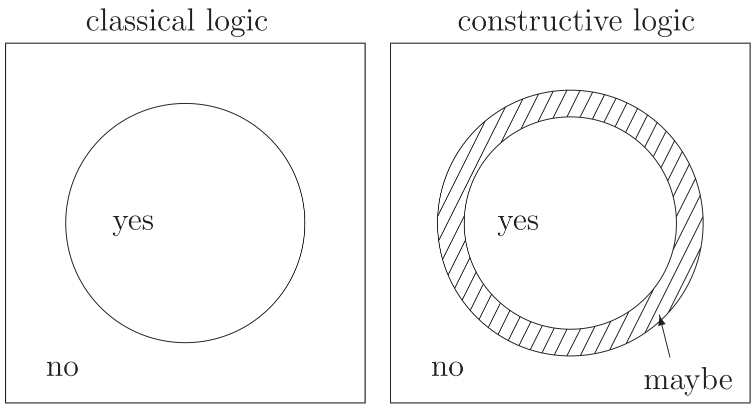
# Chapter 4

# Topologies and interiors (postponable)

*Note to instructors: This is a first draft of a replacement for part or all of Chapter 4. Chapter 4 is technically correct, but I think the presentation can be improved. Topologies can be presented in terms of open sets, interior operations, or several other approaches (closed sets, closure operations, net convergences, etc.). My original version was in terms of open sets, but I'm considering the idea that the interior operations are a better place to start. This is for two reasons: (i) We're actually going to end up using the interior operations, more directly than the open sets, to define the resulting logics. (ii) The definition of an interior operation only involves finite operations, whereas the definition of open sets involves the union of* arbitrarily *many sets — i.e., an infinite operation. That seemed to confuse a few students.*

**4.1.** *Preview.* In classical, two-valued logic, either a thing is or it is not. Thus $P \vee \overline{P}$ is a tautology of classical logic, known as the *Law of the Excluded Middle.* This viewpoint is reflected very simply by set theory: For any set $S$ and any point $x$, either $x$ is a member of $S$ or it is a member of the complement of $S$. Thus $S \cup \complement S = \Omega$.




But in constructive logic, $P \vee \overline{P}$ is *not* a theorem. Between "yes" and "no" there is a *boundary* region of "maybe" (or "we don't know yet"). The middle case is *not* excluded. The middle case is not used as a semantic value, but the cases of "yes" and "no" do not make up everything. This philosophical or logical idea can be represented readily by topology, which is like set theory with boundaries. We represent "yes" (a proposition $S$) and "no" (its negation $\neg S$) as the sets that remain after the boundary is subtracted. The negation $\neg S$ may be strictly smaller than the complement $\complement S$, and so $S \cup \neg S$ might not be equal to $\Omega$.

Topology, introduced in this chapter, is a natural continuation of set theory. However, the results in this chapter are more specialized and can be postponed; these results will not be needed until Chapter 10. And if we postpone that chapter along with this one, neither of them would be needed until 22.14.

*Remarks.* Topology is a major branch of mathematics in its own right; most students majoring in mathematics will take at least one course devoted to topology. It is studied mainly for its insights into shapes, approximations, and continuity. The use of topology that we make in this book — as an approach to constructive logic — is actually unusual. Most logicians would postpone constructive logic until a more advanced course, and then they would study it using Heyting algebras instead of topologies.

## Interiors

**4.2.** An *interior operation* on a set $\Omega$ is an operation that, in some ways, resembles $\cup, \cap, \setminus, \complement, \mathcal{P}$. However, those operations are uniquely determined. The interior operation is not uniquely determined; we can define many different interior operations on a set $\Omega$.

