Reduction of Order exercises

$$(1)\qquad y'' - \tfrac{1}{x}y' - 4x^2y = -\tfrac{1}{x} - 4x^3, \qquad y_1 = e^{x^2}$$

$$(2)\qquad y'' - (4+\tfrac{2}{x})y' + (4+\tfrac{4}{x})y = x^2 - x - \tfrac{1}{2}, \qquad y_1 = e^{2x}$$

$$(3)\qquad x^2y'' - 2xy' + (x^2+2)y = x^3, \qquad y_1 = x\sin x$$

Solution to (1). $y'' - \frac{1}{x}y' - 4x^2y = -\frac{1}{x} - 4x^3$, with $y_1 = e^{x^2}$. Substitute $y = ue^{x^2}$. Compute

$$\begin{array}{rcllll} y &=& ue^{x^2} & & & \\ y' &=& 2xue^{x^2} & & +u'e^{x^2} & \\ y'' &=& (4x^2+2)ue^{x^2} & & +4xu'e^{x^2} & +u''e^{x^2} \\ & & & & & \\ y'' - \frac{1}{x}y' - 4x^2y &=& (4x^2+2)ue^{x^2} & & +4xu'e^{x^2} & +u''e^{x^2} \\ & & -2ue^{x^2} & & -\frac{1}{x}u'e^{x^2} & \\ & & -4x^2ue^{x^2} & & & \\ \hline &=& 0ue^{x^2} & +(4x-\frac{1}{x})u'e^{x^2} & & +u''e^{x^2} \end{array}$$

which should equal $-\frac{1}{x} - 4x^3$. Divide both sides by e^{x^2}, and substitute $v = u'$, to obtain

$$(4x - \frac{1}{x})v + v' = -(\frac{1}{x} + 4x^3)e^{-x^2}.$$

That's linear in standard form (except for the order of summation on the left side). The integrating factor is

$$\mu(x) = e^{\int(4x-\frac{1}{x})dx} = e^{2x^2 - \ln x} = \frac{1}{x}e^{2x^2}.$$

Multiply the standard form equation through by that, to obtain

$$(4 - \frac{1}{x^2})ve^{2x^2} + \frac{1}{x}v'e^{2x^2} = -(\frac{1}{x^2} + 4x^2)e^{x^2}$$

$$\left(\frac{1}{x}ve^{2x^2}\right)' = -(\frac{1}{x^2} + 4x^2)e^{x^2}$$

Now we need to integrate both sides. This was apparently a very bad choice of an example problem, because that right side is difficult to integrate – perhaps impossible in elementary terms. For what it's worth, the problem I *should* have given you is $y'' - \frac{1}{x}y' - 4x^2y = -4x^2$, with $y_1 = e^{x^2}$. If that had been our problem, the procedure outlined above would now give us

$$\left(\frac{1}{x}ve^{2x^2}\right)' = -4xe^{x^2}$$

and then the right hand side is much easier to integrate. Sorry about that.

Solution to (2). $y'' - (4 + \frac{2}{x})y' + (4 + \frac{4}{x})y = x^2 - x - \frac{1}{2}$ with $y_1 = e^{2x}$.

Substitute $y = ue^{2x}$. Then

$$\begin{array}{rcllll} y &=& ue^{2x} & & \\ y' &=& 2ue^{2x} & +u'e^{2x} & \\ y'' &=& 4ue^{2x} & +4u'e^{2x} & +u''e^{2x} \end{array}$$

$$\begin{array}{rclll} y'' - (4+\frac{2}{x})y' + (4+\frac{4}{x})y &=& 4ue^{2x} & +4u'e^{2x} & +u''e^{2x} \\ & & -(8+\frac{4}{x})ue^{2x} & -(4+\frac{2}{x})u'e^{2x} & \\ & & +(4+\frac{4}{x})ue^{2x} & & \\ \hline &=& 0ue^{2x} & -\frac{2}{x}u'e^{2x} & +u''e^{2x} \end{array}$$

which should be equal to $x^2 - x - \frac{1}{2}$. Substitute $u' = v$ and multiply both sides by e^{-2x}, to obtain

$$-\frac{2}{x}v + v' = (x^2 - x - \frac{1}{2})e^{-2x}$$

which is linear in standard form. The integrating factor is

$$\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = \frac{1}{x^2}.$$

Multiply the standard form equation through by that, to obtain

$$-\frac{2}{x^3}v + \frac{1}{x^2}v' = (1 - \frac{1}{x} - \frac{1}{2x^2})e^{-2x}$$

$$\left(\frac{1}{x^2}v\right)' = (1 - \frac{1}{x} - \frac{1}{2x^2})e^{-2x}$$

Again, I seem to have messed up – the right side is too difficult to integrate.

Solution to (3). $x^2y'' - 2xy' + (x^2+2)y \;=\; x^3$, with $y_1 \;=\; x\sin x$.

Substitute

$$\begin{array}{rcrrrrr} y &=& ux\sin x & & & & \\ y' &=& u\sin x & +ux\cos x & +u'x\sin x & & \\ y'' &=& -ux\sin x & +2u\cos x & +2u'\sin x & +2u'x\cos x & +u''x\sin x \end{array}$$

$$\begin{array}{rcrrrrr} x^2y''-2xy' & & & & & & \\ +(x^2+2)y &=& -ux^3\sin x & +2ux^2\cos x & +2u'x^2\sin x & +2u'x^3\cos x & +u''x^3\sin x \\ & & -2xu\sin x & -2ux^2\cos x & -2u'x^2\sin x & & \\ & & \multicolumn{3}{l}{+(x^3+2x)u\sin x} & & \\ \hline &=& 0ux^3\sin x & +0ux^2\cos x & 0u'x^2\sin x & +2u'x^3\cos x & +u''x^3\sin x \end{array}$$

which should equal x^3. Substituting $v = u'$ yields

$$2vx^3\cos x + v'x^3\sin x \;=\; x^3$$

$$v' + 2(\cot x)v \;=\; \csc x$$

linear in standard form, with integrating factor

$$\mu(x) \;=\; e^{\int 2\cot x\,dx} \;=\; e^{2\ln\sin x} \;=\; \sin^2 x.$$

Multipying that through the standard form equation yields

$$(\sin^2 x)v' + 2(\sin x)(\cos x)v \;=\; \sin x$$

$$\Big((\sin^2 x)v\Big)' \;=\; \sin x$$

Integrating,

$$(\sin^2 x)v \;=\; -\cos x + c_1$$

Multiply through by $\csc^2 x$ to isolate v;

$$u' \;=\; v \;=\; -\csc x\cot x + c_1\csc^2 x$$

Integrating again,

$$u \;=\; \csc x - c_1\cot x + c_2$$

and finally

$$y \;=\; uy_1 \;=\; (\csc x - c_1\cot x + c_2)x\sin x$$

$$\boxed{y \;=\; x - c_1x\cos x + c_2x\sin x}$$