

## Review 3.

1. Compute the arc length of the semicubical parabola  $y^2 = x^3$  from the coordinate origin to the point  $x = 4$ .

**Solution:**  $\frac{8}{27}(10\sqrt{10} - 1)$

2. Compute the arc length of parabola  $y = 2\sqrt{x}$  from  $x = 0$  to  $x = 1$ .

**Solution:**  $\sqrt{2}\ln(1 + \sqrt{2})$

3. Find the arc length of the curve  $x = e^y$  lying between the points  $(1, 0)$ ,  $(e, 1)$ .

**Solution:**  $\sqrt{1 + e^2} - \sqrt{2} + \ln \frac{(\sqrt{1+e^2}-1)(\sqrt{2}+1)}{e}$

4. Find the arc length of the curve  $y = \ln x$  from  $y = \frac{1}{2} \ln 3$  to  $\frac{3}{2} \ln 2$ .

**Solution:**  $1 + \frac{1}{2} \ln \frac{3}{2}$

5. Find the arc length of the curve  $y = \arcsin(e^{-x})$  from  $x = 0$  to  $x = 1$ .

**Solution:**  $\ln(e + \sqrt{e^2 - 1})$

6. Find the arc length of the curve  $x = \frac{1}{4}y^2 - \frac{1}{2} \ln y$  from  $y = 1$  to  $y = e$ .

**Solution:**  $\frac{1}{4}(e^2 + 1)$

7. Find the arc length of one arc of the cycloid  $x = (t - \sin t)$ ,  $y = (1 - \cos t)$ , that is, for  $t \in [0, 2\pi]$ .

**Solution:** 8

8. Find the arc length of  $x = (\cos t + t \sin t)$ ,  $y = (\sin t - t \cos t)$  from  $t = 0$  to  $t = 4$ .

**Solution:** 2

9. Find the arc length of  $x = \frac{9}{5} \cos^3 t$ ,  $y = \frac{9}{4} \sin^3 t$  for  $t \in [0, 2\pi]$ .

**Solution:**  $\frac{9}{5}$ .

10. Find the arc length of the first turn of the spiral  $r = 2\theta$ .

**Solution:**  $2\pi\sqrt{1+4\pi^2} + \ln(2\pi + \sqrt{1+4\pi^2})$ .

11. Find the length of the cardioid  $r = 1 + \cos \theta$ .

**Solution:** 8

12. Find the length of the hyperbolic spiral  $r\theta$  from the point  $(2, \frac{1}{2})$  to the point  $(\frac{1}{2}, 2)$ .

**Solution:**  $\frac{\sqrt{5}}{2} + \ln \frac{3+\sqrt{5}}{2}$

13. Find the arc length of the curve  $\theta = \frac{1}{2} \left(r + \frac{1}{r}\right)$  from  $r = 1$  to  $r = 3$ .

**Solution:**  $\frac{1}{2}(4 + \ln 3)$

14. Find the area of a surface formed by the rotation of the curve  $9y^2 = x(3-x)^2$  for  $0 \leq x \leq 3$  around the  $x$ -axis.

**Solution:**  $3\pi$

15. Find the area of the parabolic mirror, that is rotationally symmetric with respect to  $x$  axis. In the  $xy$ -plane the mirror has the shape of a parabola that passes through the origin and it ends at the point  $(1, 8)$ .

**Solution:**  $\frac{8\sqrt{2}\pi}{3}(3^{\frac{3}{2}} - 1)$

16. Find the area of a surface formed by the rotation of the curve  $y = \sin x$  for  $0 \leq x \leq \pi$  around the  $x$ -axis.

**Solution:**  $2\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$

17. Find the area of the surface formed by rotation, about  $x$ -axis, of an arc of the curve  $y = e^{-x}$ , from  $x = 0$  to  $x = \infty$ . Notice that the integral is improper.

**Solution:**  $\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$

18. Find the area of the surface formed by the rotation of  $y = \cosh x$  about the  $x$ -axis from  $x = 0$  to  $x = 1$ .  
*Hint.*  $\cosh^2 x - \sinh^2 x = 1$ .

**Solution:**  $\frac{\pi}{4}(e^2 + e^{-2} + 4) = \frac{\pi}{2}(2 + \sinh 2)$

19. Find the area of the surface formed by rotation of the cycloid  $x = (t - \sin t)$ ,  $y = (1 - \cos t)$  with  $0 \leq t \leq 2\pi$  about  $y$ -axis.

**Solution:**  $\frac{128}{5}\pi$

20. Determine the area of the surface formed by the rotation of  $r^2 = \cos 2\phi$ ,  $0 \leq \phi \leq \frac{\pi}{4}$  about  $x$ -axis. Notice that we did not cover this formula, but you should be able to derive it.

**Solution:**  $\pi(2 - \sqrt{2})$

21. A vertical isosceles triangle with base 8 and altitude 3 is submerged vertex downwards in water so that its base is on the surface of the water. Find the overall force of the water on the triangle.

**Solution:**  $12\rho g$

22. A vertical dam has the shape of a trapezoid. Calculate the force of the water on the full dam if we know that the upper base has 70 m, the lower base has 50 m, and the height is 20 m.

**Solution:**  $\rho g \frac{170^2 20^2}{6} \approx \rho g 11.3 \cdot 10^3$

23. A trough is filled with water and its vertical ends have the shape of the parabolic region, symmetric with respect to  $y$ -axis, upper edge is 8 m long and the height is 4 m. Find the hydrostatic force on one end of the trough.

