

Syllogisms

(This material could be inserted before the treatment of Venn diagrams.)

Syllogistic reasoning was developed mainly by the Greek philosopher Aristotle (384-322 BC). Two typical syllogisms are

- (i) All humans are mortal; Socrates is a human; therefore Socrates is mortal
- (ii) All humans are mortal; all philosophers are humans; therefore all philosophers are mortal.

Aristotle was so influential that his theory of syllogisms became the main theory of logic; for over two millennia, few attempted to improve upon it.

But a whole new approach to logic appeared in 1847, with the publication of Augustus DeMorgan's *Formal Logic* and George Boole's *Mathematical Analysis of Logic*. This school of thought — now known by various names, among them “Boolean logic” — soon replaced syllogism as the mainstream of logic. Important further ingredients were added over the next few decades, particularly by Frege, Russell, and Whitehead.

Aristotelian syllogism still can be found in the introductory chapters of many logic textbooks, but it is there mainly for historical reasons. It no longer plays a central role, nor can it even be viewed as a forerunner of modern logic. (Perhaps its appearance at the beginning of logic textbooks misleads some students, who try in vain to see a connection between syllogism and modern logic.) Really, syllogism is just an old-fashioned way of describing some very elementary parts of set theory. That becomes apparent when we abbreviate the two examples about humans and mortality, given a few paragraphs ago:

- (i) $H \subseteq M$, and $s \in H$; therefore $s \in M$.
- (ii) $H \subseteq M$, and $P \subseteq H$; therefore $P \subseteq M$.

Charles Lutwidge Dodgson (1832–1898) did research in several areas of mathematics. None of his mathematical discoveries are nearly as famous as the books that he wrote under the pseudonym of “Lewis Carroll,” particularly *Alice's Adventures in Wonderland* (1865) and *Through the Looking Glass* (1872). Like those books, *Symbolic Logic* (1896) was written for the amusement of children, but this book was also intended to teach the children mathematical logic. It contains many silly syllogisms, intended to teach not-so-silly mathematical principles. Here are a couple of examples:

- (1) No ducks waltz;
- (2) No officers ever decline to waltz;
- (3) all my poultry are ducks.

Abbreviating $D = \{\text{ducks}\}$, $W = \{\text{willing to waltz}\}$, $O = \{\text{officers}\}$, $M = \{\text{my poultry}\}$; we can restate the given information as

- (1) $D \cap W = \emptyset$

$$(2) \ O \subseteq W$$

$$(3) \ M \subseteq D$$

From this information, one of the conclusions we can draw is $M \cap O = \emptyset$; that is, “none of my poultry are officers.”

Here is another example:

(1) I despise anything that cannot be used as a bridge;

(2) Anything that is worth writing an ode to would be a welcome gift to me;

(3) A rainbow will not bear the weight of a wheelbarrow;

(4) Whatever can be used as a bridge will bear the weight of a wheelbarrow;

(5) I would not take, as a gift, a thing that I despise.

Now abbreviate: $D = \{\text{despised by me}\}$, $B = \{\text{usable as bridges}\}$, $O = \{\text{worthy of an ode}\}$, $G = \{\text{I would accept as a gift}\}$, $R = \{\text{rainbows}\}$, $W = \{\text{will bear the weight of a wheelbarrow}\}$. Then the statements can be rewritten as:

$$(1) \ \complement B \subseteq D, \text{ or equivalently } \complement D \subseteq B$$

$$(2) \ O \subseteq G$$

$$(3) \ R \subseteq \complement W, \text{ or equivalently } W \subseteq \complement R$$

$$(4) \ B \subseteq W$$

$$(5) \ D \subseteq (\complement G), \text{ or equivalently } G \subseteq \complement D$$

Putting these together in the right order, we get $O \subseteq G \subseteq \complement D \subseteq B \subseteq W \subseteq \complement R$. We conclude $O \subseteq \complement R$, or equivalently $R \subseteq \complement O$ — thus Lewis Carroll’s conclusion, “Rainbows are not worth writing odes to.”

It should be understood that we are seeking the simplest conclusion that can be reached from *all* the assumptions. Thus, in the preceding example, another conclusion which follows from just *some* of the assumptions is $G \subseteq W$, or “Anything that I would accept as a gift, would bear the weight of a wheelbarrow.” But that’s not the answer we’re trying to find, because it doesn’t use all the given information. Just *why* this answer is any less “useful” than Lewis Carroll’s answer is not entirely clear to me; I’m just describing his rules, not explaining them.

Here is a final example:

(1) Every eagle can fly.

(2) Some pigs cannot fly.

Let $E = \{\text{eagles}\}$, $F = \{\text{creatures that can fly}\}$, $P = \{\text{pigs}\}$. Then the two given pieces of information can be restated:

- (1) $E \subseteq F$
- (2) $P \setminus F$ is not empty — i.e., there is some $x \in P \cap \mathbb{C}F$.

Assumption (1) can be restated as $\mathbb{C}F \subseteq \mathbb{C}E$. Combining that with (2), we conclude that there is some $x \in P \cap \mathbb{C}E$. That is, “there are some pigs that are not eagles.”

The rules of the game are such that we’re allowed to use English grammar, but not common sense. Note that, from the given information, we cannot conclude that any eagles exist! When we say “every eagle can fly,” this really means “if there are any eagles, then they can all fly.” On the other hand, we’re given that “some pigs cannot fly,” which means “there are some pigs that cannot fly” — which implies, as a corollary, that there do exist some pigs.

Exercises. Following are three more of Carroll’s examples. For each of them, rewrite the statements in set-theoretical notation, draw a conclusion in set-theoretical notation (making the simplest conclusion that uses all the given information), and then restate your conclusion in English.

Exercise A:

- (1) No birds other than ostriches are 9 feet high.
- (2) There are no birds in this aviary that belong to anyone but me.
- (3) No ostrich lives on mince-pies.
- (4) I have no birds less than 9 feet high.

Use these abbreviations: $O = \{\text{ostriches}\}$, $T = \{9 \text{ feet or taller}\}$, $A = \{\text{in this aviary}\}$, $B = \{\text{belong to me}\}$, $M = \{\text{live on mince-pies}\}$.

Exercise B:

- (1) No kitten that loves fish is unteachable.
- (2) No kitten without a tail will play with a gorilla.
- (3) Kittens with whiskers always love fish.
- (4) No teachable kitten has green eyes.
- (5) No kittens have tails unless they have whiskers.

Use these abbreviations: $L = \{\text{loves fish}\}$, $U = \{\text{teachable}\}$, $T = \{\text{has a tail}\}$, $P = \{\text{will play with a gorilla}\}$, $W = \{\text{has whiskers}\}$, $E = \{\text{green eyes}\}$.

Exercise C:

- (1) No birds other than peacocks are proud of their tails.
- (2) Some birds that are proud of their tails cannot sing.

Use these abbreviations: $P = \{\text{peacocks}\}$, $T = \{\text{proud of their tails}\}$, $S = \{\text{able to sing}\}$.