Name (please print):

Math 250 Test 2, Wednesday 22 October 2008, 4 pages, 50 points, 50 minutes

Reference section. Following are two formulas that you were told you did not need to memorize, but did need to understand how to use:

$$\text{Zadeh:} \quad [\![A \to B]\!] = 1 - \max\{0, [\![A]\!] - [\![B]\!]\} = 1 + \min\{0, [\![B]\!] - [\![A]\!]\}$$

$$\text{Sugihara:} \quad [\![A \to B]\!] = \begin{cases} \max\{-[\![A]\!],\ [\![B]\!]\} & \text{if } [\![A]\!] \le [\![B]\!] \\ \min\{-[\![A]\!],\ [\![B]\!]\} & \text{if } [\![A]\!] > [\![B]\!] \end{cases}$$

(12 points) Evaluate:

In the Zadeh interpretation, $(\frac{1}{2} \textcircled{\wedge} \frac{1}{3}) \textcircled{\rightarrow} \frac{1}{5} = \boxed{}$

Solution: $\frac{1}{2} \textcircled{\wedge} \frac{1}{3} = \min\left\{\frac{1}{2}, \frac{1}{3}\right\} = \frac{1}{3}$, and then $\frac{1}{3} \textcircled{\rightarrow} \frac{1}{5} = 1 - \max\{0, \frac{1}{3} - \frac{1}{5}\} = \boxed{13/15}$.

In the Sugihara interpretation, $(2 \textcircled{\wedge} -3) \textcircled{\rightarrow} 5 = \boxed{}$

Solution: $2 \textcircled{\wedge} -3 = \min\{2, -3\} = -3$, and then $-3 \textcircled{\rightarrow} 5 = \max\{- -3, 5\} = \boxed{5}$.

In the comparative interpretation, $(2 \textcircled{\wedge} -3) \textcircled{\rightarrow} 5 = \boxed{}$

Solution: $2 \textcircled{\wedge} -3 = \min\{2, -3\} = -3$, and then $-3 \textcircled{\rightarrow} 5 = 5 - (-3) = \boxed{8}$.

In the powerset interpretation with $\Omega = \mathbb{Z}$, evaluate

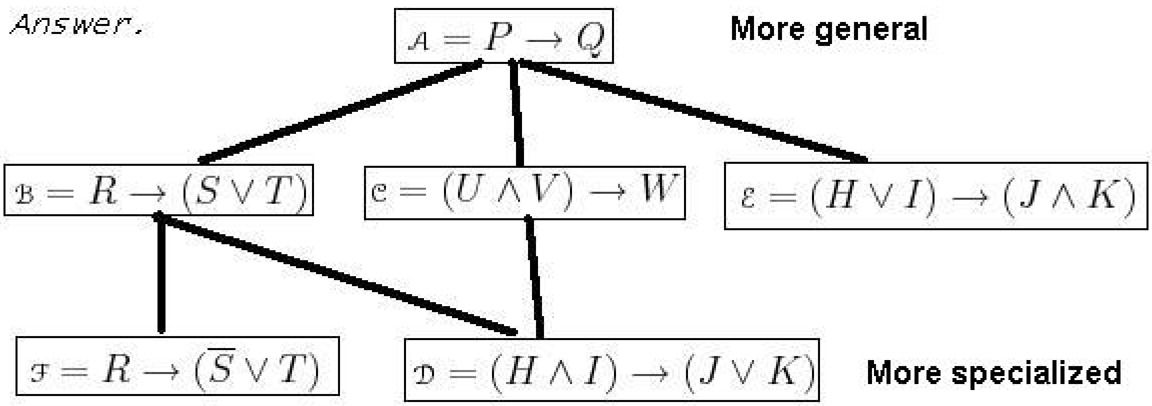
$$\Big(\{1, 2, 3\} \textcircled{\wedge} \{3, 4, 5\}\Big) \textcircled{\rightarrow} \{1, 2, 3, 4, 5, 6, 7\} = \boxed{}$$

Solution: $\{1, 2, 3\} \textcircled{\wedge} \{3, 4, 5\} = \{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$, and $\mathbf{C}\{3\} = \mathbb{Z} \setminus \{3\} = \{1, 2, 4, 5, 6, 7, \ldots\}$. Then $\{3\} \textcircled{\rightarrow} \{1, 2, 3, 4, 5, 6, 7\} = \{1, 2, 3, 4, 5, 6, 7\} \cup \mathbf{C}\{3\} = \boxed{\mathbb{Z}}$ or $\boxed{\Omega}$.