*Definition.* Let $\Omega$ be any set. By an **interior operation** on $\Omega$ we shall mean a rule that assigns to each set $S \subseteq \Omega$, a corresponding set $\text{int}(S) \subseteq \Omega$, satisfying these four conditions:

**(a)** $\text{int}(\Omega) = \Omega$
**(b)** $S \supseteq \text{int}(S)$
**(c)** $\text{int}(\text{int}(S)) = \text{int}(S)$
**(c)** $\text{int}(S \cap T) = \text{int}(S) \cap \text{int}(T)$

for all sets $S, T \subseteq \Omega$.

A set $G \subseteq \Omega$ is called **open** (for this interior operation) if it has the property that $G = \text{int}(G)$. We shall often let $\Sigma$ denote the collection of all open sets; it is called the **topology** corresponding to that interior operation. The pair $(\Omega, \Sigma)$, or just the set $\Omega$ itself, is then called a **topological space**.

**4.3.** Examples.

**a.** Let $\Omega = \{4, 7, 9\}$, and define

| $S$ | $\varnothing$ | $\{4\}$ | $\{7\}$ | $\{9\}$ | $\{4,7\}$ | $\{4,9\}$ | $\{7,9\}$ | $\{4,7,9\}$ |
|---|---|---|---|---|---|---|---|---|
| $\text{int}(S)$ | $\varnothing$ | $\varnothing$ | $\{7\}$ | $\{9\}$ | $\{4,7\}$ | $\{9\}$ | $\{7,9\}$ | $\{4,7,9\}$ |

We may verify (somewhat tediously) that this satisfies the four rules in 4.2. The resulting topology is

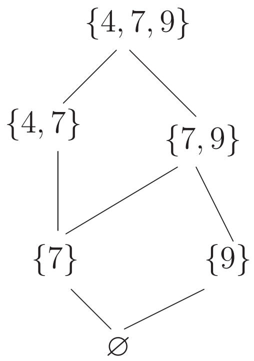
$$\Sigma = \Big\{ \varnothing,\ \{7\},\ \{9\},\ \{4,7\},\ \{7,9\},\ \{4,7,9\} \Big\}.$$

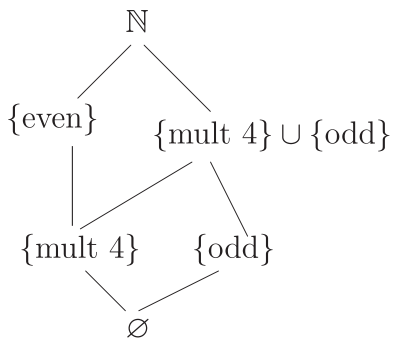
Thus, 6 of the 8 subsets of $\{4,7,9\}$ are open sets; the two sets $\{4\}$ and $\{4,9\}$ are not open. Actually, once we have determined what $\Sigma$ is, we could restate the definition of int this way:

$$\text{int}(S) \text{ is the largest member of } \Sigma \text{ contained in } S.$$

This "largest member rule" works in general; we will discuss it further in 4.6.

The **inclusion diagram**, below, shows the inclusion relations in this topology — i.e., the diagram shows all the open sets, and a downward path goes from any open set to any open subset of that set. Diagrams of this sort are particularly helpful when the topology is finite — i.e., when $\Sigma$ has only finitely many members. We can see at a glance which is the largest member of $\Sigma$ contained in a given set.




**b.** Let $\Omega = \mathbb{N} = \{\text{natural numbers}\}$, and let

$$\Sigma = \Big\{ \mathbb{N},\ \varnothing,\ \{\text{even numbers}\},\ \{\text{multiples of } 4\},\ \{\text{odd numbers}\},\ \{\text{multiples of } 4\} \cup \{\text{odd numbers}\} \Big\}.$$

For each set $S \subseteq \mathbb{N}$, define $\operatorname{int}(S)$ to be the largest member of $\Sigma$ that is contained in $S$. (The fact that there *is* a largest follows from these two properties of $\Sigma$: the empty set is a member of $\Sigma$, and any union of members of $\Sigma$ is also a member of $\Sigma$.) It can be verified that int, defined in this fashion, is indeed an interior operation — i.e., that it satisfies the rules of 4.2.

