Name: **ANSWER KEY**
Math 155B Test 4, Thurs 1 Dec 2011, 4 pages, 50 points, 75 minutes.

High score, 50/50, 3 people. (It would have been four people, but I had to penalize one student a point for omitting his name from an otherwises perfect paper!)

Median score, 43/50.

Mean score, 41.63/50. That's 83.27 percent.

(6 points) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^3}$. $R = \boxed{}$

Solution. First rewrite the series as $\sum_{n=1}^{\infty} \frac{3^n}{n^3}(x-\frac{1}{3})^n = \sum_{n=1}^{\infty} c_n(x-\frac{1}{3})^n$ where $c_n = 3^n/n^3$. Then (by the method of page 764 problem 40, which was discussed in class)

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \boxed{1/3}.$$

I also gave full credit for an answer of $\boxed{|x-\frac{1}{3}| < \frac{1}{3}}$.

The two most common errors were

- overlook the 3 inside the parenthesis in the problem, and treat the problem as though it involved $c_n = 1/n^3$ (should be $c_n = 3^n/n^3$). Result is $R = 1$.
- compute correctly to arrive at the interval $-\frac{1}{3} < x - \frac{1}{3} < \frac{1}{3}$, and then add $\frac{1}{3}$ to obtain $0 < x < \frac{2}{3}$, and incorrectly conclude that the answer is $R = \frac{2}{3}$.

I gave 4 points for either of those answers.

(6 points) Find the radius of convergence $R = \boxed{}$ for the series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Solution. You should recognize this as the power series for $\cos x$, which – as you should have memorized – has radius of convergence $\boxed{\infty}$. However, if you didn't recall that, here is a way to do this problem:

Substitute $u = x^2$. Then the series becomes

$$1 - \frac{u}{2!} + \frac{u^2}{4!} - \frac{u^3}{6!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} u^n \;=\; \sum_{n=0}^{\infty} c_n u^n$$

where $c_n = (-1)^n/(2n)!$. Hence

$$R \;=\; \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \;=\; \lim_{n\to\infty} (2n+1)(2n+2) \;=\; \boxed{\infty}.$$

I gave 5 points for an answer of "all real numbers," which is a correct statement of the *interval* of convergence but is not equal to the *radius* of convergence. Partial credit for other answers depended on how much understanding (or lack thereof) the work showed.

(8 points) Write the first 4 terms in the power series expansion (centered at $x = 0$) for the function $g(x) = 1/\sqrt{1-2x}$. That is, find c_0, c_1, c_2, c_3 in the expression $g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$.

Solution, method 1. You should know the binomial series

$$(1+u)^{-1/2} \;=\; \sum_{n=0}^{\infty} \binom{-1/2}{n} u^n .$$

Substituting $u = -2x$ yields

$$(1-2x)^{-1/2} \;=\; \sum_{n=0}^{\infty} \binom{-1/2}{n} (-2)^n x^n$$

$$=\; 1 - 2\binom{-1/2}{1} x + 4\binom{-1/2}{2} x^2 - 8\binom{-1/2}{3} x^3 + \cdots .$$

Now compute the first few values of

$$\binom{-1/2}{n} \;=\; \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!} \;=\; \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n! 2^n} .$$

We get

$$c_0 = 1, \quad c_1 = \frac{(-\frac{1}{2})}{1!}(-2), \quad c_2 = \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-2)^2, \quad c_3 = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-2)^3$$

and thus

$$(1-2x)^{-1/2} = \boxed{1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \cdots}.$$

A common error was to overlook the -2 in the problem, and instead compute something like

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

That equation is true, but it's the correct solution to the wrong problem (look carefully at the left side of the equation). I'd give 5 points for the right side of that equation, but generally there were some additional arithmetic mistakes, yielding only an approximation to that right side, worth fewer points.

Solution, method 2.

n	$g^{(n)}(x)$	$g^{(n)}(0)$	$n!$	$c_n = g^{(n)}(0)/n!$
0	$(1-2x)^{-1/2}$	1	1	1
1	$(1-2x)^{-3/2}$	1	1	1
2	$3(1-2x)^{-5/2}$	3	2	$3/2$
3	$15(1-2x)^{-7/2}$	15	6	$5/2$

and thus

$$\boxed{c_0 = 1, \qquad c_1 = 1, \qquad c_2 = \frac{3}{2}, \qquad c_3 = 5/2}$$

(8 points) Sketch the cardioid $r = 1 + \sin\theta$. Then find the Cartesian coordinates (i.e., xy coordinates) of the three points where the tangent to the curve is horizontal.