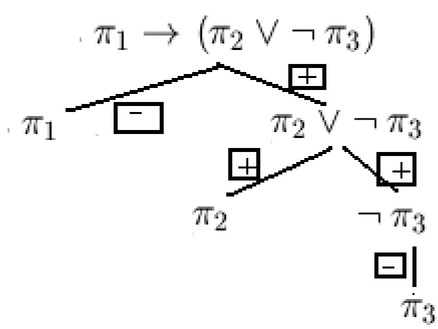
(6 points) Draw a diagram showing which of the following formula schemes are specializations or generalizations of which others:

$$\mathcal{A} = P \to Q \qquad \mathcal{B} = R \to (S \vee T) \qquad \mathcal{C} = (U \wedge V) \to W$$

$\mathcal{D} = (H \wedge I) \to (J \vee K)$ $\mathcal{E} = (H \vee I) \to (J \wedge K)$ $\mathcal{F} = R \to (\overline{S} \vee T)$



(4 points) Let X be the formula $\pi_1 \to (\pi_2 \vee \neg\, \pi_3)$. Draw a tree diagram of X.



(The plus and minus symbols were not required for this problem but were helpful for the next problem.)

(5 points) For the same formula X given above, list all of the subformulas of X. Classify each subformula's relation to X by listing it in one of the following two columns.

order-preserving/isotone/increasing	order-reversing/antitone/decreasing
$\pi_1 \to (\pi_2 \vee \neg\pi_3)$, $\pi_2 \vee \neg\pi_3$, $\neg\pi_3$, π_2	π_1, π_3

The most common error was to omit the formula X, which is an isotone subformula of itself; one-point penalty for that.

(8 points) Let $\Omega = \{0, 1, 2, 3\}$ and

$$\Sigma = \Big\{\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1, 2\},\ \{1, 3\},\ \{2, 3\}\ \{1, 2, 3\},\ \Omega\Big\}.$$

Then Σ is a topology on Ω (you don't have to prove that fact; you can take my word for it). Now let $[\![A]\!] = \{1\}$ and $[\![B]\!] = \{2\}$ and $[\![C]\!] = \{3\}$. Evaluate

$[\![\, \overline{C} \to (A \vee B)\,]\!] =$ $\boxed{}$ in the topological interpretation.

Answer. $\boxed{\Omega}$ or $\boxed{\{0, 1, 2, 3\}}$.
Solution and partial credit. Reason as follows:

Noting $[\![\, \overline{C} \to (A \vee B)\,]\!] = \operatorname{int}\big([\![A \vee B]\!] \cup \mathbf{C}[\![\, \overline{C}\,]\!]\big)$ is worth 1 point.

$[\![A \vee B]\!] = \{1\} \cup \{2\} = \{1, 2\}$. That's worth 1 point.

$\mathbf{C}[\![C]\!] = \Omega \setminus \{3\} = \{0, 1, 2\}$ is worth 2 points, or $[\![\neg C]\!] = \operatorname{int}(\mathbf{C}[\![C]\!]) = \operatorname{int}(\{0, 1, 2\}) = \{1, 2\}$ is worth 4 points, or $\mathbf{C}[\![\, \overline{C}\,]\!] = \{0, 3\}$ is worth 5 points.

Finally, showing $[\![A \vee B]\!] \cup \mathbf{C}[\![\, \overline{C}\,]\!] = \{0, 1, 2, 3\}$ or $= \Omega$ in your computations is worth 7 points. Putting either of those answers in the answer box (i.e., realizing that you've arrived at the answer) is the 8th point.

The answer, in any case, should be a member of Σ. I gave zero partial credit for any answer that was not a member of Σ.

