

For each power series determine the radius of convergence and the interval of convergence.

1.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

**Solution:** Radius = 1, Interval =  $(-1, 1]$

2.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n$$

**Solution:** Radius = 1, Interval =  $[-1, 1]$

3.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} x^n$$

**Solution:** Radius = 3, Interval =  $(-3, 3)$

4.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} x^n$$

**Solution:** Radius =  $\infty$ , Interval =  $(-\infty, \infty)$

5.

$$\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$$

**Solution:** Radius = 2, Interval =  $(-2, 2)$

6.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^n} x^n$$

**Solution:** Radius =  $\frac{2}{3}$ , Interval =  $(-\frac{2}{3}, \frac{2}{3})$

7.

$$\sum_{n=1}^{\infty} (1 + 2/n)^n x^n$$

**Solution:** Radius = 1, Interval =  $(-1, 1)$

8.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}} (x - 2)^n$$

**Solution:** Radius = 1, Interval =  $(1, 3]$

9.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}} (x - 2)^n$$

**Solution:** Radius = 1, Interval =  $(1, 3]$

10.

$$\sum_{n=1}^{\infty} 3^{n^2} x^{n^2}$$

*Hint.* Use directly Root, or Ratio test, do not use the formula for  $R$ .

**Solution:** Radius =  $\frac{1}{3}$ , Interval =  $(-\frac{1}{3}, \frac{1}{3})$

11.

$$\sum_{n=1}^{\infty} x^{n!}$$

**Solution:** Radius = 1, Interval =  $(-1, 1)$

12.

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} x^n$$

**Solution:** Radius =  $\frac{1}{e}$ , Interval =  $(-\frac{1}{e}, \frac{1}{e})$

13. Suppose that the power series  $\sum_{n=0}^{\infty} a_n x^n$  is convergent at  $x = -3$  and divergent at  $x = 5$ . What can be said about

- (a) convergence at  $x = -2$ ?
- (b) absolute convergence at  $x = 2$ ?
- (c) convergence at  $x = 6$ ?
- (d) convergence at  $x = 3$ ?
- (e) divergence at  $x = -5$ ?

**Solution:** (a) Convergent, (b) convergent, (c) divergent, (d) Nothing, (e) Nothing.

14. Expand the function  $\sin x$  into power series, find the interval of convergence, and prove that  $\sin x$  is equal to its power series.

**Solution:**  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ ,  $R = \infty$ ,  $|R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!} \rightarrow 0$  as  $N \rightarrow \infty$ .

15. Expand the function  $\cosh x$  into power series, find the interval of convergence, and prove that  $\cosh x$  is equal to its power series.

**Solution:**  $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ ,  $R = \infty$ ,  $|R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!} \max\{\cosh x, |\sinh x|\} \rightarrow 0$  as  $N \rightarrow \infty$ .

Expand the following functions into power series and find the radius of convergence. You can either use geometric series method, known expansions, or derivatives. You don't have to analyze the remainder.

16.  $f(x) = \frac{2x-3}{(x-1)^2}$ .

**Solution:**  $f(x) = -\sum (n+3)x^n$ ,  $R = 1$ .

17.  $f(x) = \frac{3x-5}{x^2-4x+3}$ .

**Solution:**  $f(x) = -\sum \left(1 + \frac{2}{3^{N+1}}\right) x^n$ ,  $R = 1$ .

18.  $f(x) = xe^{-2x}$ .

**Solution:**  $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{n-1}}{(n-1)!} x^n$ ,  $R = \infty$ .

19.  $f(x) = e^{x^2}$ .

**Solution:**  $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$ ,  $R = \infty$ .

20.  $f(x) = \cos 2x$ .

**Solution:**  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$ ,  $R = \infty$ .

21.  $f(x) = \frac{x}{9+x^2}$ .

**Solution:**  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} x^{2n+1}$ ,  $R = 3$ .

22.  $f(x) = \ln(1+x-2x^2)$ .

**Solution:**  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^n - 1}{n} x^n, R = \frac{1}{2}.$

23.  $f(x) = (1+x) \ln(1+x)$  *Hint.* Differentiate twice.

**Solution:**  $f(x) = x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} x^n, R = 1.$

24.  $f(x) = \arctan x.$

**Solution:**  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, R = 1.$

25. Using the power series expansions find

$$\int \frac{\sin x}{x} dx$$

**Solution:**  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1} + C.$

26. Using the power series expansions find

$$\int \frac{\ln(1+x)}{x} dx$$

**Solution:**  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} x^n + C.$

Write the first three nonzero terms of the expansion of the following functions in powers of  $x$ .

27.  $f(x) = \tan x$

**Solution:**  $x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$

28.  $f(x) = e^x \sin x$

**Solution:**  $x + x^2 + \frac{x^3}{3} + \cdots$ .

29. What is the magnitude of the error if we put approximately

$$e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}.$$

*Hint.*  $e = e^x$  when  $x = 1$ . Use the estimate for remainder to estimate the error.

**Solution:**  $|R_4(1)| \leq \frac{e}{5!} < \frac{1}{40}$ .

30. How many terms do we have to take of the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

to calculate  $\sin \frac{\pi}{12}$  to four decimal places?

**Solution:** Since  $|R_4(\frac{\pi}{12})| \leq \frac{1}{5!}(\frac{\pi}{12})^5 < \frac{1}{10^5}$ , we need two terms, i.e.  $x - \frac{x^3}{3!}$ .

31. For what value of  $x$  does the approximate formula  $\sin x \approx x$  yield an error that does not exceed 0.01?

**Solution:** Since  $|R_2(x)| \leq \frac{1}{3!}|x|^3$ , we want  $\frac{1}{3!}|x|^3 < 0.01$ , that is,  $|x| < 0.39$ .