Calculus exam. There were 150 points possible. The high score was 143 (that's about 95.33 percent); that score was achieved by two members of the class. The class average on this exam was 77.52 percent.

Answers:

Page 1:

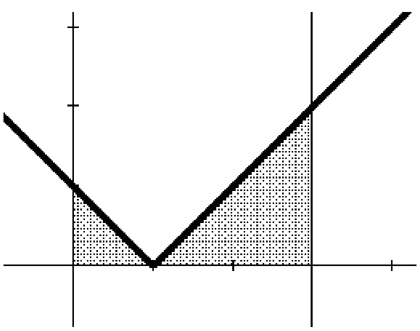
(8 points) $\displaystyle\int_0^3 |x-1|dx =$

Solution. We have $|x-1| = \begin{cases} 1-x & \text{when} \quad 0 < x < 1, \\ x-1 & \text{when} \quad 1 < x < 3, \end{cases}$ and so the answer is

$$\int_0^1 (1-x)dx + \int_1^3 (x-1)dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3 = \boxed{\frac{5}{2}} = \boxed{2\tfrac{1}{2}} = \boxed{2.5}.$$

Here is a shortcut to reduce the work in computing those two integrals: Substitute $u = 1-x$ in the first one, and $v = x-1$ in the second one. This yields $\int_0^1 udu + \int_0^2 vdv$, which yields the same numerical answer with slightly less effort.

Alternative method: Sketch the graph. The two triangles have areas equal to $\frac{1}{2}$ and 2, respectively. (See diagram.)



About 3/4 of the class got this one right. The wrong answers were all different. A couple of the most basic wrong answers were

- $\left[\frac{x^2}{2} - x\right]_0^3$, which yields an answer of $\frac{3}{2}$, for which I gave 4 points; and
- $\left[\,\left|\frac{x^2}{2} - x\right|\,\right]_0^3$, which yields an answer of 3, for which I gave 4 points.

(6 points) $\displaystyle\sum_{n=2}^{5} \frac{1}{n+1} =$

Solution. $= \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6} = \frac{20+15+12+10}{60} = \frac{57}{60} = \frac{60-3}{60} = \boxed{\dfrac{19}{20}} = \boxed{0.95}$.

Most students got full credit on this problem. A few students gave an answer of $\frac{57}{60}$, for which I gave only 5 points (the instructions at the beginning of the test said to simplify).

(6 points) $1+2+3+\cdots+998+999+1000 =$

Solution. $= \frac{1}{2}\cdot 1000 \cdot 1001 = 500 \cdot 1,001 = \boxed{500{,}500}$. That's the answer I had in mind. However, my instructions weren't explicit enough, and some students interpreted the problem as "rewrite this expression using sigma notation," since we did have a homework problem or two of that sort. Consequently, I also gave full credit for an answer of $\boxed{\displaystyle\sum_{n=1}^{1000} n}$.

(8 points) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1-x}}\,dx =$

Solution. Only about 1/4 of the class got this one right.

Substitute $u = 1-x$. Then $du = -dx$. (That much is already worth 3 points.) When x goes from 0 to 1, then u goes from 1 to 0. Thus the integral is $-\int_1^0 \frac{(1-u)^3}{\sqrt{u}}du = \int_0^1 (1-u)^3 u^{-1/2} du$ (now we're up to 5 points). Continuing, $\int_0^1 (1-3u+3u^2-u^3)u^{-1/2}du = \int_0^1 (u^{-1/2} - 3u^{1/2} + 3u^{3/2} - u^{5/2})du$ — and an expression like that one was worth 6 points. Finally, for full credit, evaluate: $\left[2u^{1/2} - 2u^{3/2} + \frac{6}{5}u^{5/2} - \frac{2}{7}u^{7/2}\right]_0^1 = 2 - 2 + \frac{6}{5} - \frac{2}{7} = \boxed{\dfrac{32}{35}} \approx \boxed{0.914286}$.

I deducted 1 point for sign errors, and 2 points for most other arithmetic errors; those accounted for most wrong answers.

This problem could also be done using a substitution of $u = \sqrt{1-x}$, but it's harder that way. The rule of thumb discussed in class is to look for a composition of the form $f(g(x))$ appearing somewhere in the integral, and then try substituting $u = 1-x$.

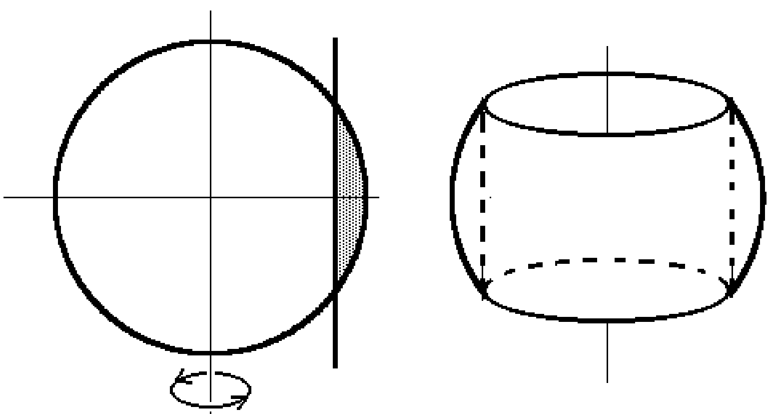
A couple of students used the formula $\int uv dx$ " $=$" $\left(\int u dx\right)\left(\int v dx\right)$, which yields $\int x^3(1-x)^{-1/2}dx$ " $=$" $\left(\frac{x^4}{4}\right)\left(-2\sqrt{1-x}\right)$. But I've put that equation in quote marks because that formula is *wrong*, as I mentioned in class several times. No sympathy and no partial credit on this error.

