Name: **ANSWER KEY**
Math 155B Test 3, Thurs 3 Nov 2011, 4 pages, 50 points, 75 minutes.

Class results –
Median score, 34/50 = 68%
Mean score, 34.06/50 = 68.12%
High score, 50/50 (1 person)

Note: In this answer key, I have tried to show far more than what a fully knowledgeable student would *write* — I have also tried to show what the student would *think*. As you will see, most of the solutions involve relatively little actual computation, and so if you knew what you were doing, this could all be done with relatively little writing.

Except where otherwise indicated, answer the problem by **circling** the appropriate UNDERLINED, CAPITALIZED word or phrase. Partial credit on wrong answers may in some cases be given for additional work shown.

(5 points) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n}{1+e^n}$ is DIVERGENT or ABSOLUTELY CONVERGENT or CONDITIONALLY CONVERGENT.

Solution. $\boxed{\text{ABSOLUTELY CONVERGENT}}$, because $\displaystyle\sum_{n=0}^{\infty} \left|\frac{\cos n}{1+e^n}\right| \leq \sum_{n=0}^{\infty} \frac{1}{1+e^n}$ $\displaystyle\leq \sum_{n=0}^{\infty} \frac{1}{e^n}$ which is a convergent geometric series.

Some students tried to apply the alternating series test, but that is inapplicable for two reasons. First, the sign of $\cos n$ is not alternating — rather, it goes

$\pm$ in a somewhat erratic fashion. Indeed, the first few values are

n	$\cos n$
0	$1.00\cdots$
1	$0.54\cdots$
2	$-0.41\cdots$
3	$-0.98\cdots$
4	$-0.65\cdots$
5	$0.28\cdots$

(Perhaps you were thinking of $\cos(n\pi)$, which *is* alternating in sign.) And second, even though e^n is increasing, we don't always have $|\cos n|/(1+e^n)$ decreasing. Indeed, once in a while $\cos n$ is near 0, and then we're going to have

$$\frac{|\cos(n+1)|}{1+e^{n+1}} \quad > \quad \frac{|\cos(n)|}{1+e^n}\,.$$

For example,

$$\cos(11) = 0.0044\cdots \qquad e^{11} \approx 5.98\times 10^4 \qquad \frac{|\cos(11)|}{1+e^{11}} \approx 7.39\times 10^{-8}$$

$$\cos(12) = -.843\cdots \qquad e^{12} \approx 1.62\times 10^5 \qquad \frac{|\cos(12)|}{1+e^{12}} \approx 5.18\times 10^{-6}$$

and the latter is much bigger.

(6 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt[3]{n}}$ is DIVERGENT or ABSOLUTELY CONVERGENT or CONDITIONALLY CONVERGENT.

Solution. $\boxed{\text{CONDITIONALLY CONVERGENT}}$.

Some students noted that $\lim_{n\to\infty}(\ln n)/\sqrt[3]{n} = 0$; that by itself is worth 2 points.

For n sufficiently large we have

$$\ln n > 1, \qquad \text{hence} \quad \frac{\ln n}{\sqrt[3]{n}} > \frac{1}{\sqrt[3]{n}},$$

and $\sum n^{-1/3}$ is divergent by the p-test with $p = 1/3$. Some students reasoned that far correctly, and then incorrectly decided "divergent" is the answer to the problem — I gave 2 points if they showed that reasoning. But the conclusion we actually reach from the computation above is that the given series is not absolutely convergent.

Is it conditionally convergent? It's an alternating series, so we just have to show that the absolute values of the terms, $(\ln n)/\sqrt[3]{n}$, are decreasing to 0 when n gets large enough. We know that they are positive (for $n > 1$) and they are tending to 0 in the limit, since $\ln n$ grows more slowly than any power of n. Are they monotone?

Here is a computation-free way to see that they are: Let $g(x) = (\ln x)/\sqrt[3]{x}$. Without actually computing $g'(x)$, think in very general terms about what kind of function it is. It might be a mess — we don't really want to compute it — but we can see that it involves nothing except a few powers of x and perhaps a few $\ln x$ terms. It doesn't involve anything periodic, like $\sin x$. And so there might conceivably be a few places where $g'(x) = 0$, but there are only finitely many of them. Once x gets past those, $g'(x)$ stays positive or stays negative, so $g(x)$ is going to be monotone from then on.

If you don't find that argument convincing, go ahead and compute $g'(x) = (1-\frac{1}{3}\ln x)x^{-4/3}$. That's negative for all $x > e^3$, so the sequence $g(n)$ is decreasing for sufficiently large n. That fact by itself was worth 3 points.