The resulting topology has only 6 members, so we call it a *finite topology*, even though some of those members are themselves infinite sets. Note that the inclusion diagram for this example is "the same as" the diagram of 4.3.a — i.e., it has the same arrangement of lines. It will turn out that, in later chapters, when we use topologies to determine logics, these two topologies will yield the same logic; the logic is really determined by the pattern of inclusions. The particular choice of sets merely gives us a convenient concretization for that pattern.

**c.** Let $\Omega$ be any set, and define $\operatorname{int}(S) = S$ for every set $S \subseteq \Omega$. This defines an interior operation. The resulting topology is $\Sigma = \mathcal{P}(\Omega)$; it is called the **discrete topology**.

**d.** Let $\Omega$ be any set. Verify that

$$\operatorname{int}(S) = \begin{cases} \varnothing & \text{if } \ S \subsetneqq \Omega, \\ \Omega & \text{if } \ S = \Omega \end{cases}$$

defines an interior operation. It yields the **indiscrete topology**, $\Sigma = \{\varnothing, \Omega\}$. Its inclusion diagram has just two vertices, connected by a vertical line segment.

**e.** (Optional.) Let $\Omega$ be any set. A set $S \subseteq \Omega$ is called **cofinite** if the set $\Omega \setminus S$ is finite. Verify that

$$\operatorname{int}(S) = \begin{cases} S & \text{if } S \text{ is cofinite, i.e., if } \Omega \setminus S \text{ is finite,} \\ \varnothing & \text{otherwise} \end{cases}$$

is an interior operation on $\Omega$. Then $\Sigma = \{S \subseteq \Omega \ : \ S = \varnothing \text{ or } S \text{ is cofinite}\}$ is the resulting **cofinite topology**.

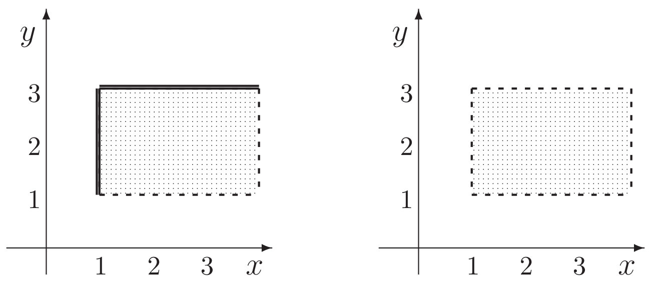
**f.** Let $\Omega$ be any of $\mathbb{R}$ (the real line) or $\mathbb{R}^2$ (the plane) or $\mathbb{R}^3$ (three-dimensional space). Different topologies on $\Omega$ are possible, but the one that is used most often is the **Euclidean topology**, so it is also known as the **usual topology**; it is defined as follows. Let any point $x \in \Omega$ and any set $S \subseteq \Omega$ be given; we say that $x$ is in the interior of $S$ if $x$ has a strictly positive distance from $\Omega \setminus S$ — that is, if

> there exists some strictly positive number $r$ (which may depend on both $x$ and $S$) such that $\{u \in \Omega : \ \mathrm{dist}(x, u) < r\} \subseteq S$.

We then define the interior of $S$ to be all such points.

Here is another way to describe the same interior operation. Say that a point $z$ is in the **boundary** of a set $S$ if there are points in $S$ that are arbitrarily close to $z$ and there are also points in $\Omega \setminus S$ that are arbitrarily close to $z$. Denote the set of boundary points by $\mathrm{bdry}(S)$. Then we obtain $\mathrm{int}(S) = S \setminus \mathrm{bdry}(S)$. That is, if you subtract from $S$ all of its boundary points, what remains is the interior of $S$. Geometrically or graphically, then the interior of $S$ is the "inside" of $S$ — hence the name.