I was more sympathethic about a rather different type of error. A couple of students thought that $\int_0^1 \dfrac{x^3}{\sqrt{1-x}}\,dx$ does not exist, because $\frac{x^3}{\sqrt{1-x}}$ is undefined at $x=1$. That mistake is understandable (and so I gave full credit to these two students), because I did not discuss integrals of this type very much in class; when I was making up the test, I forgot that I had not discussed them.

In some cases, if the integrand is nonexistent or infinite at some point, the integral does not exist; that is the case in Example 7 on page 343 in section 5.3 of your textbook. However, that discussion is not really adequate. $\int_0^1 \frac{x^3}{\sqrt{1-x}}\,dx$ is an example of what is called an *improper integral*, but those aren't discussed until section 8.8 of your textbook, next semester.

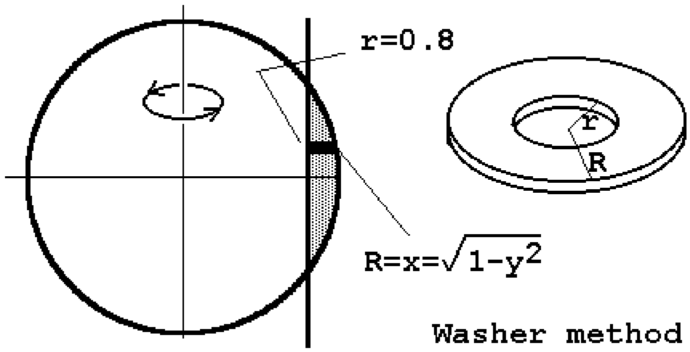
Page 2:

(12 points) Consider the region that lies inside the circle $x^2 + y^2 = 1$, and lies to the right of the line $x = 0.8$. By rotating that region around the y-axis, we get a solid shaped like a wooden bead or an unornamented wedding ring (see diagram). Find the volume of that solid.



Solution. About half the class got this right. This problem can be done by washers or shells; most students chose washers.

Washer method: The highest and lowest values of y are $\pm\sqrt{1-0.8^2} = \pm\sqrt{1-0.64} = \pm\sqrt{0.36} = \pm 0.6$. For each value of y, we get a thin washer with thickness dy, inner radius $r = 0.8$, and outer radius $R = x = \sqrt{1-y^2}$. The volume of the washer is $\pi(R^2 - r^2)dy = \pi(x^2 - 0.8^2)dy = \pi(1 - y^2 - 0.64)dy = \pi(0.36 - y^2)dy$.



Hence the total volume is

$$\int_{-0.6}^{0.6} \pi(R^2 - r^2)dy = \int_{-0.6}^{0.6} \pi\left[(1-y^2) - (0.8)^2\right] dy = \int_{-0.6}^{0.6} \pi(0.36 - y^2)dy.$$

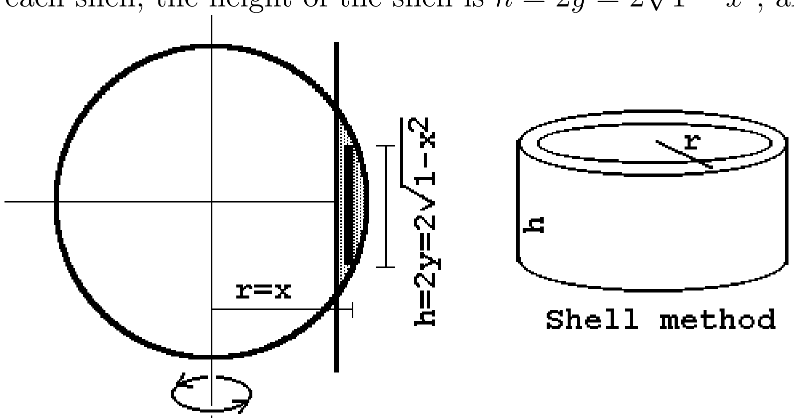
Many students came close to that formula, but inadvertently modified it in some little way — e.g., omitting the π, interchanging the roles of 0.6 and 0.8, forgetting to square something, or omitting the $-r^2$ term altogether. I deducted 3 points for each such conceptual error, and 2 points for each arithmetic error.