**Solution:**  $\frac{512}{15}\rho g$

24. Find the center of gravity of an arc of the semicircle  $x^2 + y^2 = 9$ ,  $y \leq 0$ .

**Solution:**  $\bar{x} = 0; \bar{y} = -\frac{6}{\pi}$

25. Find the center of gravity of an area bounded by the curves  $y = x^2$  and  $y = \sqrt{x}$ .

**Solution:**  $\bar{x} = \bar{y} = \frac{9}{20}$

26. Find the coordinates of the center of gravity of an area bounded by the cycloid  $x = (t - \sin t)$ ,  $y = (1 - \cos t)$ ,  $(0 \leq t \leq 2\pi)$  and the  $x$ -axis. We did not cover this formula, but you should be able to solve the problem anyway.

**Solution:**  $\bar{x} = \pi, \bar{y} = \frac{5}{6}$

27. Find the area contained inside Bernulli's lemniscate  $r^2 = \cos 2\phi$ ,

**Solution:**  $\frac{1}{4}$

28. Find the area of the cardioid  $r = (1 + \cos \phi)$ .

**Solution:**  $\frac{3}{2}\pi$

29. Find the area contained between the first and second turn of Archimedes' spiral  $r = \phi$ .

**Solution:**  $8\pi^3$

30. Find the area that lies inside  $r = 3 \sin \theta$  and outside  $r = 2 - \sin \theta$ .

**Solution:**  $3\sqrt{3}$

31. Find the area that lies inside  $r^2 = 3 \sin 2\theta$  and  $r^2 = 3 \cos 2\theta$ .

**Solution:**  $\frac{1}{8}\pi - \frac{1}{4}$

32. Find the area inside the larger loop and outside the smaller loop of  $r = \frac{1}{2} + \cos \theta$ .

**Solution:**  $\frac{1}{4}(\pi + 3\sqrt{3})$

33. You should be able to convert between polar and Cartesian coordinates, and plot curves polar coordinates.

**Solution:** No answer.

34. You should be able to match curve with its equation.

**Solution:** No answer.

35. You should be able to convert parametric equation into nonparametric one and sketch the curve.

**Solution:** No answer.

36. Find the tangent line to the curve  $x = \tan \theta$ ,  $y = \frac{1}{\cos \theta}$  at the point  $(0, \sqrt{2})$

**Solution:**  $y - \sqrt{2} = (\sqrt{2}/2)(x - 1)$

37. Find  $dy/dx$  and  $d^2y/dx^2$  for  $x = \cos 2t$ ,  $y = \cos t$ ,  $0 < t < \pi$ .

**Solution:**  $dy/dx = \frac{1}{4 \cos t}$ ,  $d^2y/dx^2 = -\frac{1}{16 \cos^3 t}$

38. Find the equations of tangents to the curve  $x = 3t^2 + 1$ ,  $y = 2t^3 + 1$  that pass through the point  $(4, 3)$ .

**Solution:**  $y = x - 1$ ,  $y = -2x + 11$

39. Solve the differential equation  $xy' - y = y^3$ .

**Solution:**  $x = \frac{Cy}{\sqrt{1+y^2}}$

40. Find the solution of the differential equation  $(1 + e^x)yy' = e^x$  such that  $y = 1$  when  $x = 0$ .

**Solution:**  $y(x) = \sqrt{2 \ln \frac{\sqrt{e}(1+e^x)}{2}}$

41. Find the solution of the differential equation  $y' \sin x = y \ln y$  such that  $y = 1$  when  $x = \frac{\pi}{2}$ .

**Solution:**  $y(x) = 1$ .

42. Solve the differential equation  $y' = x + 5y$ .

**Solution:**  $y(x) = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$

43. Solve the differential equation  $xy' - 4y = x^4e^4$ .

**Solution:**  $y(x) = x^4 \left( \int \frac{e^x}{x} dx + C \right)$

44. Solve the differential equation  $2xy' + y = 6x$  with  $y(4) = 20$ .

**Solution:**  $y(x) = 2x + \frac{24}{\sqrt{x}}$

45. A bacteria culture contains 20 cells initially and grows at a rate proportional to its size. After an hour the population has increased to 360 cells.

- (a) Find the number of bacteria after  $t$  hours.
- (b) Find the number of bacteria after 4 hours.
- (c) Find the rate of growth after 4 hours.
- (d) When will the population reach 10,000?

**Solution:** (a)  $200(3.24)^t$ ; (b)  $\approx 22,040$ ; (c)  $\approx 25,910$  bacteria/hour; (d)  $(\ln 50)/(\ln 3.24)$

46. Cobalt-60 has a half life of 5.24 years.

- (a) Find the mass that remains from a 100-mg sample after 20 years.
- (b) How long would it take for the mass to decay to 1mg?

**Solution:** (a)  $100 \cdot 2^{-20/5.24}$ ; (b)  $5.24 \frac{\ln 100}{\ln 20}$

47. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.

- (a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after  $t$  years.
- (b) How long will it take for the population to increase to 5000?

**Solution:** (a)  $\frac{10,000}{1+24(11/36)^t}$ ;  $t = \frac{\ln 24}{\ln(36/11)}$

48. Assume that the carrying capacity for US population is 4 billion.

- (a) Use the fact that the population was 250 million in 1990 to formulate a logistic model for the US population.

- (b) Determine the value  $k$  in your model by using the fact that the population in 2000 was 275 million.
- (c) Use your model to predict US population in the years 2100 and 2200.
- (d) Use your model to predict the year in which the US population will exceed 350 million.

**Solution:** (a)  $\frac{4000}{1+15e^{-kt}}$  (in millions); (b)  $k = \frac{1}{10} \ln \frac{165}{149}$ ; (c)  $2100 \approx 680$ ,  $2200 \approx 1449$ ; (d) 2026