For example, let $S$ be the rectangle $[1, 4) \times (1, 3] = \{(x, y) : \ 1 \le x < 4, \ 1 < y \le 3\}$. The boundary of the set $S$ consists of the four line segments along its sides, but only two of those line segments are part of $S$ itself. We get the interior by deleting those two line segments. This leaves the inside of the rectangle with none of its sides: $\mathrm{int}(S) = (1, 4) \times (1, 3) = \{(x, y) : \ 1 < x < 4, \ 1 < y < 3\}$.




$S = [1, 4) \times (1, 3]$ $\qquad$ $\mathrm{int}(S) = (1, 4) \times (1, 3)$

**4.4.** *Exercises*: Using the Euclidean topology,

**a.** Show that the only open finite set is the empty set.

**b.** Show that any cofinite set is open.

**c.** Give an example of an open set that is not empty or cofinite.

For the exercises below, use the Euclidean topology on $\mathbb{R}$.

**d.** Show that if $x \in \mathrm{int}(S)$, then $x \in (a, b) \subseteq S$ for some interval $(a, b)$ of positive length.

**e.** Show that $\mathrm{int}(\mathbb{Q}) = \varnothing$.

**f.** Give the interior of each of the following sets:

$$(1,5), \quad \{1,5\}, \quad [1,5], \quad \mathbb{Z}, \quad [1,5]\cup(7,11), \quad \mathbb{R}\backslash\mathbb{Q}.$$

**4.5.** *Miscellaneous properties of topologies.* Let $\Omega$ be a set, and let int be an interior operation on $\Omega$. Prove these properties. (Each proof should consist of a few complete sentences, and should make use of one or more of the four properties listed in 4.2.)

**a.** Both $\Omega$ and $\varnothing$ are open sets.

**b.** $G$ is open if and only if there is at least one set $S$ for which $G = \text{int}(S)$.

**c.** If $S$ and $T$ are open sets, then $S \cup T$ and $S \cap T$ are open sets.

**d.** If $S_1, S_2, S_3, \ldots, S_n$ are open sets (where $n$ is some positive integer), then both of the sets $S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n$ and $S_1 \cap S_2 \cap S_3 \cap \cdots \cap S_n$ are also open.

**e.** (This one is harder and is postponable.) Suppose $\{S_\alpha : \alpha \in A\}$ is a collection of open sets. Show that $\bigcup_{\alpha\in A} S_\alpha$ is also open. Here the index set $A$ may be arbitrarily large: it may be infinite; it may even be uncountable.

**f.** (This one is harder and is postponable.) Give an example in which $\{S_\alpha : \alpha \in A\}$ is a collection of open sets and $\bigcap_{\alpha\in A} S_\alpha$ is *not* open. Hint: Among the topologies we've already studied, this is probably easiest using the Euclidean topology on $\mathbb{R}$.

**g.** If $A \subseteq B$ then $\text{int}(A) \subseteq \text{int}(B)$.

**h.** Give an example in which we have $\text{int}(A) \subseteq \text{int}(B)$ but not $A \subseteq B$.

**i.** $\text{int}(S \cup T) \supseteq \text{int}(S) \cup \text{int}(T)$.

**j.** Give an example in which $\text{int}(S \cup T) = \text{int}(S) \cup \text{int}(T)$.

**k.** Give an example in which $\text{int}(S \cup T) \supsetneqq \text{int}(S) \cup \text{int}(T)$.

**4.6.** In 4.2 we defined the topology in terms of the interior. That can be done in the other order, as well:

Let $\Omega$ be any set, and let $\Sigma$ be any collection of subsets of $\Omega$. Then $\Sigma$ might or might not satisfy the following three properties:

(i) $\varnothing, \Omega \in \Sigma$.

(ii) The intersection of any two members of $\Sigma$ is also a member of $\Sigma$.

(iii) The union of arbitrarily many members of $\Sigma$ is also a member of $\Sigma$. That is, if $\{S_\alpha : \alpha \in A\} \subseteq \Sigma$, then $\bigcup_{\alpha\in A} S_\alpha \in \Sigma$.