Finishing the computation,

$$2\int_{0}^{0.6} \pi(0.36 - y^2)dy = 2\pi\left[0.36y - \frac{1}{3}y^3\right]_0^{0.6} = 2\pi\left[(0.6)^3(1 - \frac{1}{3})\right]$$

$$= 2\pi(0.216)(\frac{2}{3}) = \boxed{0.288\pi} = \boxed{\tfrac{36\pi}{125}} \approx \boxed{0.9048}.$$

Shell method: The integration proceeds from radius $r = x = 0.8$ to radius $r = x = 1$. For each shell, the height of the shell is $h = 2y = 2\sqrt{1-x^2}$, and the shell has volume $2\pi r h \Delta r$.



Thus the whole volume is

$$\int_{0.8}^{1} 2\pi r h dr = \int_{0.8}^{1} 4\pi x\sqrt{1-x^2}dx.$$

To evaluate this, substitute $u = 1 - x^2$; then $du = -2xdx$. When x goes from 0.8 to 1, then u goes from 0.36 to 0. Thus the answer is

$$\int_{0}^{0.36} 2\pi\sqrt{u}du = 2\pi \cdot \frac{2}{3}\left[u^{3/2}\right]_0^{0.36} = \frac{4\pi}{3}\cdot(0.36)^{3/2}$$

$$= \frac{4\pi}{3} \cdot 0.216 = 4\pi \cdot 0.072 = \boxed{0.288\pi} = \boxed{\tfrac{36\pi}{125}} \approx \boxed{0.9048}.$$

Page 3:

(5 points) $\displaystyle\int (x^2+5)(x^2-7)dx =$

Solution. $= \int (x^4 - 2x^2 - 35)dx = \boxed{\tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 - 35x + C}$.

I was worried that someone would try to use a "product rule" for integrals, such as $\int uv = \int u \int v$, but no one made that error on this problem. The only errors on this problem were as follows:

- Omitting the "$+C$" — penalty of 2 points.
- Not simplifying — penalty 1 point.
- Writing fractions incorrectly, e.g., writing $-2/3x^3$ or writing $-\frac{2}{3x^3}$ — penalty 1 point. (These misrepresentations of fractions were discussed a few times in class and also on the "common errors" web page, which was recommended repeatedly in class.)

(9 points) Find the equation for the line that is tangent to the curve

$$y^3 \cos x + x \sin\left(\frac{\pi y}{2}\right) = 1$$

at the point $(0, 1)$.

Solution. I gave no points at all to anyone who still thinks that the derivative of the product equals the product of the derivatives — i.e., that $(uv)' = u'v'$, or more specifically in this instance that $(y^3 \cos x)' = -3y^2 y' \sin x$. I mentioned that non-rule early in the semester, and anyone who hasn't learned about it by now should not go on to a higher math course.

Differentiate both sides of that equation with respect to x (using implicit differentiation). Thus we get

$$3y^2 y' \cos x - y^3 \sin x + \sin\left(\frac{\pi y}{2}\right) + x\cos\left(\frac{\pi y}{2}\right) \cdot \frac{\pi}{2} y' = 0$$

or, if you prefer,

$$y' = \frac{y^3 \sin x - \sin\left(\frac{\pi y}{2}\right)}{3y^2 \cos x + \frac{\pi x}{2}\cos\left(\frac{\pi y}{2}\right)}.$$

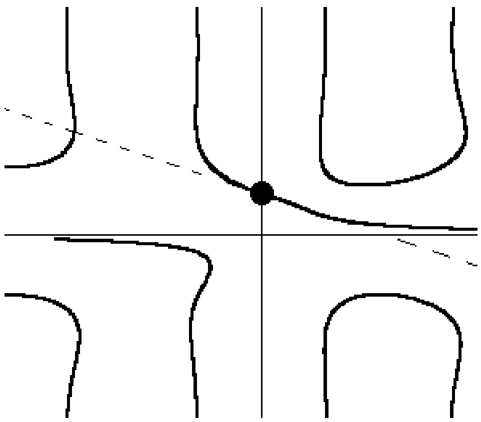
Getting either of those formulas correctly was worth 6 of the 9 points (and you could get a 7th point if you wrote down an equation for *some* straight line that passes through the point $(0, 1)$).

Now plug in $x = 0$, $y = 1$, noting that $\cos(0) = 1$, $\sin(0) = 0$, $\sin(\frac{\pi}{2}) = 1$, $\cos(\frac{\pi}{2}) = 0$. Thus we obtain

$$(3 \cdot 1^2 \cdot y' \cdot 1) - (1^3 \cdot 0) + 1 + (0 \cdot 0 \cdot \frac{\pi}{2}) = 0$$

which simplifies to $3y' + 1 = 0$, or $y' = -1/3$. Thus we want the straight line that has slope $-1/3$ and that passes through the point $(0, 1)$. That is the line $\boxed{y - 1 = -x/3}$, or $\boxed{y = -\frac{x}{3} + 1}$, or $\boxed{\frac{x}{3} + y = 1}$, or $\boxed{x + 3y = 3}$.

