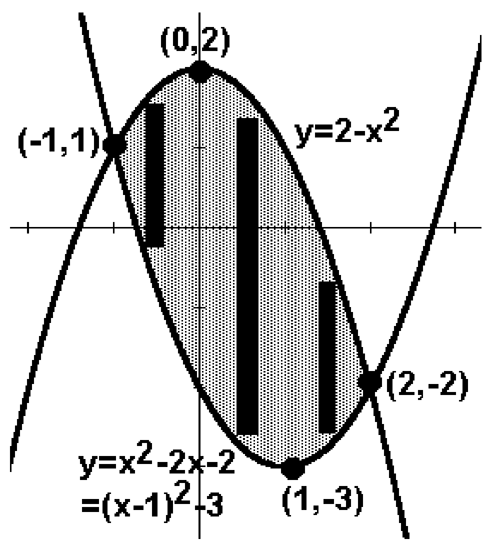
We're interested in finding the area of the shaded region between the two parabolas, shown at the bottom of this page.

The parabola $y = x^2 - 2x - 2 = (x-1)^2 - 3$ has lowest point $(1, -3)$; its left and right halves are $x = 1 - \sqrt{y+3}$ and $x = 1 + \sqrt{y+3}$ respectively.

The parabola $y = 2 - x^2$ has highest point at $(0, 2)$; its left and right sides halves are $x = -\sqrt{2-y}$ and $x = \sqrt{2-y}$ respectively.

(8 points) The easier way to do this problem is by using tall, narrow rectangles, and writing everything in terms of x. Then you'll get just one integral. **Write the integral, but do not evaluate it.**

Solution: **All of the tall, thin rectangles have their tops along the curve $y = 2 - x^2$ and their bottoms along the curve $y = x^2 - 2x - 2$. (See diagram.) So the rectangles all have height equal to $(2 - x^2) - (x^2 - 2x - 2)$.**



As we vary from one rectangle to another, the value of x varies from -1 to 2. Thus the desired integral is $\boxed{\int_{-1}^{2} [(2-x^2) - (x^2 - 2x - 2)]\,dx}$. I did not require further simplification for full credit. It is *not* correct to omit the brackets — i.e., to write $\int_{-1}^{2} (2-x^2) - (x^2-2x-2)dx$ — but I decided to overlook that minor error this time and give full credit anyway. I also gave full credit for several other correct forms of the answer — for instance,

$$= \boxed{\int_{-1}^{2} [(2-x^2) - ((x-1)^2 - 3)]\,dx} = \boxed{\int_{-1}^{2} [-2x^2 + 2x + 4]\,dx}$$

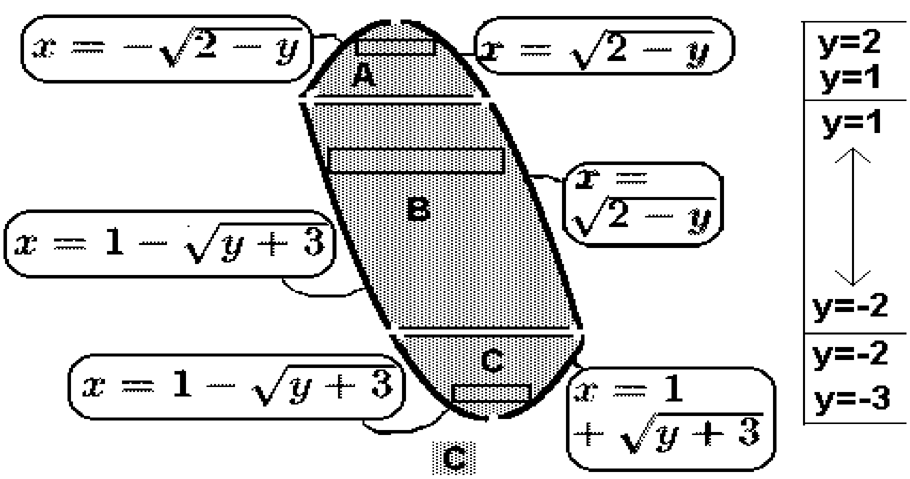
$$= \boxed{2\int_{-1}^{2} [-x^2 + x + 2]\,dx} = \boxed{-2\int_{-1}^{2} [x^2 - x - 2]\,dx}\,.$$

The most common errors were sign errors and other minor arithmetic errors; I generally deducted 2 points for those.