Solution. Sketch is as on page 679 of your textbook (worth 2 points) – it's an upside-down heart; I won't try to reproduce it here. And from the picture it's easy to see that one of the horizontal points is at $(0,2)$; that's worth another 2 points. As for other two geometric points (and the remaining 4 scoring points) –

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}\,.$$

For that to vanish, we must have $dy/d\theta = 0$. (That's only a necessary condition, not a sufficient one, since it's possible for $dx/d\theta$ to be zero at the same point.) Compute

$$x = r\cos\theta = \cos\theta + \cos\theta\sin\theta, \qquad y = r\sin\theta = \sin\theta + \sin^2\theta$$

$$dy/d\theta = \cos\theta + 2\sin\theta\cos\theta = (\cos\theta)(1+2\sin\theta)$$

Thus, we need $(\cos\theta)(1+2\sin\theta) = 0$. That requires $\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$. For $\cos\theta = 0$, the solution at $\theta = 3\pi/2$ is extraneous, but the solution at $\theta = \pi/2$ yields $\sin\theta = 1$ and $(x,y) = \boxed{(0,2)}$.

For $\sin\theta = -\frac{1}{2}$, we have $\theta = -\pi/6$ or $-5\pi/6$ (worth 2 points), or, if you prefer, $\theta = 7\pi/6$ or $11\pi/6$ (worth 2 points). That gives us we have $\cos\theta = \pm\frac{1}{2}\sqrt{3}$, hence the two solutions $\boxed{(\pm\frac{1}{4}\sqrt{3},\,-\frac{1}{4})}$.

(8 points) The curve $r = \sqrt{\cos 2\theta}$ is shaped like the symbol "∞" and is called a **lemniscate**. Calculate the area it encloses. *Hint*: You'll need to figure out the angle where the curve goes through the origin.

Solution. The curve goes through the origin when $r = 0$, which is when 2θ is an odd number times $\pi/2$ – i.e., when θ is $\pm\pi/4$, $\pm 3\pi/4$, $\pm 5\pi/4$, etc. We'll find the area of the right half of the figure, and then multiply by 2. The area of the right half is

$$\int_{-\pi/4}^{\pi/4} \frac{1}{2}r^2\,d\theta \;=\; \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos 2\theta\,d\theta \quad \text{(worth 5 points)}$$

$$= \left[\frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2} \quad \text{(worth 7 points)}$$

and so the area of both halves is $\boxed{1}$.

(8 points) The three-leaved rose $r = \cos 3\theta$ and the circle $r = -\cos\theta$ intersect in the origin and in two other points. Sketch the figures, and find the Cartesian (xy) coordinates for those two other points. *Hint:* $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ is a trigonometric identity (i.e., true for all θ).

Solution. For a picture of the 3-leaved rose, see the back of our textbook, answer to section 11.4 problem 13. (That's actually a picture of $r = 2\cos 3\theta$, so it's the same shape but twice as big.) The circle is $(x+\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$, which is to the left of the y-axis, and tangent to the y-axis at the origin. Correct picture is worth 3 points.

At places where the two radii are equal, we get

$$\begin{aligned} \cos 3\theta &= -\cos\theta \\ 4\cos^3\theta - 3\cos\theta &= -\cos\theta, \\ 0 = 4\cos^3\theta - 2\cos\theta &= 4(\cos\theta)(\cos^2\theta - \frac{1}{2}), \end{aligned}$$

so either $\cos\theta = 0$ (which gives us the point at the origin) or $\cos\theta = \pm\frac{1}{2}\sqrt{2}$ (worth 2 points). With the latter case, we have θ equal to an odd multiple of $\pi/4$, and $\sin\theta = \pm\frac{1}{2}\sqrt{2}$ as well, and $r = \pm\frac{1}{2}\sqrt{2}$ too, so x and y are both $\pm\frac{1}{2}$. But two of those points are extraneous, since the circle which has no points in the right halfplane. Thus we get the answers $\boxed{(-\frac{1}{2}, \pm\frac{1}{2})}$. The problem asked for Cartesian coordinates, but I'll give 7 points for the correct polar coordinates: $r = \frac{1}{2}\sqrt{2}$ and $\theta = \pm 3\pi/4$.

(We've now found as many points as we were told there are, so we don't need to look any further. But if we weren't given that hint, we could now also check whether there are any points of intersection where one r is minus one times the other r, and the two angles differ by π. That is, $\cos 3\theta = \cos(\theta + \pi)$. But that just reduces to the same equation, $\cos 3\theta = -\cos\theta$, so it yields no additional solutions.)

(6 points) Set up and simplify, but do not evaluate, an integral that expresses the arclength of the curve $r = \theta$ $(0 \leq \theta \leq 1)$.

Solution. The formula at the bottom of textbook page 688 gives us $L = \boxed{\int_0^1 \sqrt{\theta^2 + 1}\, d\theta}$.

(In case you're interested, we can evaluate that by the substitution $\theta = \tan v$, with $d\theta = \sec^2 v\, dv$. Then the integral becomes $\int_0^{\pi/4} \sec^3 v\, dv$, which can be carried out as in page 500 example 8. Or see formula 21 in the table of integrals in the back inside cover of the textbook.)

Total number of points is 50 out of 50.