Just in case you're wondering what that curve looks like, I'm including a picture of it below. It's a rather complicated curve, made up of many pieces. I used some mathematical software to draw it — I did not draw it by hand. The tangent line is also included in my graph, as a dashed line.



Page 4:

(11 points) Graph $y = x + 2 + \dfrac{1}{x+1}$.

Be sure that your graph shows all intercepts, local maxima, local minima, inflection points, vertical tangents, horizontal tangents, and all asymptotes — horizontal, vertical, or slant. Label all those objects on the graph. Be sure to give the x-coordinates of the points (you may omit the y-coordinates), and the equations of the asymptotes.

Solution. From the form of the equation, we can immediately see that it has a vertical asymptote at $x = -1$ and a slant asymptote at $y = x + 2$. The y-intercept is at $(0, 3)$.

For the x-intercept, solve $y = 0$; that gives us $x + 2 + \frac{1}{x+1} = 0$. Multiply through by $x + 1$ to get $(x+1)(x+2) + 1 = 0$. Rewrite as $x^2 + 3x + 3 = 0$. The quadratic formula then yields $x = \frac{-3 \pm \sqrt{9-12}}{2}$ —- which has no real solution. So there is no x-intercept — i.e., the graph does not have any points on the x-axis. We can rewrite the function as $y = \frac{(x+\frac{3}{2})^2 + \frac{3}{4}}{x+1}$; this is positive when $x > -1$ and negative when $x < -1$.

Differentiate to obtain $y' = 1 - \frac{1}{(x+1)^2}$ and $y'' = \frac{2}{(x+1)^3}$. Thus y'' is positive when $x > -1$ and negative when $x < -1$. As for y', we have y' undefined when $x = -1$, and elsewhere we have

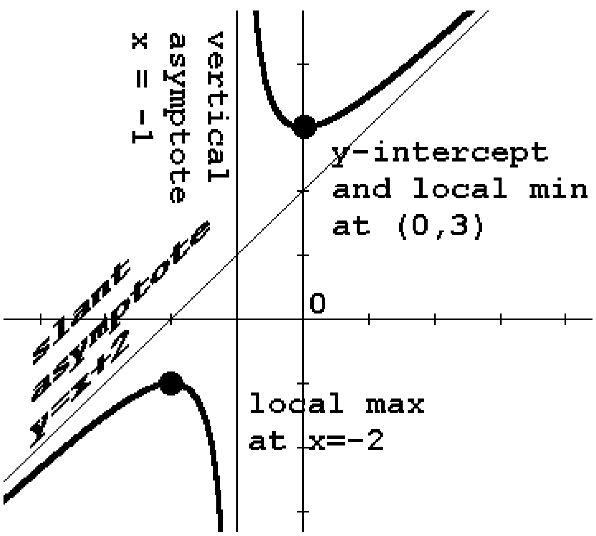
$$\begin{aligned} y' < 0 \quad &\Leftrightarrow\; 1 < \frac{1}{(x+1)^2} \\ &\Leftrightarrow (x+1)^2 < 1 \\ &\Leftrightarrow |x+1| < 1 \\ &\Leftrightarrow -1 < x+1 < 1 \\ &\Leftrightarrow -2 < x < 0 \end{aligned}$$

and similarly we have $y' > 0$ when x is outside the interval $[-2, 0]$; we have $y' = 0$ at $x = 0$ and at $x = -2$.

That information yields this chart:

x	.	-2	.	$(-1)-$	$(-1)+$	.	0	.
y	$-$	$-$	$-$	$-\infty$	$+\infty$	$+$	$+$	$+$
y'	$+$	0	$-$	$-\infty$	$-\infty$	$-$	0	$+$
y''	$-$	$-$	$-$	$-\infty$	$+\infty$	$+$	$+$	$+$

and finally this graph:



Grading: I gave full credit to anyone who had these five features:

- Correct general shape of the curve (a hyperbola)
- Local min labeled at $x = 0$
- Local max labeled at $x = -2$
- Vertical asymptote labeled $x = -1$
- Slant asymptote labeled $y = x + 2$

If the curve had the right general shape, but one of those other four features was not right, I took off 2 points if the feature was inadequately labeled (for instance, if the local min was labeled as $x = 1$ or was not labeled at all), or 3 points if the feature was entirely missing (e.g., if there was no indication that the curve even has a slant asymptote).

I also deducted 1 point for a nearly-correct general shape — e.g., if the curve looked more like two parabolas or two semicircles than two branches of a hyperbola.

For students with entirely or mostly the wrong shape, I started at 0 points and added for correct features — e.g., 3 points for correctly identifying $x = -1$ as a vertical asymptote.

Page 5:

(6 points) Find y, if $y'' = 6x^2 + 2$, $y(0) = 2$, $y'(0) = 1$.

Solution. Integrating once yields $y' = 2x^3 + 2x + A$. Plug in $1 = y'(0) = A$. Thus $y' = 2x^3 + 2x + 1$. Integrating again yields $y = \frac{1}{2}x^4 + x^2 + x + B$. Plug in $2 = y(0) = B$. Thus $y = \boxed{\frac{1}{2}x^4 + x^2 + x + 2}$. Nearly everyone got this right.