(6 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{4/3}}$ is <u>DIVERGENT</u> or <u>ABSOLUTELY CONVERGENT</u> or <u>CONDITIONALLY CONVERGENT</u>.

Solution. For $n > e$ the terms are positive and decreasing, so we can use the integral test — i.e., analyze $\int_3^\infty x^{-1}(\ln x)^{-4/3}\,dx$; getting this far was worth 2 points. Probably the best approach is to substitute $u = \ln x$ and $du = \frac{1}{x}dx$; that yields $\int_{\ln 3}^\infty u^{-4/3}\,du$ — worth 4 points, if you get this far correctly. That integral converges, so the given series is convergent (worth 5 points). Since its terms are positive, it is $\boxed{\text{ABSOLUTELY CONVERGENT}}$ (6 points).

Some students tried working with $\int_1^\infty x^{-1}(\ln x)^{-4/3}\,dx = \int_0^\infty u^{-4/3}\,du$ – i.e., with the wrong lower limit of integration – but that integral diverges, giving the wrong answer.

(6 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ is <u>DIVERGENT</u> or <u>ABSOLUTELY CONVERGENT</u> or <u>CONDITIONALLY CONVERGENT</u>. *Hint:* First find an expression v such that $(\ln n)^{\ln n} = n^v$.

Solution. I'm surprised that some students answered "conditionally convergent." That can't happen in a series whose terms are all positive. Evidently those students did not understand the definitions.

Take ln on both sides of the hint equation, to get $(\ln n)(\ln \ln n) = v \ln n$, and therefore $v = \ln \ln n$ (worth 2 points). The reason for the hint is that the problem is now transformed to deciding whether $\sum \frac{1}{n^v}$ is convergent. Of course, v is not a constant, so the p-test is not directly applicable. But v is *getting larger*, and we can make use of that fact. Indeed, for all n sufficiently large we have $v > 2$, and therefore $\frac{1}{n^v} < \frac{1}{n^2}$, and we know $\sum \frac{1}{n^2}$ is convergent. Thus the given series is $\boxed{\text{ABSOLUTELY CONVERGENT}}$.

(In case you're wondering, "for all n sufficiently large" in this instance means for all $n > e^{e^2}$, but we don't actually need to compute that.)

(5 points) For what value of x is $\displaystyle\sum_{n=0}^{\infty} 2^{nx}$ equal to 5 ?

Solution. We have a geometric series, $a + ar + ar^2 + ar^3 + \cdots = a/(1-r)$. In this case $a = 1$ and $r = 2^x$, so we have the equation $1/(1-2^x) = 5$; getting that far was worth 3 points. Solving algebraically yields $2^x = 4/5$, which is worth 4 points. For full credit, then, $x = \boxed{\log_2(4/5)}$. That answer can also be written in a few other forms, such as $\boxed{2 - \log_2 5}$ or $\boxed{\frac{\ln(4/5)}{\ln 2}}$ or $\boxed{2 - \frac{\ln 5}{\ln 2}}$.

(5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n\sqrt{n+2}}{n^5+3n+7}$ is <u>DIVERGENT</u> or <u>ABSOLUTELY CONVERGENT</u> or <u>CONDITIONALLY CONVERGENT</u>.

Solution. $\boxed{\text{ABSOLUTELY CONVERGENT}}$. Use the limit comparison test, comparing with

$$\sum_{n=1}^{\infty} \frac{n\sqrt{n}}{n^5} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^{7/2}}$$

which converges by the p-test, since $7/2 > 1$.

Most got this one right. I couldn't understand the work of the two students who wrote "divergent." But I was more puzzled by the four students who wrote either "conditionally convergent" or just "divergent" — they evidently did not understand the definitions. No series of positive terms can ever be conditionally convergent. If it is convergent at all, then it is absolutely convergent.

(5 points) $\displaystyle\sum_{n=1}^{\infty} \sin\left(n^2\right) \sin\left(\frac{1}{n^2}\right)$ is <u>DIVERGENT</u> or <u>ABSOLUTELY CONVERGENT</u> or <u>CONDITIONALLY CONVERGENT</u>.

Solution. We first analyze the two ingredients separately:

behavior of $\sin\left(n^2\right)$	behavior of $\sin\left(n^{-2}\right)$
$\sin\left(n^2\right)$ does *not* behave like n^2. Rather, it oscillates between positive and negative, in a somewhat erratic fashion, not a very regular pattern. And $\lvert\sin\left(n^2\right)\rvert$ is not monotone. So we're not going to be able to use the alternating series test. But in any case we always have $\left\lvert\sin\left(n^2\right)\right\rvert \le 1$, which is worth 2 points by itself.	$\sin\left(n^{-2}\right)$ behaves much like n^{-2}. It is a sequence of positive terms converging to 0, at about the same rate as n^{-2} is — that is, $\lim_{n\to\infty} \sin\left(n^{-2}\right)/n^{-2} = 1$ (worth 2 points), since $\lim_{x\to 0}\sin(x)/x = 1$. And we know $\sum n^{-2}$ is convergent (p-series with p=2). Hence $\sum \sin\left(n^{-2}\right)$ is convergent (worth 3 points).