(10 points) The hard way to do this area problem would be by writing everything as a function of y, and using wide, short rectangles. You'd then have to express the area as the sum of three integrals. **Write the three integrals, but do not evaluate them.**

Solution. (This explanation may seem long, but almost all of the explanation is conceptual, not computational. Nearly all of the actual work just consists of copying the right expression from the first few lines of the quiz, into the integral. It is not necessary to draw the picture or the first three columns of the table below.)

We must subdivide the region into three regions, here denoted by A, B, and C:



$\left[\begin{array}{c}\text{region}\\ \text{bottom}\\ \text{\& top}\end{array}\right]$	$\left[\begin{array}{c}\text{rectangle}\\ \text{right \&}\\ \text{left ends}\end{array}\right]$	$\left[\begin{array}{c}\text{integral}\\ \text{for}\\ \text{region}\end{array}\right]$
(A) $1 \le y \le 2$	$x = \sqrt{2-y}$ $x = -\sqrt{2-y}$	$\boxed{\int_1^2 \left[(\sqrt{2-y}) - (-\sqrt{2-y})\right] dy}$
(B) $-2 \le y \le 1$	$x = \sqrt{2-y}$ $x = 1-\sqrt{y+3}$	$\boxed{\int_{-2}^1 \left[\sqrt{2-y} - (1-\sqrt{y+3})\right] dy}$
(C) $-3 \le y \le -2$	$x = 1+\sqrt{y+3}$ $x = 1-\sqrt{y+3}$	$\boxed{\int_{-3}^{-2} \left[(1+\sqrt{y+3}) - (1-\sqrt{y+3})\right] dy}$

The total area, then, is the sum of those three integrals. I deducted 1 point for each minor error (e.g., a sign change) from the answers above. Several students made sign errors, especially in the integral for region B.

Some explanation: Each rectangle has height equal to Δy, and width equal to the difference of its right and left ends; hence the rectangle has area equal to

$$[\text{right end} \; - \; \text{left end}]\,\Delta y.$$

In the limit, as we take more and more rectangles that are shorter and shorter, Δy is replaced by dy. The sum of all the rectangles in one region is then equal to

$$\int_{y=\text{bottom of region}}^{y=\text{top of region}} [\text{right end} \; - \; \text{left end}]\,dy\ .$$

Some students simply omitted the right end or left end from this expression, which shows a real lack of understanding. (By analogy: We can talk about the distance between Nashville and Chattanooga, but we can't talk about the distance between Nashville.)

A number of simplifications were possible and were permitted (but not required) for full credit. For instance, the integral for region A obviously can be rewritten as $2\int_1^2 \sqrt{2-y}\,dy$, and the integral for region C can be rewritten as $2\int_{-3}^{-2} \sqrt{y+3}\,dy$. The integral for region B does not simplify in any analogous fashion. Thus we obtain this answer, which is also correct:

$$\boxed{2\int_1^2 \sqrt{2-y}\,dy + \int_{-2}^1 \left[\sqrt{2-y} - (1-\sqrt{y+3})\right] dy + 2\int_{-3}^{-2} \sqrt{y+3}\,dy}\,.$$

Still another approach: Group together the left ends of regions B and C, and group together the right ends of regions A and B. That yields this expression (which also gets full credit):

$$\boxed{\int_1^2 \sqrt{2-y}dy + \int_{-2}^2 \sqrt{2-y} - \int_{-3}^1 (1-\sqrt{y+3})dy + \int_{-3}^{-2}(1+\sqrt{y+3})dy}.$$