(9 points) Recall that $[\![x]\!]$ denotes the greatest integer less than or equal to x. Find (or write "DNE" for "Does Not Exist"):

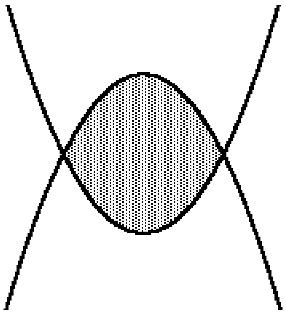
$\lim_{x\to 3+}[\![2x]\!] = \boxed{6}$ $\qquad$ $\lim_{x\to 3-}[\![2x]\!] = \boxed{5}$ $\qquad$ $\lim_{x\to 3}[\![2x]\!] = \boxed{\text{DNE}}$

$\lim_{x\to 3+}[\![\frac{1}{2}x]\!] = \boxed{1}$ $\qquad$ $\lim_{x\to 3-}[\![\frac{1}{2}x]\!] = \boxed{1}$ $\qquad$ $\lim_{x\to 3}[\![\frac{1}{2}x]\!] = \boxed{1}$

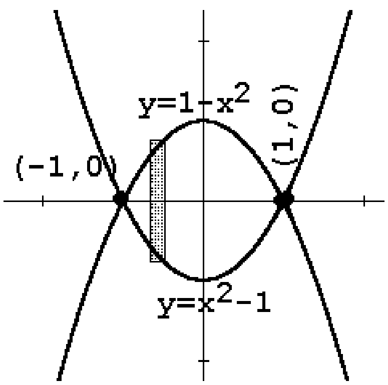
Solution. see boxed answers above.

Explanations: When x is slightly greater than 3, then $2x$ is some number slightly higher than 6, so $2x$ is between 6 and 7; hence $[\![2x]\!]$ is equal to 6. When x is slightly lower than 3, then $2x$ is slightly lower than 6, so $2x$ is between 5 and 6; hence $[\![2x]\!]$ is equal to 5. And so on. When x is near 3 (on either side), then $\frac{1}{2}x$ is near 1.5, and so $\frac{1}{2}x$ is between 1 and 2; thus $[\![\frac{1}{2}x]\!] = 1$, etc.

(9 points) Find the area between the curves $y = 1 - x^2$ and $y = x^2 - 1$.



Solution. Those curves intersect at the points where $1 - x^2 = x^2 - 1$. That simplifies to $2(x^2 - 1) = 0$, or $2(x-1)(x+1) = 0$, or $x = \pm 1$. For $-1 < x < 1$ we have $0 < x^2 < 1$ and so $1 - x^2 > 0 > x^2 - 1$. Thus the thin tall vertical rectangles have height equal to $(1 - x^2) - (x^2 - 1) = 2 - 2x^2$, and width equal to$\Delta x$.



The total area is $\int_{-1}^{1}(2-2x^2)dx = 2\int_0^1(2-2x^2)dx = \left[4x - \frac{4}{3}x^3\right]_0^1 = (4 - \frac{4}{3}) = \boxed{\frac{8}{3}} = \boxed{2\frac{2}{3}} = \boxed{2.666\cdots}$.

This problem would be harder with short, wide rectangles, since the region above the x-axis and the region below the x-axis would require different descriptions of the left and right ends of the rectangle — i.e., it would require two integrals. Actually, you could get by with one integral, by symmetry — just use the top half of the region, and then double the resulting area. The top parabola is $y = 1 - x^2$, or $x = \pm\sqrt{1-y}$. Thus we get $2\int_0^1(\sqrt{1-y} - -\sqrt{1-y}\,dy = 4\int_0^1 \sqrt{1-y}\,dy = 4\int_0^1 \sqrt{u}\,du = \frac{8}{3}$.

Most students got this right. A few made sign errors or other arithmetic errors. How much I deducted for those errors varied, depending on their nature. For instance, a sign error that results in an area of 0 is a big error, since a glance at the picture shows that the area is *not* 0, and should lead to the student backtracking and looking for his or her error.

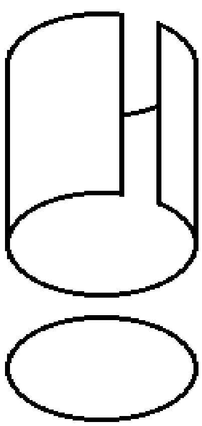
Page 6:

(6 points) $\displaystyle\int x^2 \sin(x^3)\, dx =$

Solution. Substitute $u = x^3$; then $du = 3x^2 dx$. So the given integral is equal to $\frac{1}{3}\int \sin(u)du = -\frac{1}{3}\cos(u) + C = \boxed{\frac{-1}{3}\cos(x^3) + C}$.

I deducted 1 point for omitting "$+C$", and 2 points for getting some other number than $\frac{-1}{3}$ at the front of the answer.