The most common error was as follows: $[\![\, \neg C\,]\!]$ should be equal to $\operatorname{int}(\mathbf{C}[\![C]\!])$, but some students simply took $[\![\, \neg C\,]\!]$ to be $\mathbf{C}[\![C]\!]$, which is $\{0, 1, 2\}$. That's not a member of Σ, so it can't be the semantic value of a formula, but let's continue the computation as though it were. Now its complement is $\{3\}$, and so some students took $[\![\, \overline{C} \to (A \vee B)\,]\!] = \operatorname{int}([\![A \vee B]\!] \cup \mathbf{C}[\![\neg C]\!])$ to be equal to $\operatorname{int}(\{1, 2\} \cup \{3\}) = \{1, 2, 3\}$. I gave 5 points for this incorrect answer.

(7 points) Show that $(A \to B) \to (A \to (A \to B))$ is not a tautology in the

comparative interpretation, by giving suitable values of

$$[\![A]\!] = \boxed{} \qquad \text{and} \qquad [\![B]\!] = \boxed{}.$$

Solution. Recall that in the comparative interpretation, $x \ominus\!\!\rightarrow y = y - x$. Therefore

$$[\![\ (A \to B) \to (A \to (A \to B))\ ]\!] = (([\![B]\!] - [\![A]\!]) - [\![A]\!]) - ([\![B]\!] - [\![A]\!]) = -[\![A]\!].$$

To get that to be less than 0 (and thus false), we just need

$$\boxed{[\![A]\!] = \text{any positive integer}; \quad [\![B]\!] = \text{any integer}}.$$

(8 points) **Church's chain** is an interpretation that can be described as follows: For semantic values use the numbers ± 1 and ± 2, with $\Sigma_+ = \{+2, +1, -1\}$ and $\Sigma_- = \{-2\}$. Let $\circledvee$ be the maximum and let $\circledwedge$ be the minimum of two numbers. Let $\ominus x = -x$. Finally, define implication by

$S \ominus\!\!\rightarrow T =$

T=	−2	−1	+1	+2
$S = -2$	+2	+2	+2	+2
$S = -1$	−2	−1	+1	+2
$S = +1$	−2	−2	−1	+2
$S = +2$	−2	−2	−2	+2

$$= \begin{cases} -2 & \text{if} \quad S > T \\ -1 & \text{if} \quad S = T = -1 \text{ or } S = T = +1 \\ +1 & \text{if} \quad S = -1 \text{ and } T = +1 \\ +2 & \text{if} \quad S = -2 \text{ or } T = +2 \end{cases}$$

(Thus $S \ominus\!\!\rightarrow T$ is true if and only if $S \le T$.) This logic has enough in common with familiar logics such as classical logic that it is worthy of study. But it has at least a few properties of relevant logic. For instance, $(A \wedge \neg A) \to B$ is tautological in classical logic, but you are now to show that $(A \wedge \neg A) \to B$ is **not** a tautology in Church's chain, by giving suitable values of

$$[\![A]\!] = \boxed{} \qquad \text{and} \qquad [\![B]\!] = \boxed{}.$$

Hint: First figure out what are the possible values of $A \wedge \neg A$.

Solution. We can compute

$$\begin{array}{lrrrr} [\![A]\!] & -2 & -1 & +1 & +2 \\ [\![\neg A]\!] & +2 & +1 & -1 & -2 \\ [\![A \wedge \neg A]\!] & -2 & -1 & -1 & -2 \end{array}$$

We're trying to find an example in which $[\![(A \wedge \neg A) \to B]\!]$ is false; that requires $[\![A \wedge \neg A]\!] > [\![B]\!]$. Since $[\![A \wedge \neg A]\!]$ is either -2 or -1, the only way it can be greater than $[\![B]\!]$ is if $[\![A \wedge \neg A]\!] = -1$ and $[\![B]\!] = -2$. Thus we have either $\boxed{[\![A]\!] = -1 \text{ and } [\![B]\!] = -2}$ or $\boxed{[\![A]\!] = +1 \text{ and } [\![B]\!] = -2}$.