Then

$$\sum_{n=1}^{\infty} \lvert\sin(n^2)\sin(1/n^2)\rvert \quad \le \quad \sum_{n=1}^{\infty} \sin(1/n^2) \quad = \quad \text{convergent},$$

and so $\sum \sin(n^2)\sin(1/n^2)$ is $\boxed{\text{ABSOLUTELY CONVERGENT}}$.

(6 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \tan\left(\frac{1}{\sqrt{n}}\right)$ is DIVERGENT or ABSOLUTELY CONVERGENT or CONDITIONALLY CONVERGENT.

Solution. $\boxed{\text{CONDITIONALLY CONVERGENT}}$.

Partial credit for wrong answers was based on how much was shown of the following two parts to this problem; each part was worth 3 points.

Use the alternating series test to show that the series is convergent (and thus it is either absolutely or conditionally convergent).	Show that $\sum_{n=1}^{\infty} \tan\left(\frac{1}{\sqrt{n}}\right)$ is divergent, and therefore the series given in the problem is not absolutely convergent.
Proof or computation – It's clear that as n increases to ∞, then $1/\sqrt{n}$ decreases to 0, and therefore the numbers $\tan(1/\sqrt{n})$ are positive numbers decreasing to 0.	Proof or computation – For small x, we know that $\cos x$ is near 1 and $\sin x$ behaves like x; hence $\tan x$ also behaves like x. That is, $\lim_{x\to 0} \frac{\tan x}{x} = 1$. Therefore, by the limit comparison test, $\sum \tan(1/\sqrt{n})$ has the same convergence behavior as $\sum 1/\sqrt{n}$, which diverges by the p-test with $p = 1/2$.

Some students tried to determine whether $\sum_{n=1}^{\infty} \tan\left(\frac{1}{\sqrt{n}}\right)$ converges by figuring that it has the same behavior as $\int_1^{\infty} \tan(1/\sqrt{x})\,dx$. That's correct reasoning, but a difficult integral to work with. That integral turns out to be divergent, but the only ways I've found so far for showing that are by comparing $\tan(1/\sqrt{x})$ with $1/\sqrt{x}$.

(6 points) Find the sum (i.e., a number) for $\dfrac{1}{5} + \dfrac{1}{21} + \dfrac{1}{45} + \dfrac{1}{77} + \cdots$

Solution. Begin by factoring: the given series is

$$\frac{1}{1\cdot 5}+\frac{1}{3\cdot 7}+\frac{1}{5\cdot 9}+\frac{1}{7\cdot 11}+\cdots$$

– I gave 2 points for noticing that much of a pattern. You can analyze* this like examples covered in class; this yields the telescoping series

$$\frac{1}{4}\left\{\left(1-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{7}-\frac{1}{11}\right)+\cdots\right\}$$

All the terms cancel out except $\frac{1}{4}\left\{1+\frac{1}{3}\right\} \quad = \quad \boxed{\frac{1}{3}}$. Some students misplaced the factor of 1/4, and came up with an answer of 4/3 or 16/3, for either of which I gave 4 points.

If you need more explanation of the analysis* step, here it is:

$$\frac{1}{1\cdot 5}+\frac{1}{3\cdot 7}+\frac{1}{5\cdot 9}+\frac{1}{7\cdot 11}+\cdots \quad = \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+3)}$$

(3 points for getting that far) and you can use the method of partial fractions to decompose

$$\frac{1}{(2x-1)(2x+3)} \quad = \quad \frac{1/4}{2x-1}-\frac{1/4}{2x+3} \quad = \quad \frac{1}{4}\left(\frac{1}{2x-1}-\frac{1}{2x+3}\right)$$

(4 points for getting that far).

Some students decided this had to be a geometric series, whose first two terms are 1/5 and 1/21. If that were so, the common ratio would be $r = 5/21$, and so the next term in the series would be 25/441. With $a = 1/5$, the total would then be

$$\frac{a}{1-r} \quad = \quad \frac{1/5}{1-(5/21)} \quad = \quad \frac{1/5}{16/21} \quad = \quad \frac{21}{80},$$

an answer for which I gave 2 points — or 1 point for answers attempting to arrive at that.