(10 points) A cup with volume 1000π cubic centimeters is to be made of plastic. The cost is proportional to the cup's surface area, which we therefore want to minimize. The cup is shaped like a cylinder, with the top open. Thus the area consists of one circle (for the cup's bottom) plus one rectangle (for the side, unrolled). (See diagram.) What is the smallest possible area?



Solution. I intentionally made the number in the problem (1000π) unrealistic in order to make the computations in the solution simple. If you did this problem without error, the answer was not very messy. Unfortunately, many students made a wide variety of arithmetic and algebra errors, defeating that effort of mine. Typical errors were to change 1000π to 1000, to change $\pi r^2 h$ to $\pi r h$, and so on.

Correct solution: Say the height is h and the radius of the bottom is r. Then the volume is $1000\pi = \pi r^2 h$, and the area is $A = \pi r^2 + 2\pi r h$. Rewrite that first equation as $h = 1000 r^{-2}$. Then the second equation becomes

$$A = (r^2 + 2000 r^{-1})\pi.$$

Getting that far was worth 5 points.

Then $A'(r) = (2r - 2000r^{-2})\pi$ (now we're up to 6 points). Then $A''(r) = \pi(2 + 4000r^{-3}) > 0$, so the function $A(r)$ takes a minimum when $A'(r) = 0$ (but that step is optional and gains no points, just assurance that you've got your signs pointed the wrong way). The equation

$A'(r) = 0$ simplifies to $2r = 2000r^{-2}$ (worth 7 points), hence $r^3 = 1000$, hence $r = 10$ (worth 8 points). Now plug that into the formula for A, to get $A(10) = \pi(100 + 200) = \boxed{300\pi \text{ square cm}} = \boxed{942.4778 \text{ sq cm}}$.

Page 7:

(6 points) $\dfrac{d}{dx}\Big(\sin(\cos^2 x)\Big) =$

Solution. Use the chain rule, with

$$y = \sin u, \qquad u = v^2, \qquad v = \cos x.$$

Then

$$\frac{dy}{du} = \cos u, \qquad \frac{du}{dv} = 2v, \qquad \frac{dv}{dx} = -\sin x.$$

The product of those is

$$\frac{dy}{dx} = -2(\sin x)v\cos u = \boxed{-2(\sin x)(\cos x)\cos(\cos^2 x)}$$

which can also be written more briefly as $\boxed{-(\sin 2x)\cos(\cos^2 x)}$.

Most students got this right. *Partial credit*:

Some students had no understanding of the chain rule whatsoever, and I couldn't make any sense out of their answers, so they got 0 points.

Some students wrote down the right answer and then transformed it, via incorrect trigonometric identities, to a wrong answer, and circled that. I gave these students 4 points.

Some students understood that the answer should be of the form $\cos(\cos^2 x)$ times some function of x, but they fouled up the "some function of x." I gave these students 3 points. (Fewer points, if they replaced $\cos(\cos^2 x)$ with something slightly different, or if they replaced the function of x with something that isn't quite a function of x.)

(8 points) Find the third derivative of $\tan x$.

Solution. You should already know that $\frac{d}{dx}\tan x = \sec^2 x$ (worth 2 points, by itself) and $\frac{d}{dx}\sec x = \sec x \tan x$. Also the product rule $(uv)' = u'v + uv'$ and the power rule $(u^n)' = nu^{n-1}u'$. Combining those facts:

$$\frac{d^2}{dx^2}\tan x = \frac{d}{dx}\left(\sec^2 x\right) = 2\sec x \frac{d}{dx}\sec x = 2(\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$$

(worth 4 points if you got this far correctly). Then, finally,

$$\frac{d^3}{dx^3}\tan x = \frac{d}{dx}\left(2\sec^2 x\tan x\right) = 2\left(\sec^2 x\right)'\tan x + 2\sec^2 x\,(\tan x)'$$

$$= 2\left(2\sec^2 x\tan x\right)\tan x + 2(\sec^2 x)(\sec^2 x) = \boxed{4\sec^2 x\tan^2 x + 2\sec^4 x}\,.$$

That can also be written as $\boxed{(2\sec^2 x)(2\tan^2 x + \sec^2 x)}$ or $\boxed{\dfrac{4\sin^2 x + 2}{\cos^4 x}}$.

I think the method above is probably easiest, but there are other ways to do this problem correctly. Those methods yield a wide variety of other answers that are equal to the answer above but different in appearance. To see some of the other answers, keep in mind that $\tan^2 x + 1 = \sec^2 x$. Here are a few other correct answers:

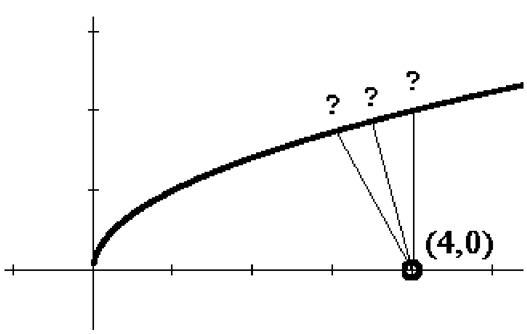
$$\boxed{(2\sec^2 x)(3\tan^2 x + 1)} = \boxed{6\sec^2 x\tan^2 x + 2\sec^2 x}$$

$$= \boxed{(2\sec^2 x)(3\sec^2 x - 2)} = \boxed{6\sec^4 x - 4\sec^2 x}\,.$$

That last answer actually has some advantages: It expresses everything in terms of just *one* trigonometric function.

