Reference section. Following are two formulas that you were told you did not need to memorize, but did need to understand how to use:

Zadeh:  
\[ [A \rightarrow B] = 1 - \max\{0, [A] - [B]\} = 1 + \min\{0, [B] - [A]\} \]

Sugihara:  
\[ [A \rightarrow B] = \begin{cases} 
\max\{-[A], [B]\} & \text{if } [A] \leq [B] \\
\min\{-[A], [B]\} & \text{if } [A] > [B] 
\end{cases} \]

(12 points) Evaluate:

In the Zadeh interpretation,  
\((\frac{1}{2} \otimes \frac{1}{3}) \oplus \frac{1}{5} = \)

Solution:  
\(\frac{1}{2} \otimes \frac{1}{3} = \min\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{3}\), and then  
\(\frac{1}{3} \oplus \frac{1}{5} = 1 - \max\{0, \frac{1}{3} - \frac{1}{5}\} = \frac{13}{15}\).

In the Sugihara interpretation,  
\((2 \otimes -3) \oplus 5 = \)

Solution:  
\(2 \otimes -3 = \min\{2, -3\} = -3\), and then  
\(-3 \oplus 5 = \max\{- -3, 5\} = 5\).

In the comparative interpretation,  
\((2 \otimes -3) \oplus 5 = \)

Solution:  
\(2 \otimes -3 = \min\{2, -3\} = -3\), and then  
\(-3 \oplus 5 = 5 - (-3) = 8\).

In the powerset interpretation with  \(\Omega = \mathbb{Z}\), evaluate

\([\{1, 2, 3\} \otimes \{3, 4, 5\}] \oplus \{1, 2, 3, 4, 5, 6, 7\} = \)

Solution:  
\(\{1, 2, 3\} \otimes \{3, 4, 5\} = \{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}\), and  
\(\mathcal{C}\{3\} = \mathbb{Z} \setminus \{3\} = \{1, 2, 4, 5, 6, 7, \ldots\}\). Then  
\(\{3\} \oplus \{1, 2, 3, 4, 5, 6, 7\} = \{1, 2, 3, 4, 5, 6, 7\} \cup \mathcal{C}\{3\} = \mathbb{Z}\)  
or  \([\Omega]\).

(6 points) Draw a diagram showing which of the following formula schemes are specializations or generalizations of which others:

\[ A = P \rightarrow Q \quad B = R \rightarrow (S \lor T) \quad C = (U \land V) \rightarrow W \]
\[ D = (H \land I) \rightarrow (J \lor K) \quad \varepsilon = (H \lor I) \rightarrow (J \land K) \quad \mathcal{F} = R \rightarrow (\overline{S} \lor T) \]

\[ \text{Answer.} \]

\[ A = P \rightarrow Q \quad \text{More general} \]

\[ B = R \rightarrow (S \lor T) \quad \varepsilon = (U \land V) \rightarrow W \quad \varepsilon = (H \lor I) \rightarrow (J \land K) \]

\[ \mathcal{F} = R \rightarrow (\overline{S} \lor T) \quad \mathcal{D} = (H \land I) \rightarrow (J \lor K) \quad \text{More specialized} \]

(4 points) Let \( X \) be the formula \( \pi_1 \rightarrow (\pi_2 \lor \neg \pi_3) \). Draw a tree diagram of \( X \).

(5 points) For the same formula \( X \) given above, list all of the subformulas of \( X \). Classify each subformula's relation to \( X \) by listing it in one of the following two columns.

<table>
<thead>
<tr>
<th>order-preserving/isotone/increasing</th>
<th>order-reversing/antitone/decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_1 \rightarrow (\pi_2 \lor \neg \pi_3) ), ( \pi_2 \lor \neg \pi_3 ), ( \neg \pi_3 ), ( \pi_2 )</td>
<td>( \pi_1 ), ( \pi_3 )</td>
</tr>
</tbody>
</table>

The most common error was to omit the formula \( X \), which is an isotone subformula of itself; one-point penalty for that.
Let \( \Omega = \{0, 1, 2, 3\} \) and
\[
\Sigma = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \Omega\}.
\]
Then \( \Sigma \) is a topology on \( \Omega \) (you don’t have to prove that fact; you can take my word for it). Now let \([A] = \{1\}\) and \([B] = \{2\}\) and \([C] = \{3\}\). Evaluate \([C \rightarrow (A \lor B)]\) in the topological interpretation.