I was puzzled by the fact that five different students had *the exact same complicated combination of errors*; they all wrote this wrong answer:

$$\int_1^2 \sqrt{2-y}\,dy + \int_{-2}^1 (\sqrt{2-y} - 1 - \sqrt{y+3}))dy + \int_{-3}^{-2}(1+\sqrt{y+3})dy.$$

Evidently they all made the same conceptual error. If I could figure out what that error was, I would explain how to avoid it; but so far I haven't been able to figure out how this answer was arrived at. (If anyone can tell me, please do so.) It bears a slight resemblance to the last correct answer that I have listed (in the box preceding this paragraph). Or, it could be arrived at from the first correct answer that I listed, if you make a sign error for the left end of region B, and omit altogether the regions A and C. At first I thought maybe it was the right answer to the wrong area problem, but it isn't even that — it evaluates to $-2/3$, and an area can't be negative. — I gave 6 points for this answer, because it showed at least some understanding of what are the right ingredients of the integral. I gave more than 6 points for answers that showed more understanding than this, and fewer points for answers that indicated less understanding.

(7 points) Finally, **find the area**. Be sure to $\boxed{\text{circle your answer}}$. If you're pressed for time, you can just do the problem one way; but if you have time, you can check your work by doing the problem two different ways.

Solution using tall, thin rectangles.

$$\int_{-1}^2 \left[-2x^2+2x+4\right] dx = \left[-\frac{2}{3}x^3 + x^2 + 4x\right]_{-1}^2$$

$$= -\frac{2}{3}(8 - -1) + (4-1) + 4(2 - -1) = \boxed{9}.$$

I also gave full credit, albeit reluctantly, for an answer of $\frac{27}{3}$, because I was so pleased when the answer was not even worse. Note that I simplified $\int_{-1}^2 [(2-x^2) - (x^2-2x-2)]\,dx$ down to $\int_{-1}^2(-2x^2+2x+4)dx$ *before* integrating; this makes the computation much easier and less prone to errors.

Many students, however, did not simplify before integrating — i.e., they attempted to calculate

$$\int_{-1}^{2} \left[(2-x^2)-(x^2-2x-2)\right] dx = \left[2x - \frac{1}{3}x^3 - \frac{1}{3}x^3 + x^2 + 2x\right]_{-1}^{2}$$

$$= 2(2--1) - \frac{1}{3}(8--1) - \frac{1}{3}(8--1) + (4-1) + 2(2--1) = 9.$$

But that's a much longer computation, and and so it's not surprising that many of these students made arithmetic errors. Remember, **simplify <u>before</u> you do computations**, whenever possible.

Solution using short, wide rectangles.

$$\int_{-3}^{-2} 2\sqrt{y+3}\,dy + \int_{-2}^{1} \left[\sqrt{2-y} - 1 + \sqrt{y+3})\right] dy + \int_{1}^{2} 2\sqrt{2-y}\,dy$$

$$= \int_{-3}^{-2} 2\sqrt{y+3}\,dy + \int_{-2}^{1} \sqrt{2-y}dy - 3 + \int_{-2}^{1} \sqrt{y+3}dy + \int_{1}^{2} 2\sqrt{2-y}\,dy$$

(Now use the substitutions $u = y+3$ and $v = 2-y$.)

$$= \int_{0}^{1} 2u^{1/2}du - \int_{4}^{1} v^{1/2}dv - 3 + \int_{1}^{4} u^{1/2}du - \int_{1}^{0} 2v^{1/2}dv$$

$$= \int_{0}^{1} 2u^{1/2}du + \int_{1}^{4} v^{1/2}dv - 3 + \int_{1}^{4} u^{1/2}du + \int_{0}^{1} 2v^{1/2}dv$$

$$= 4\int_{0}^{1} u^{1/2}du + 2\int_{1}^{4} u^{1/2}du - 3 = \boxed{9}.$$