Several students carried through with the right ideas, and then either (i) made an arithmetic error in the last step, or (ii) wrote down the right answer and then "simplified" it to a wrong answer. For either of these types of errors, I deducted two points.

(8 points) Find the point on the graph of $y = \sqrt{x}$ that is closest to the point $(4, 0)$. (See diagram.)



Solution. The distance from $(4, 0)$ to a point (x, y) on the curve $y = \sqrt{x}$ is

$$s = \sqrt{(x-4)^2 + (y-0)^2} \qquad \text{(worth 2 points)}$$

$$= \sqrt{(x-4)^2 + x} \qquad \text{(worth 3 points)}$$

That number s is minimized at the same value of x where the number $q = s^2$ is minimized. That's easier to work with. Compute $q(x) = (x-4)^2 + x = x^2 - 7x + 16$ (worth 5 points);

then $q'(x) = 2x - 7$ (now worth 6 points). (Optional: Note that $q''(x) = 2 > 0$, so it is indeed a minimum that we'll find, not a maximum.)

Then $q(x)$ has its minimum where $q'(x) = 0$, at $\boxed{x = \frac{7}{2}}$. I gave full credit (8 points) for that answer, but a little reluctantly; the question asks what is the *point on the curve.* Thus the correct answer actually is

$\boxed{\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)} = \boxed{(3.5,\ 1.871)}$. About 3/5 of the class got this right.

The solution above does everything in terms of x. What if you try to do everything in terms of y instead? If you do it correctly, it's not much harder. We have $y = \sqrt{x}$, so $x = y^2$. Then the distance is $s = \sqrt{(x-4)^2 + y^2} = \sqrt{(y^2-4)^2 + y^2} = \sqrt{y^4 - 7y^2 + 16}$, which is minimized at the same value of y as $q = s^2 = y^4 - 7y^2 + 16$ is minimized. Compute $q'(y) = 4y^3 - 14y = 4y(y^2 - \frac{7}{2})$, which vanishes at $y = 0$ and at $y = \pm\sqrt{\frac{7}{2}}$. A glance at the graph shows that, among those, the only feasible solution is $y = \sqrt{\frac{7}{2}}$. Also check that $q''(y) = 12y^2 - 14$ is positive in the vicinity of $y = \sqrt{\frac{7}{2}}$, so we're talking about a minimum for q, not a maximum. Finally, $x = y^2 = \frac{7}{2}$.

One student had another interesting (and correct) way to analyze the problem: At the point on the curve which is closest to $(4, 0)$, the tangent to the curve will be perpendicular to the line connecting that point to $(4, 0)$. If the point is at $(u, u^{1/2})$ for some number u, then the slope of the tangent line is $\frac{1}{2}u^{-1/2}$. The line perpendicular to that has slope equal to its negative reciprocal, which is $-2u^{1/2}$; that must be the slope of the line through $(u, u^{1/2})$ and $(4, 0)$. Those requirements tell us $-2u^{1/2} = \frac{u^{1/2}}{u-4}$. That equation simplifies to $-2(u-4) = 1$, or $u - 4 = \frac{-1}{2}$, or $u = 4 - \frac{1}{2} = 3.5$. Thus the desired point has $x = 3.5$, etc. (Unfortunately, the student in question did not carry through this interesting analysis without errors.)

A surprisingly large number of students (three) began with exactly the same peculiar error: They thought the distance from $(4, 0)$ to (x, y) is $s = \sqrt{(x-4)^2 - (y-0)^2}$. If we plug in $y = \sqrt{x}$, then we're trying to minimize $s = \sqrt{(x-4)^2 - x}$. That's minimized at the same location as $q = s^2 = (x-4)^2 - x = x^2 - 9x + 16$. Compute $q'(x) = 2x - 9$, so the minimum occurs at $x = 9/2 = 4.5$. This is obviously wrong — a glance at the diagram given on the test shows that $(4, 0)$ is closer to $(4, \sqrt{4})$ than it is to $(4.5, \sqrt{4.5})$. Nevertheless, I gave 5 points for an answer of $(4.5, \sqrt{4.5})$ (and slightly less if this computation was carried out with additional errors).

