

Given a sequence  $(s_n)_{n \in \mathbb{N}}$  calculate the limit  $\lim_{n \rightarrow \infty} s_n$  if

1.  $s_n = \frac{2n+3}{5n-1}$ .

**Solution:**  $\frac{2}{5}$

2.

$$s_n = \begin{cases} n & n \leq 100 \\ \frac{1}{n} & n > 100. \end{cases}$$

**Solution:** 0

3.  $s_n = \frac{n + \sin(n)}{\sqrt{n^3}}$ .

**Solution:** 0

4.  $s_n = \frac{2n + (-1)^n}{n - (-1)^n}$ .

**Solution:** 2

5.  $s_n = \frac{e^{-\cos \sqrt{n}} \sin \sqrt{n}}{\ln n}$ .

**Solution:** 0

6.  $s_n = \sqrt{n^2 + 3n + 1} - n$ .

**Solution:**  $\frac{3}{2}$

7.  $s_n = (-1)^n \sin(\pi n)$ .

**Solution:** 0

8.

$$s_n = \left( \frac{n+1}{n} \right)^n .$$

**Solution:**  $e$

9.

$$s_n = n^{\frac{1}{n}} .$$

**Solution:** 1

10.

$$s_n = n \ln \left( \sin \frac{1}{n} \right) .$$

**Solution:**  $-\infty$

11.

$$s_n = \frac{3^n}{n!} .$$

**Solution:** 0

Determine whether the series is convergent or divergent. If convergent, find its sum.

12.

$$\sum_{n=3}^{\infty} \frac{1+5^n}{7^n} .$$

**Solution:** Convergent and equal to  $\frac{188}{147}$

13.

$$\sum_{n=2}^{\infty} \frac{n}{n^2 - 1}.$$

**Solution:** Divergent

14.

$$\sum_{n=2}^{\infty} \frac{3}{n(n+3)}.$$

**Solution:** Convergent and equal to  $\frac{11}{6}$

Determine whether the infinite series is convergent, or divergent. Show your reasoning. In particular, make clear which of the several tests you are using.

15.

$$\sum_{n=5}^{\infty} \frac{n-4}{n^2+2}$$

**Solution:** Divergent.

16.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+3}}{n^2+1}$$

**Solution:** Convergent.

17.

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)(n+3)}$$

**Solution:** Convergent.

18.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 5n + 6}}$$

**Solution:** Divergent.

19.

$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

**Solution:** Convergent.

20.

$$\sum_{n=1}^{\infty} \frac{2^{n/2}}{n!}$$

**Solution:** Convergent.

21.

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$$

**Solution:** Divergent.

22.

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^3$$

**Solution:** Divergent.

23.

$$\sum_{n=2}^{\infty} \frac{(1 + \frac{1}{n})^n}{n \ln n}$$

**Solution:** Divergent.

24.

$$\sum_{n=1}^{\infty} \frac{|\sin(n^2)|}{n^2}$$

**Solution:** Convergent.

25.

$$\sum_{n=1}^{\infty} \frac{3^{(n+1)/4}}{4^n}$$

**Solution:** Convergent.

26.

$$\sum_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^n}{n^2}$$

**Solution:** Convergent.

27.

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

**Solution:** Divergent.

28.

$$\sum_{n=1}^{\infty} \sin \left( \frac{1}{n^2} \right)$$

**Solution:** Convergent.

29.

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

**Solution:** Divergent.

30.

$$\sum_{n=1}^{\infty} \sqrt{\frac{n!}{2^n}}$$

**Solution:** Divergent.

31.

$$\sum_{n=1}^{\infty} \cos \left( \frac{1}{n^2} \right)$$

**Solution:** Divergent.

32.

$$\sum_{n=1}^{\infty} n \sin \left( \frac{1}{n^2} \right)$$

**Solution:** Divergent.

33.

$$\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{2} - \frac{1}{n}\right)$$

**Solution:** Divergent.

34.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n!}}{n^{n/2}}$$

**Solution:** Convergent.

35.

$$\sum_{n=1}^{\infty} \frac{\ln(n!)}{n^3}$$

**Solution:** Convergent.

36.

$$\sum_{n=1}^{\infty} \frac{n^{10}}{1 + 2^n}$$

**Solution:** Convergent.

37.

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n^2)}$$

**Solution:** Divergent.

38.

$$\sum_{n=1}^{\infty} \frac{1}{(\ln(n))^2}$$

**Solution:** Divergent.

39.

$$\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^2}$$

**Solution:** Convergent.

Determine whether the series is absolutely convergent, or conditionally convergent. Also, determine whether the series is convergent, or divergent. Show your reasoning. If you use the Alternating-Series Test, make sure that you verify that the requirements for the Alternating-Series Test are satisfied.