We saw in some earlier exercises that the topology determined by an interior operation must satisfy these three conditions. We now assert the converse (and leave the proof as an exercise):

> Suppose that $\Sigma$ is a collection of sets satisfying the three conditions above. Then $\Sigma$ is the topology determined by an interior operation. Moreover, that interior operation is given by this rule: $\text{int}(S)$ is equal to the union of all the members of $\Sigma$ that are subsets of $S$.

(The fact that we are taking the union of *at least one* thing follows from the fact that $\varnothing \in \Sigma$.)

Thus $\mathrm{int}(S)$ is the *largest* open subset of $S$, not only in the sense of having more members than any other open subset of $S$, but in the stronger sense that $\mathrm{int}(S)$ is a superset of any other open subset of $S$.

**4.7.** *Example.* Let $\Omega = \mathbb{N} = \{\text{natural numbers}\}$. Define

$$\Sigma \quad = \quad \Big\{\varnothing,\ \{1\},\ \{1,2\},\ \{1,2,3\},\ \{1,2,3,4\},\ \ldots,\ \mathbb{N}\Big\}.$$

That is, a set belongs to $\Sigma$ if either it is all natural numbers, or it is all the natural numbers that lie below some particular natural number.

Verify that this collection $\Sigma$ satisfies the three conditions in 4.6, and thus it is a topology. It is called the **lower set topology** on $\mathbb{N}$. For any set $S \subseteq \mathbb{N}$, then, $\mathrm{int}(S)$ is the largest member of $\Sigma$ that is contained in $S$.

**4.8.** An arbitrarily chosen collection of sets might or might not be a topology. *Exercises*:

**a.** Show that $\Big\{\varnothing, \{1\}, \{2\}, \{1,3\}, \{1,2,3\}\Big\}$ is not a topology on the set $\Omega = \{1,2,3\}$. That is, give an example of this collection violating one of the three rules of 4.6.

**b.** Show that $\Big\{\ \varnothing,\ \{1\},\ \{2\},\ \{1,2\},\ \{2,3\},\ \{3,4\},\ \{1,2,3\},\ \Omega\ \Big\}$ is not a topology on the set $\Omega = \{1,2,3,4\}$.

**c.** $\Sigma = \Big\{\ \varnothing,\ \{1\},\ \{2\},\ \{1,2\},\ \{2,3\},\ \{1,2,3\},\ \{1,2,3,4\}\ \Big\}$ is a topology on $\Omega = \{1,2,3,4\}$. The seven members of $\Sigma$ are the open sets, and thus they are equal to their own interiors. The other nine subsets of $\Omega$ are not open. *Exercise*: List the non-open sets, and their interiors. Also, draw the inclusion diagram for $\Sigma$.

**d.** How many different topologies $\Sigma$ are there on the set $\Omega = \{1,2\}$? List them, and draw the inclusion diagram for each.

**e.** How many different topologies are there on the set $\Omega = \{1,2,3\}$? List them.

**4.9.** *Exercises on comparing interiors*

**a.** Consider $\Omega = \mathbb{R}$ as a topological space; then $S = [0,1)$ is a subset of that space. Find $\mathrm{int}(S)$ if the topology $\Sigma$ is

(i) the discrete topology (4.3.c),

(ii) the indiscrete topology (4.3.d),

(iii) the Euclidean topology on $\mathbb{R}$ (4.3.f).

(Those three questions have different answers.)

**b.** Consider $\Omega = \mathbb{N}$ as a topological space; then $S = \{1,2,5\}$ is a subset of that space. Find $\mathrm{int}(S)$ if the topology $\Sigma$ is

(i) the discrete topology (4.3.c),

(ii) the indiscrete topology (4.3.d),

(iii) the lower set topology (4.7).

(Those three questions have different answers.)