Page 8:

(9 points) $\lim\limits_{x\to\infty} \left(\sqrt{x^2 + 6x - 1} - \sqrt{x^2 - 2x + 5}\right) =$

Solution.

$$= \lim_{x\to\infty} \frac{(x^2+6x-1)-(x^2-2x+5)}{\sqrt{x^2+6x-1}+\sqrt{x^2-2x+5}} = \lim_{x\to\infty} \frac{x\left(8-\frac{6}{x}\right)}{x\sqrt{1+\frac{6}{x}-\frac{1}{x^2}}+x\sqrt{1-\frac{2}{x}+\frac{5}{x^2}}}$$

$$= \frac{8-0}{\sqrt{1+0-0}+\sqrt{1-0+0}} = \boxed{4}.$$

Several students wrote, for their very first step, that the given expression is equal to $\left(\lim\limits_{x\to\infty}\sqrt{x^2+6x-1}\right) - \left(\lim\limits_{x\to\infty}\sqrt{x^2-2x+5}\right)$. But that's wrong. That gives an answer of $\infty - \infty$, which is meaningless. What we have is (something very large) minus (something else very large), but the two large things are not getting very far apart; we have to analyze how far apart. We can only analyze that *while* they're getting large, not after they've already both gone to ∞. To see the inadequacy of that "$\infty - \infty$" approach, think about what it would say about these similar problems (all of which have different answers):

- $\lim_{x\to\infty}[(x^2+5x)-(x^2)]$
- $\lim_{x\to\infty}[(x^2+5)-(x^2)]$
- $\lim_{x\to\infty}[(x^2)-(x^2)]$
- $\lim_{x\to\infty}[(x)-(x^2)]$

Also, some students began with $\lim\limits_{x\to\infty} x\left(\sqrt{1+\dfrac{6}{x}-\dfrac{1}{x^2}}-\sqrt{1-\dfrac{2}{x}+\dfrac{5}{x^2}}\right)$, which is correct but not helpful. What we have there is (something very big) times (something very small), and that's not informative enough to tell us the answer. To see the inadequacy of that "$\infty \cdot 0$" approach, think about what it would say about these similar problems (all of which have different answers):

- $\lim_{x\to\infty}\left[(x^2)\cdot\left(\frac{1}{x}\right)\right]$
- $\lim_{x\to\infty}\left[(x^2)\cdot\left(\frac{1}{x^2}\right)\right]$
- $\lim_{x\to\infty}\left[(x)\cdot\left(\frac{1}{x^2}\right)\right]$

Students who arrived at an answer of 0 or ∞, using either the $\infty-\infty$ or $\infty\cdot 0$ argument, missed the whole point of the problem (introduced in section 4.4), and so I gave no partial credit or answers of that sort. You may get some additional understanding of this topic when you get to section 7.7, but that section is primarily concerned with harder problems that require more advanced techniques.

(6 points) $\quad \lim\limits_{x\to 0} \dfrac{(\sin^2 3x)(\cos 2x)}{x\tan 4x} =$

Solution. Only about 1/4 the class got this right; I was surprised at how many students forgot how to do problems of this sort, which were covered near the beginning of the semester. This problem is similar to Examples 4–5 and problems 35–44 on pages 174–176, using equation 2 from page 171. Begin by rewriting the problem as

$$= \lim_{x\to 0} \left(\frac{\sin 3x}{3x}\right)(3)\left(\frac{\sin 3x}{3x}\right)(3)(\cos 2x)(\cos 4x)\left(\frac{4x}{\sin 4x}\right)(\frac{1}{4})$$

— I gave 5 points for a correct factorization, or 3 points for an incorrect one that resembles the correct one. Now use the formula $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, covered early this semester. Continuing,

$$= 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot \frac{3\cdot 3}{4} = \boxed{\frac{9}{4}} = \boxed{2\tfrac{1}{4}} = \boxed{2.25}.$$

Some students apparently tried to apply l'Hopital's rule, though we won't cover it until next semester. It says that, under certain circumstances, $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$. (See Section 7.7 of the textbook.) But that's not necessarily helpful, since in some cases (e.g., this one) $\frac{f'(x)}{g'(x)}$ is even more complicated than $\frac{f(x)}{g(x)}$.

(8 points) $\displaystyle\int_0^8 \frac{x^2+3x}{\sqrt[3]{x}}\,dx =$

Solution. $= \int_0^8 (x^{2-1/3} + 3x^{1-1/3})dx = \int_0^8 (x^{5/3} + 3x^{2/3})dx$. Getting that far was worth 4 points. Continuing, $\left[\frac{3}{8}x^{8/3} + \frac{3}{5}\cdot 3x^{5/3}\right]_0^8 = \frac{3}{8}\cdot 2^8 + \frac{9}{5}\cdot 2^5 = 32\cdot(3+\frac{9}{5}) = 32\cdot\frac{24}{5} = \boxed{\frac{768}{5}} = \boxed{153\frac{3}{5}} = \boxed{153.6}$.

Most errors on this problem were just arithmetic — e.g., an exponent copied incorrectly; I only charged 2 points per arithmetic error. However, a few students committed the much more grievous sin of thinking that $\int(uv)$ is equal to $\left(\int u\right)\left(\int v\right)$; those students received no partial credit at all.