40.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$$

**Solution:** Conditionally convergent.

41.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{1 + 2^{-n}}$$

**Solution:** Divergent.

42.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

**Solution:** Conditionally convergent.

43.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{2n}}{n!}$$

**Solution:** Absolutely convergent.

44.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{1 + \frac{1}{n}}$$

**Solution:** Divergent.

45.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{e^n}$$

**Solution:** Absolutely convergent.

46.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{n^n}$$

**Solution:** Absolutely convergent.

47.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\ln n)^2}{n!}$$

**Solution:** Absolutely convergent.

48.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\ln n)^2}{\sqrt{n}}$$

**Solution:** Conditionally convergent.

For each power series determine the radius of convergence and the interval of convergence.

49.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

**Solution:** Radius = 1, Interval =  $(-1, 1]$

50.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n$$

**Solution:** Radius = 1, Interval =  $[-1, 1]$

51.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} x^n$$

**Solution:** Radius = 3, Interval =  $(-3, 3)$

52.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} x^n$$

**Solution:** Radius =  $\infty$ , Interval =  $(-\infty, \infty)$

53.

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n}$$

**Solution:** Radius = 2, Interval =  $(-2, 2)$

54.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^n} x^n$$

**Solution:** Radius =  $\frac{2}{3}$ , Interval =  $(-\frac{2}{3}, \frac{2}{3})$

55.

$$\sum_{n=1}^{\infty} (1 + 2/n)^n x^n$$

**Solution:** Radius = 1, Interval =  $(-1, 1)$

56.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}} (x - 2)^n$$

**Solution:** Radius = 1, Interval =  $(1, 3]$

57. Suppose that the power series  $\sum_{n=0}^{\infty} a_n x^n$  is convergent at  $x = -3$  and divergent at  $x = 5$ . What can be said about

- (a) convergence at  $x = -2$ ?
- (b) absolute convergence at  $x = 2$ ?
- (c) convergence at  $x = 6$ ?
- (d) convergence at  $x = 3$ ?
- (e) divergence at  $x = -5$ ?

**Solution:** (a) Convergent, (b) convergent, (c) divergent, (d) Nothing, (e) Nothing.

58. Simplify

$$(1 - x)(1 + x + x^2 + \cdots + x^N)$$

and afterwards show that  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  whenever  $|x| < 1$ .

**Solution:**  $(1 - x)(1 + x + x^2 + \cdots + x^N) = 1 - x^{N+1}$ , then pass to the limit.

59. Suppose that  $\sum_{n=0}^{\infty} a_n$  is known to be convergent and  $a_n \neq 0$  for each  $n$ . Show that  $\sum_{n=0}^{\infty} \frac{1}{a_n}$  diverges.

**Solution:** By divergence test  $\lim_{n \rightarrow \infty} a_n = 0$ , and therefore  $\lim_{n \rightarrow \infty} \frac{1}{a_n}$  diverges. Hence  $\sum_{n=0}^{\infty} \frac{1}{a_n}$  diverges.

60. Show that if  $\lim_{n \rightarrow \infty} n a_n = 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Solution:** Since  $\lim_{n \rightarrow \infty} n a_n = 1$ , there exists sufficiently large  $N$  such that  $n a_n \geq \frac{1}{2}$  for each  $n \geq N$ . Equivalently,  $a_n \geq \frac{1}{2n}$  for each  $n \geq N$ . Since  $\sum_{n=1}^{\infty} \frac{1}{2n}$  diverges by  $p$ -test,  $\sum_{n=1}^{\infty} a_n$  diverges by the comparison test.

Show that if  $\sum_{n=1}^{\infty} a_n$  with  $a_n \geq 0$  converges, then  $\sum_{n=1}^{\infty} a_n^2$  converges as well.

**Solution:** Since  $\sum_{n=1}^{\infty} a_n$  converges,  $\lim_{n \rightarrow \infty} a_n = 0$  by the divergence test. Thus, there is sufficiently large  $N$  such that  $0 \leq a_n < 1$  for any  $n \geq N$ . After multiplying these inequalities by  $a_n \geq 0$ , we have  $0 \leq a_n^2 \leq a_n$ . Since  $\sum_{n=1}^{\infty} a_n$  converges,  $\sum_{n=1}^{\infty} a_n^2$  converges by the comparison test.