**Answer.** \( \Omega \) or \( \{0, 1, 2, 3\} \).

**Solution and partial credit.** Reason as follows:

Noting \([C \rightarrow (A \lor B)] = \text{int} \ (\ [A \lor B] \cup C \ [C])\) is worth 1 point.

\([A \lor B] = \{1\} \cup \{2\} = \{1, 2\}\). That’s worth 1 point.

\(C[C] = \Omega \setminus \{3\} = \{0, 1, 2\}\) is worth 2 points, or \([-C] = \text{int} \ (C[C]) = \text{int} \ (\{0, 1, 2\}) = \{1, 2\}\) is worth 4 points, or \(C[C] = \{0, 3\}\) is worth 5 points.

Finally, showing \([A \lor B] \cup C[C] = \{0, 1, 2, 3\}\) or = \( \Omega \) in your computations is worth 7 points. Putting either of those answers in the answer box (i.e., realizing that you’ve arrived at the answer) is the 8th point.

The answer, in any case, should be a member of \( \Sigma \). I gave zero partial credit for any answer that was not a member of \( \Sigma \).

The most common error was as follows: \([\neg C]\) should be equal to \(\text{int} \ (C[C])\), but some students simply took \([\neg C]\) to be \(C[C]\), which is \(\{0, 1, 2\}\). That’s not a member of \( \Sigma \), so it can’t be the semantic value of a formula, but let’s continue the computation as though it were. Now its complement is \(\{3\}\), and so some students took \([C \rightarrow (A \lor B)] = \text{int} \ (\ [A \lor B] \cup C[C])\) to be equal to \(\text{int} \ (\{1, 2\} \cup \{3\}) = \{1, 2, 3\}\). I gave 5 points for this incorrect answer.

(7 points) Show that \((A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))\) is not a tautology in the
comparative interpretation, by giving suitable values of $[A] = \quad$ and $[B] = \quad$.

Solution. Recall that in the comparative interpretation, $x \odot y = y - x$. Therefore

$$[[ (A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B)) ]] = ((([B] - [A]) - [A]) - ([B] - [A])) = -[A].$$

To get that to be less than 0 (and thus false), we just need

$$[A] = \text{any positive integer}; \quad [B] = \text{any integer}.$$

(8 points) **Church’s chain** is an interpretation that can be described as follows: For semantic values use the numbers $\pm 1$ and $\pm 2$, with $\Sigma_+ = \{+2, +1, -1\}$ and $\Sigma_- = \{-2\}$. Let $\bigvee$ be the maximum and let $\bigwedge$ be the minimum of two numbers. Let $\ominus x = -x$. Finally, define implication by

$$S \ominus T = \begin{array}{c|cccc}
 & T = -2 & -1 & +1 & +2 \\
\hline
S = -2 & +2 & +2 & +2 & +2 \\
S = -1 & -2 & -1 & +1 & +2 \\
S = +1 & -2 & -2 & -1 & +2 \\
S = +2 & -2 & -2 & -2 & +2 \\
\end{array} = \begin{cases} 
-2 \text{ if } S > T \\
-1 \text{ if } S = T = -1 \text{ or } S = T = +1 \\
+1 \text{ if } S = -1 \text{ and } T = +1 \\
+2 \text{ if } S = -2 \text{ or } T = +2 \\
\end{cases}$$

(Thus $S \ominus T$ is true if and only if $S \leq T$.) This logic has enough in common with familiar logics such as classical logic that it is worthy of study. But it has at least a few properties of relevant logic. For instance, $(A \land \neg A) \rightarrow B$ is tautological in classical logic, but you are now to show that $(A \land \neg A) \rightarrow B$ is not a tautology in Church’s chain, by giving suitable values of $[A] = \quad$ and $[B] = \quad$.

Hint: First figure out what are the possible values of $A \land \neg A$. 
Solution. We can compute

\[
\begin{array}{c|c|c|c|c}
[A] & -2 & -1 & +1 & +2 \\
[\neg A] & +2 & +1 & -1 & -2 \\
[A \land \neg A] & -2 & -1 & -1 & -2 \\
\end{array}
\]

We’re trying to find an example in which \[ (A \land \neg A) \rightarrow B \] is false; that requires \[ [A \land \neg A] > [B] \]. Since \[ [A \land \neg A] \] is either \(-2\) or \(-1\), the only way it can be greater than \[ [B] \] is if \[ [A \land \neg A] = -1 \] and \[ [B] = -2 \]. Thus we have either \[ [A] = -1 \text{ and } [B] = -2 \] or \[ [A] = +1 \text{ and } [B] = -2 \].